Coupled oscillators: symmetries, dynamics and dead zones

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Oscillator networks and weak chimeras

We will consider systems of N coupled phase oscillators described as an ODE on the torus $\theta \in \mathbb{T}^N = [0, 2\pi)^N$:

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} A_{ij} g(\theta_i - \theta_j) \tag{1}$$

where A_{ij} is the strength of coupling, ω_i is the natural frequency of the ith oscillator and $g(\varphi)$ is a smooth 2π-periodic coupling function.

Chimera states have been described in various ways:

- "an array of identical oscillators splits into two domains: one coherent and phase locked, the other incoherent and desynchronized" [Abrams and Strogatz]
- " some fraction of the oscillators perfectly synchronized, while the remainder are desynchronized" [Laing]
- "two coexisting subpopulations, one with synchronized oscillations and the other with unsynchronized oscillations, even though all of the oscillators are coupled to each other in an equivalent manner" [Tinsley et al]
- "a hybrid spatial structure, partially coherent and partially incoherent, which can develop in networks of identical oscillators without any sign of inhomogeneity." [Omelchenko et al]
- *(add your own favourite definition from last week here)*

Small chimera questions

- Q0 What exactly is a chimera state?
- Q1 What are the limits on how small a network can be to have chimeras?
- Q2 Are there limits on the stability of chimeras in small networks?

We say oscillators i and j on a trajectory of the system (1) are *frequency synchronized* if

$$\Omega_{ij} := \lim_{T\to\infty} \frac{1}{T}[\theta_i(T) - \theta_j(T)] = 0.$$

We say $A \subset \mathbb{T}^N$ is a weak chimera state for a coupled indistinguishable phase oscillator system if it is a connected chain-recurrent flow-invariant set such that on each trajectory within A there are i, j and k such that $\Omega_{ij} \neq 0$ and $\Omega_{ik} = 0$.

(Franke & Selgrade (1976) show that any ω-limit set of a flow is flow-invariant, connected and chain-recurrent.)

A, Burylko [2015]

Theorem

For global coupling of N identical phase oscillators with $A_{ij} = K$, all trajectories of (1) are frequency synchronized. Hence no weak chimera states are possible in such a system, for any N or $g(\varphi)$.

Chimeras can be found in globally coupled systems of higher dimension.

Modular network examples

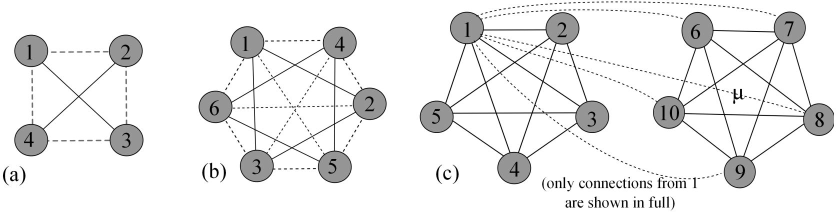


Figure: Example modular networks of (a) four, (b) six and (c) ten indistinguishable oscillators that permit robust weak chimera states.

A four oscillator example

$$\begin{aligned}
\dot{\theta}_1 &= \omega + (g(\theta_1 - \theta_3) + g(0)) + \epsilon(g(\theta_1 - \theta_2) + g(\theta_1 - \theta_4)) \\
\dot{\theta}_2 &= \omega + (g(\theta_2 - \theta_4) + g(0)) + \epsilon(g(\theta_2 - \theta_3) + g(\theta_2 - \theta_1)) \\
\dot{\theta}_3 &= \omega + (g(\theta_3 - \theta_1) + g(0)) + \epsilon(g(\theta_3 - \theta_2) + g(\theta_3 - \theta_4)) \\
\dot{\theta}_4 &= \omega + (g(\theta_4 - \theta_2) + g(0)) + \epsilon(g(\theta_4 - \theta_1) + g(\theta_4 - \theta_3))
\end{aligned} \tag{2}$$

For this system and a particular coupling function $g(\varphi)$ considered by Hansel, Mato and Meunier [1991]:

$$\begin{aligned}
g(\varphi) &:= -\sin(\varphi - \alpha) + r\sin(2\varphi) \\
&= \cos(\varphi + \beta) + r\sin(2\varphi)
\end{aligned} \tag{3}$$

where $\alpha := \pi/2 - \beta$.

Theorem

For Hansel-Mato-Meunier coupling (3) there is an open set of (r, α) such that the four-oscillator system (2) has an attracting weak chimera state for $\epsilon = 0$ that persists for all ϵ with $|\epsilon|$ sufficiently small.

Weak chimeras for a six oscillator network: existence and stability

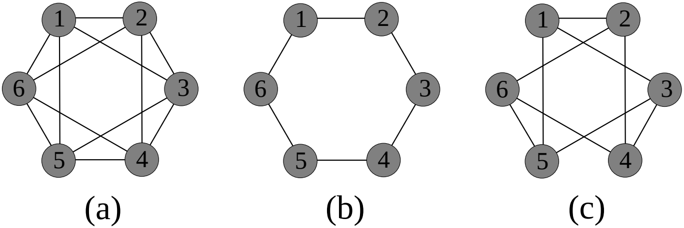


Figure: (a) Six oscillators with nearest and next-nearest neighbour coupling. (b) Six oscillators with nearest neighbour coupling only. (c) Six oscillator system with three inputs to each oscillator; each of these networks has six indistinguishable oscillators and supports weak chimera states.

Consider the system

$$\frac{d\theta_i}{dt} = \omega + \sum_{|j-i|=1,2} g(\theta_i - \theta_j). \tag{4}$$

for $i = 1, \ldots, 6$ where indices are considered modulo $N = 6$. For coupling (3) this supports a number of weak chimera solutions

- A, Burylko [2015]: numerical exploration
- Mary Thoubaan, PhD thesis [2018]: existence and stability

Subspace Σ	Typical point $(\theta_1, \dots, \theta_6)$	Dim	Reduced system
$\mathbb{D}_6$	(a, a, a, a, a, a)	1	
$\mathbb{D}_6^-$	$(a, a+\pi, a, a+\pi, a, a+\pi)$	1	
$\mathbb{Z}_6^1$	$(a, a+\zeta, a+2\zeta, a+3\zeta, a+4\zeta, a+5\zeta)$	1	
$\mathbb{Z}_6^2$	$(a, a+2\zeta, a+4\zeta, a, a+2\zeta, a+4\zeta)$	1	
$\mathbb{D}_3$	(a, b, a, b, a, b)	2	
$\mathbb{Z}_3$	$(a, b, a+2\zeta, b+2\zeta, a+4\zeta, b+4\zeta)$	2	
$\mathbb{D}_2$	(a, b, a, a, b, a)	2	
$\mathbb{D}_2^-$	$(a, b, a, a+\pi, b+\pi, a+\pi)$	2	
$\mathbb{Z}_2^1$	(a, b, c, a, b, c)	3	I
$\mathbb{Z}_2^2$	$(a, b, c, a+\pi, b+\pi, c+\pi)$	3	II
A_0	(a, b, c, a, d, e)	5	
A_1	(a, b, c, a, c, b)	3	III
A_2	(a, b, b, a, c, c)	3	III
A_3	$(a, b, c, a+\pi, c+\pi, b+\pi)$	3	IV
A_4	$(a, b, b+\pi, a+\pi, c+\pi, c)$	3	IV
A_5	$(a, a+\pi, b, a, a+\pi, b)$	2	
A_6	$(a, a+\pi, b, a, a+\pi, b+\pi)$	2	
A_7	$(a, a+\pi, b, a+\pi, a, b)$	2	

Table: Invariant subspaces for the six oscillator system (a) where $\zeta := \pi/2$ and a, b, c, d, e, f are arbitrary phases.

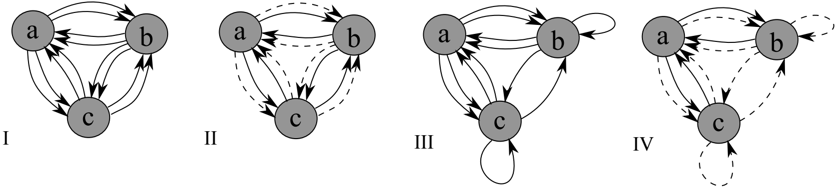


Figure: Three-cell quotient networks of the network (a). Dashed arrows indicate an input that includes a phase shift of the phase by π. Note that I, II have a quotient symmetry of $\mathbb{D}_3$. III, IV have only $\mathbb{Z}_2$ symmetry but nonetheless fully synchronized solutions.

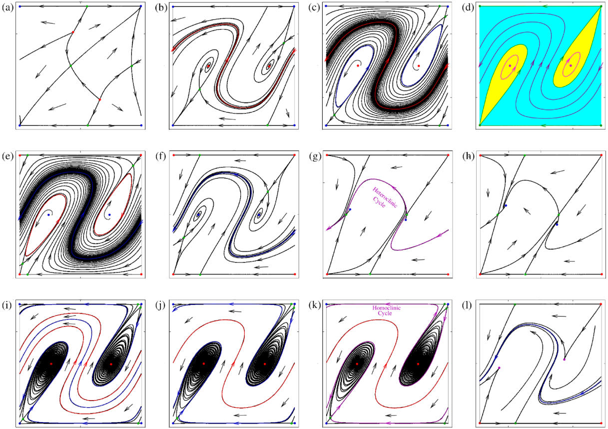
Reduction to dynamics in A_1: set

$$\xi = \phi_1 - \phi_3,\ \ \eta = \phi_2 - \phi_3,\ \ \ \xi - \eta = \phi_1 - \phi_2$$

and write in terms of phase differences:

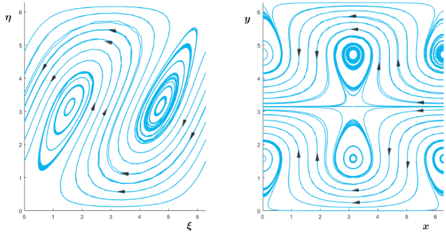
$$\begin{aligned} \dot{\xi} &= 2g(\xi - \eta) + 2g(\xi) - 2g(-\xi) - g(-\eta) - g(0) \\ \dot{\eta} &= 2g(\eta - \xi) + g(\eta) - 2g(-\xi) - g(-\eta). \end{aligned} \tag{5}$$

Phase portraits in A_1 for $\xi, \eta \in [0, 2\pi)$ plane. (a) $r = 0$, $\alpha = 0.5$, (b) $r = 0$, $\alpha = 1.3$, (c) $r = 0$, $\alpha = 1.5$, (d) $r = 0$, $\alpha = \pi/2$, (e) $r = 0$, $\alpha = 1.64$, (f) $r = 0$, $\alpha = 1.84$, (g) $r = 0$, $\alpha = 2.16205$, (h) $r = 0$, $\alpha = 2.22$, (i) $r = -0.01$, $\alpha = 1.561$, (j) $r = -0.01$, $\alpha = 1.558$, (k) $r = -0.01$, $\alpha = 1.5517$, (l) $r = -0.01$, $\alpha = 1.97794$.



Integrability and persistence of solutions for a six oscillator system

Changing to coordinates x, y such that $\xi = x + y$, $\eta = 2y$ gives a more convenient way to represent the system on A_1.



We use $\beta = \pi/2 - \alpha$.

In these coordinates:

$$
\begin{aligned}
\dot{x} &= 24r\sin x\cos x\cos^2 y - 6\sin x\cos y\sin\beta \\
&+ 2\cos x\cos y\cos\beta - 12r\sin x\cos x - 2\cos\beta\cos^2 y, \\
\dot{y} &= 2\sin y(4r\cos^2 x\cos y + 4r\cos^3 y + \sin x\cos\beta \\
&- \cos x\sin\beta - \cos y\sin\beta - 4r\cos y).
\end{aligned} \tag{6}
$$

There is an integrable structure in the invariant subspace A_1 for the special case $r = \beta = 0$: we use this to prove existence of weak chimeras.

For $r = \beta = 0$ we have

$$\begin{aligned} \dot{x} &= 2\cos x \cos y - 2\cos^2 y, \\ \dot{y} &= 2\sin y \sin x. \end{aligned} \tag{7}$$

Lemma

This system within A_1 has an integral of motion

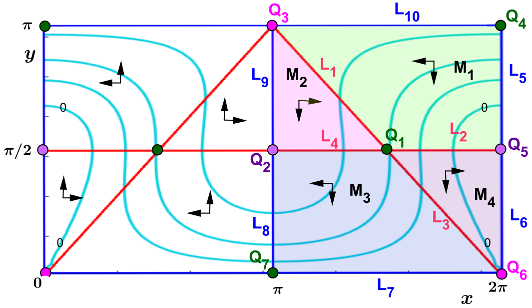
$$E(x, y) := y + \cos y \sin y - 2\sin y \cos x. \tag{8}$$

for $r = \beta = 0$.

Level curves

$$C(E_0) = \{(x, y) \ : \ E(x, y) = E_0\}$$

as preserved by the flow for $r = \beta = 0$:



Centres at Q_2, Q_5. Degenerate saddles at Q_6, Q_3.

We describe the motion of trajectories by considering monotonicity and limiting behaviour of trajectories on M_1, M_2, M_3 and M_4.

Lemma

- *For any* $0 < E_0 < \pi$ *there is an initial condition* $(x(0), y(0)) \in C(E_0)$ *and a* $T = T(E_0) > 0$ *such that if* $(x(t), y(t))$ *is a trajectory of the system on* A_1 *for* $\beta = r = 0$ *then* $x(T) = x(0) - 2\pi$ *and* $y(T) = y(0)$.
- *For* $E_0 = 0$ *or* π *then* $C(E_0)$ *consist of the nonhyperbolic saddle* Q_6 *or* Q_3, *and homoclinic orbits to these saddles.*

Lemma

For any $0 < E_0 < \pi$, *then the level curve* $C(E_0)$ *of the system on* A_1 *for* $\beta = r = 0$ *contains a trajectory* $(x, y) \in \mathbb{R}^2$ *such that* $x(T) = x(0) - 2\pi$ *and* $y(T) = y(0)$ *for some* $T = T(E_0) > 0$. *More precisely, if* $(x, y) \in C(E_0)$ *then*

$$\lim_{t\to\infty} \frac{1}{t}\, y(t) = 0 \qquad \textit{and} \qquad \lim_{t\to\infty} \frac{1}{t}\, x(t) = \frac{2\pi}{T} \neq 0.$$

This can be used to show:

Theorem

The system (4, 3) of six oscillators with $\beta = r = 0$ *has an infinite number of chimera states within* A_1 *that are neutrally stable.*

The period of the integrable chimera solution $T(E_0)$ can be computed as

$$\int_{t=0}^{T(E_0)} dt \quad = \quad 2\int_{t=0}^{T(E_0)/2} \left(\frac{dy}{dt}\right)^{-1} \ dy. \tag{9}$$

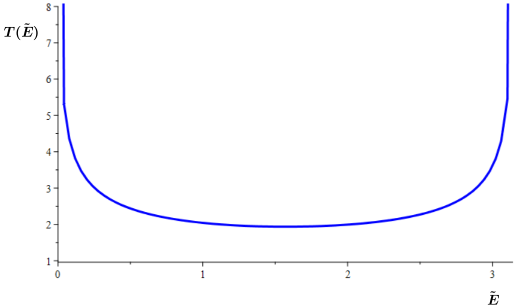
Note that for $0 < y < \pi$ and $0 < E_0 < \pi$ there is a unique x such that $E(x, y) = E_0$, namely:

$$x = \Delta_{E_0}(y) := 2\pi - \arccos\left[\frac{\cos y \sin y + y - E_0}{2\sin y}\right] \tag{10}$$

Writing $x = \Delta_{E_0}(y)$ and changing coordinates gives

$$T(E_0) = 2\int_{y_{min}(E_0)}^{y_{max}(E_0)} \frac{1}{\sqrt{4\sin(y)^2 - (\cos(y)\sin(y) + y - E_0)^2}}\, dy \qquad (11)$$

where $y_{min}(E_0)$ and $y_{max}(E_0)$ are upper and lower limits of the level curve E_0.



Period $T(E_0)$ of the weak chimera solution for $E_0 \in (0, \pi)$ in the integrable case $\beta = r = 0$: the period tends to infinity as the level curve approaches the heteroclinic orbits at $E_0 = 0$ and π

Weak chimera chimera solutions near integrability

To understand the near integrable case, consider a Poincaré section

$$\Sigma_p = \{(x,y) \in \mathbb{T}^2 : x = 2\pi \text{ and } y \in (0,\pi)\}.$$

parametrized by E_0 and define a first return map $\tilde{P} : (0,\pi) \to (0,\pi)$,

$$E_{n+1} = \tilde{P}(E_n). \tag{12}$$

For $r = \beta = 0$ note that each $0 < E_0 < \pi$ is a fixed point with return time $T(E_0)$.

We consider $(r, \beta) = \epsilon(\tilde{r}, \tilde{\beta})$ which gives a near integrable system for $0 < \epsilon \ll 1$.

We parametrize the dynamics by $(x, y) = (\Delta_{\tilde{E}}(y), y)$: if $E_0 \in (0, \pi)$ and $(x(0), y(0)) = (2\pi, y_{max}(E_0))$ then for small t we have

$$\frac{dE}{dt}(\Delta_{\tilde{E}(t)}(y(t)), y) = \epsilon[G_{E_0}(y)\tilde{\beta} + F_{E_0}(y)\tilde{r}] + O(\epsilon^2). \tag{13}$$

The next intersection is at $(0, y_{max}(E_1))$ after time $T = T(E_0) + O(\epsilon)$ and

$$\tilde{P}(E_0) \quad = \quad E_0 + \epsilon\Lambda(E_0)) + O(\epsilon^2)$$

where

$$\Lambda(E_0) = -2\epsilon(\Lambda_1(E_0)\tilde{\beta} + \Lambda_2(E_0)\tilde{r})$$

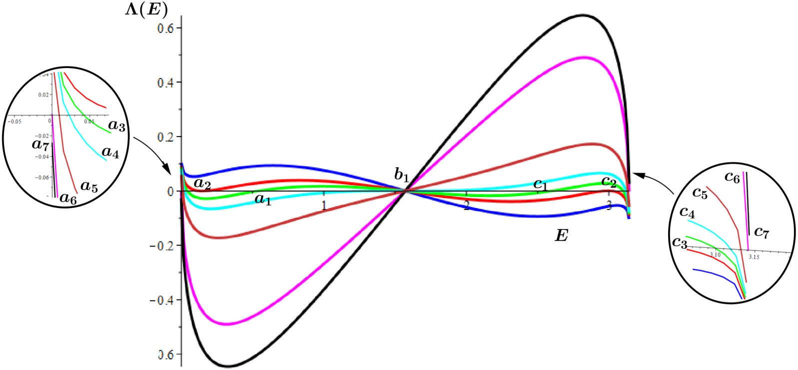
for some integrals Λ_1 and Λ_2 depending only on E_0.

There is a symmetric weak chimera state near $E_0 = \pi/2$ and $(\beta, r) \neq (0, 0)$:

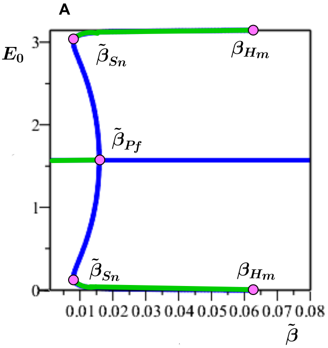
Theorem

For almost all $(\tilde{\beta}, \tilde{r})$, if ϵ is small enough then $(\beta, r) = (\epsilon\tilde{\beta}, \epsilon\tilde{r})$ has a weak chimera periodic orbit that is close to the level curve $C(\pi/2)$.

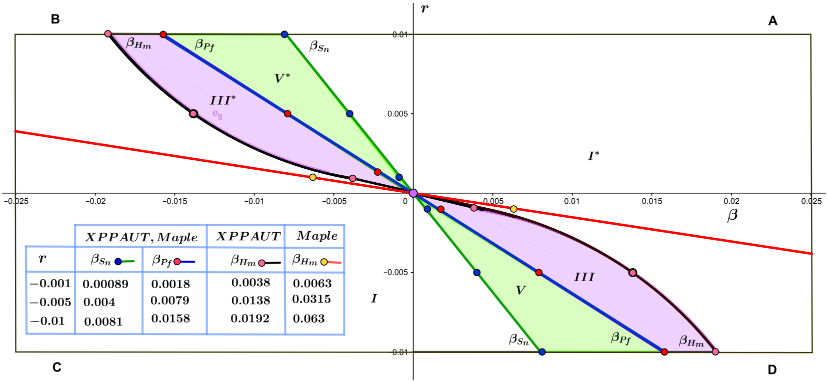
Graph of $\Lambda(E) = -2[\Lambda_1(E)\tilde{\beta} + \Lambda_2(E)\tilde{r}]$ for various values of $\tilde{\beta}$ and $\tilde{r} = -0.01$. Zeros correspond to fixed points of the approximate Poincaré map:



Bifurcation digrams (A) $\tilde{\beta}$ against E_0 for system within A_1 when $\tilde{r} = -0.01$ approximated (using Maple) from limit Poincare map for $\epsilon \to 0$, (B) β against y computed numerically (using XPPAUT) for $r = -0.01$.

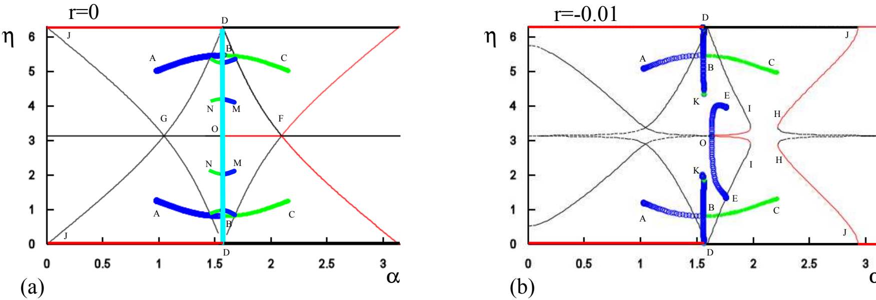


Bifurcation curves for chimeras in the parameter space (β, r) close to the integrable case $(0, 0)$

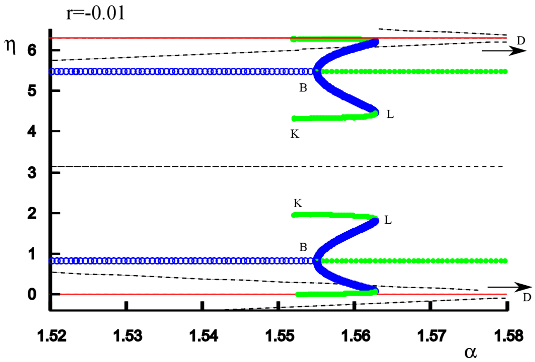


	$XPPAUT, Maple$		$XPPAUT$	$Maple$
r	β_{Sn}	β_{Pf}	β_{Hm}	β_{Hm}
-0.001	0.00089	0.0018	0.0038	0.0063
-0.005	0.004	0.0079	0.0138	0.0315
-0.01	0.0081	0.0158	0.0192	0.063

Bifurcation diagram within A_1 for (a) $r = 0$ and (b) $r = -0.01$. Red: stable equilibria, black: unstable equilibria. Green/blue/cyan lines: periodic orbits.



Close-up of branches for $r = -0.01$.



Other weak chimeras for the six-oscillator system

There are other weak chimeras in this six-oscillator system, within the invariant subspace

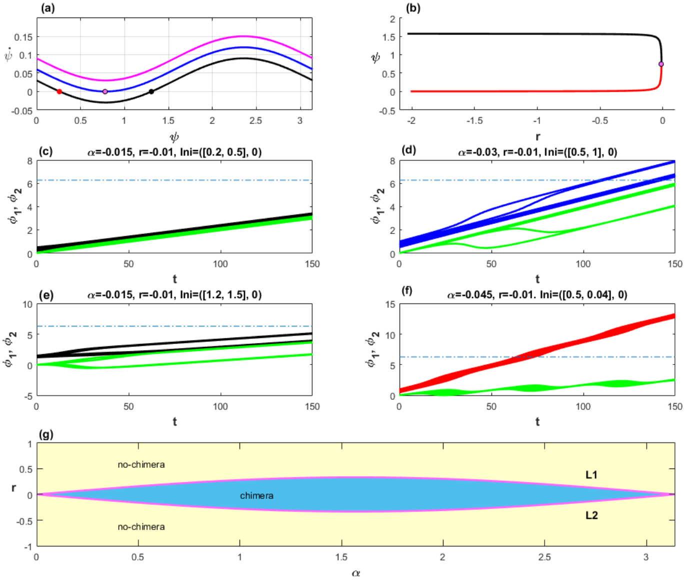
$$A_6 = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6) = (\phi_1, \phi_1 + \pi, \phi_2, \phi_1, \phi_1 + \pi, \phi_2 + \pi).$$

In these coordinates we have

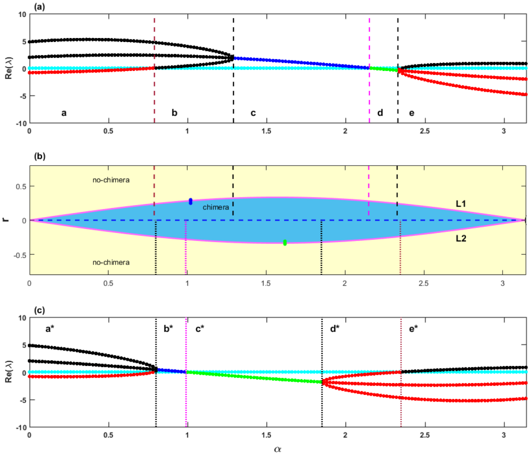
$$\begin{aligned} \dot{\phi}_1 &= w - 2\sin(\alpha) + 2r\sin(2\phi_1 - 2\phi_2), \\ \dot{\phi}_2 &= w - 4r\sin(2\phi_1 - 2\phi_2), \end{aligned} \tag{14}$$

and in term of phase difference $\psi := \phi_1 - \phi_2$ we have

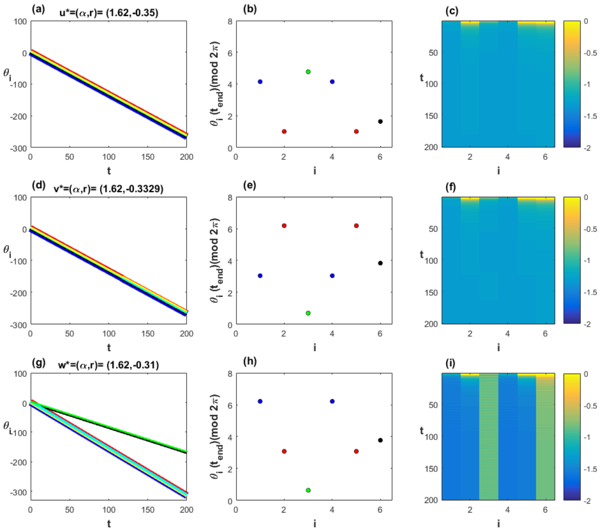
$$\dot{\psi} = -2\sin(\alpha) + 6\ r\sin(2\psi). \tag{15}$$


(a)
(b)
(c) α=-0.015, r=-0.01, Ini=([0.2, 0.5], 0)
(d) α=-0.03, r=-0.01, Ini=([0.5, 1], 0)
(e) α=-0.015, r=-0.01, Ini=([1.2, 1.5], 0)
(f) α=-0.045, r=-0.01. Ini=([0.5, 0.04], 0)
(g)
no-chimera
chimera
no-chimera
L1
L2
ψ
r
t
α
φ1, φ2

Eigenvalues (a,c) and bifurcation diagram (b) for weak chimeras in A_6.



Weak chimeras within A_6 at parameter point w^*.



Dead zones for phase oscillators

Suppose that $\theta_k \in \mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$ for $k \in \{1, \ldots, N\}$ evolves according to

$$\dot{\theta}_k = \omega + \sum_{j=1}^{N} A_{jk} \mathrm{g}(\theta_j - \theta_k), \tag{16}$$

where ω is the fixed intrinsic frequency of all oscillators, $A_{jk} \in \{0, 1\}$ gives the coupling topology between oscillators (we assume $A_{kk} = 0$), and the (non-constant) *coupling function* $\mathrm{g} : \mathbb{T} \to \mathbb{R}$ determines how the oscillators influence each other.
The adjacency matrix (A_{jk})— defines a *structural network graph* **A** encodes these connections.

[A, Bick, Poignard, arXiv:1904.00626]

Suppose the coupling function g has *dead zones*, i.e., if it is zero over some interval of phase differences. In the presence of dead zones, we will define an *effective coupling graph* of (16) as a subgraph of **A**, which encodes the effective interactions between oscillators at a particular point in phase space.

Such coupling will appear in neural systems where "pulsatile coupling" hits a "refactory zone".

We mostly restrict (16) to the case where the coupling is all-to-all (and thus fully symmetric), i.e. $A_{kj} = 1$ for all $j \neq k$, and the phase $\theta_k \in \mathbb{T}$ evolves according to

$$\dot{\theta}_k = \omega + \sum_{j=1, j\neq k}^{N} \mathrm{g}(\theta_j - \theta_k) \tag{17}$$

for $k = 1, \ldots, N$.

Some dead zone questions:

Q0: Given any subgraph of the structural network graph, is there a coupling function such that this subgraph is *realised* as the effective coupling graph for some point in the phase space?

Q1: What is the relation between the coupling function, the set of possible subgraphs that can be realised, and the points where these realisations happen?

Q2: How do the dynamics and effective couplings influence each other?

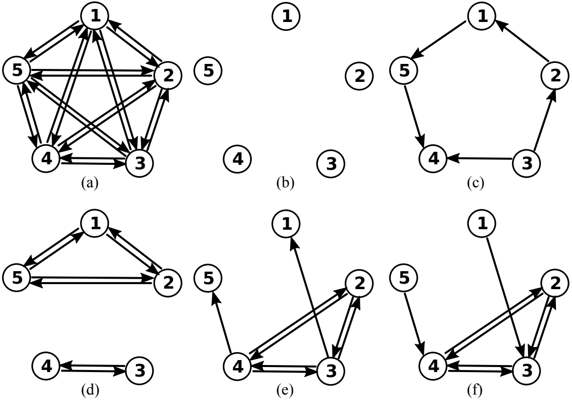


Figure: (a) Coupling for the graph $\mathbf{K}_5$ corresponding to the fully connected network (17) with $N = 5$. (b-f) show five examples of the $2^{5\times 4} = 1048576$ possible embedded subgraphs of (a), i.e., having the same number of nodes as (a): all of these can be realised as effective coupling graphs for a coupling function g with dead zones. Panels (b) and (d) shows graphs with more than one component: (b) the "empty" graph with no edges, (c) a cycle of length 5, (d,e,f) have nontrivial structure. While (e) and (f) are similar, only (e) can be realised in a dynamically stable manner as it contains a spanning diverging tree.

Let $\mathbf{G}$ be a graph with $V(\mathbf{G}) = V_N$ and let $\mathbb{S}_N$ be the symmetric group of all permutations of $V_N = \{1, \ldots, N\}$. The *automorphisms of* $\mathbf{G}$, denoted by

$$\Gamma(\mathbf{G}) = \left\{ \gamma \in \mathbb{S}_N \;\middle|\; A_{\gamma(k)\gamma(j)} = A_{jk} \text{ for all } j, k \in V_N \right\},$$

form a subgroup of $\mathbb{S}_N$ under composition. Define the set of embedded subgraphs

$$\mathcal{H}(\mathbf{G}) = \{ \mathbf{H} = (V_N, E') \mid \mathbf{H} \subset \mathbf{G} \}$$

and write $\mathcal{H}_N = \mathcal{H}(\mathbf{K}_N)$. Note that the group $\Gamma(\mathbf{G})$ naturally acts on $\mathcal{H}(\mathbf{G})$: For $\mathbf{H} \in \mathcal{H}(\mathbf{G})$ and $\gamma \in \Gamma$ the image $\gamma\mathbf{H}$ is the graph with vertices V_N and edges

$$E(\gamma\mathbf{H}) = \{ (\gamma j, \gamma k) \mid (j, k) \in E(\mathbf{H}) \}$$

for $\gamma \in \Gamma$. For this action, the *isotropy group of the graph* $\mathbf{H} \subset \mathbf{G}$ is

$$\Sigma_{\mathbf{H}} = \{ \gamma \in \Gamma \mid \gamma\mathbf{H} = \mathbf{H} \}.$$

Let the group $\mathbb{S}_N$ act on $\mathbb{T}^N$ by permuting components. Consider some $\mathbf{G} \in \mathcal{H}_N$ and let $\Gamma = \Gamma(\mathbf{G})$ be automorphisms of $\mathbf{G}$. For $\Sigma \subset \Gamma \subset \mathbb{S}_N$ we define the *fixed point space* $\mathrm{Fix}(\Sigma) = \{\, \theta \in \mathbb{T}^N \mid \gamma(\theta) = \theta \text{ for all } \gamma \in \Sigma \,\}$. For a given $\theta \in \mathbb{T}^N$, the *isotropy subgroup* of θ is the group action is $\Sigma_\theta = \{\, \gamma \in \Gamma \mid \gamma(\theta) = \theta \,\}$.

Note that (16) is equivariant with respect to the action of $\Gamma \times \mathbb{T}$ ia permutation of the oscillators and phase shifts

$$(\theta_1, \ldots \theta_N) \mapsto (\theta_1 + \phi, \ldots \theta_N + \phi). \tag{18}$$

The fixed point space of any isotropy subgroup of $\Gamma \times \mathbb{T}$ is dynamically invariant. It is often useful to consider behaviour of (16) in terms of the group orbits of $\mathbb{T}$.

The quotient by $\mathbb{T}$ corresponds to considering the dynamics in phase difference coordinates, and relative equilibria (equilibria for the quotient system) typically correspond to periodic orbits for the original system.

The all-to-all coupled oscillator network (17) has structural coupling graph $\mathbf{A} = \mathbf{K}_N$, and is $\Gamma(\mathbf{K}_N) \times \mathbb{T} = \mathbb{S}_N \times \mathbb{T}$ equivariant. In this case, the dynamics on the full phase space $\mathbb{T}^N$ are completely determined by the dynamics on the *canonical invariant region (CIR)*

$$\mathcal{C} = \{ \theta = (\theta_1, \ldots, \theta_N) \mid \theta_1 < \theta_2 < \cdots < \theta_N < 2\pi \} . \tag{19}$$

The full synchrony and splay phase configurations

$$\Theta^{\mathrm{sync}} = (\phi, \ldots, \phi), \qquad \Theta^{\mathrm{splay}} = \left(\phi, \phi + \frac{2\pi}{N}, \ldots, \phi + \frac{(N-1)2\pi}{N}\right) \in \mathcal{C}$$

are relative equilibria of the dynamics. There is a residual action of $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$ on the canonical invariant region and Θ^{splay} is the fixed point of this action.

Definition

Suppose that $\mathrm{g} : \mathbb{T} \to \mathbb{R}$ is a smooth 2π-periodic function.

- A coupling function g is *locally constant at* $\theta^0 \in \mathbb{T}$ *with value* $c \in \mathbb{R}$ if there is an open set U with $\theta_0 \in U \subset \mathbb{T}$ such that $\mathrm{g}(U) \equiv c$. Define $\mathrm{LC}(\mathrm{g})$ to be the set of locally constant points of g.
- A coupling functions g is *locally null at* $\theta^0 \in \mathbb{T}$ if it is locally constant with $c = 0$. Let $\mathrm{DZ}(\mathrm{g}) \subset \mathrm{LC}(\mathrm{g})$ denote the set of locally null points of g.
- A coupling function g has *simple dead zones* if $\mathrm{DZ}(\mathrm{g})$ has finitely many connected components and $\mathrm{LC}(\mathrm{g}) = \mathrm{DZ}(\mathrm{g})$, i.e., if there is a finite set of locally constant regions, and all are locally null.
- Let g be a coupling function with simple dead zones. Any connected component of $\mathrm{DZ}(\mathrm{g})$ is a *dead zone* of g. Connected components of the complements $\mathrm{LZ}(\mathrm{g}) = \mathbb{T} \setminus \mathrm{DZ}(\mathrm{g})$ are *interaction* or *live zones*.

Here, we will only consider the case of *simple dead zones*.

Definition

The *effective coupling graph* $\mathcal{G}_{\mathrm{g}}(\theta)$ of (16) with coupling function g at $\theta \in \mathbb{T}^N$ is the graph on N vertices with edges

$$E(\mathcal{G}_{\mathrm{g}}(\theta)) = \{\, (j,k) \mid A_{jk} \neq 0 \text{ and } \theta_j - \theta_k \notin \mathrm{DZ}(\mathrm{g}) \,\}\,.$$

Conversely, an edge $(j,k) \notin E(\mathcal{G}_{\mathrm{g}}(\theta))$ if $A_{jk} = 0$ (the edge is not contained in $\mathbf{A}$) or $\theta_j - \theta_k \in \mathrm{DZ}(\mathrm{g})$ (the phase difference is in a dead zone).

Note that for the special case (17) the edges of the effective coupling graph are simply given by $E(\mathcal{G}_{\mathrm{g}}(\theta)) = \{\, (j,k) \mid \theta_j - \theta_k \notin \mathrm{DZ}(\mathrm{g}) \,\}$.

Clearly $\mathcal{G}_{\mathrm{g}}(\theta) \subset \mathbf{A} \subset \mathbf{K}_N$, and this will be a proper subgraph (that is, it differs from $\mathbf{A}$ by at least one edge) for some $\theta \in \mathbb{T}^N$ if g has at least one dead zone. For the system (16) with coupling function g and given $\mathbf{H} \subset \mathbf{K}_N$, define

$$\Theta_{\mathrm{g}}(\mathbf{H}) = \left\{ \theta \in \mathbb{T}^N \mid \mathcal{G}_{\mathrm{g}}(\theta) = \mathbf{H} \right\}. \tag{20}$$

Definition

If $\Theta_{\mathrm{g}}(\mathbf{H})$ is not empty, then $\mathbf{H}$ is *realised as an effective coupling graph for* (16) *with coupling function* g. Moreover, a graph $\mathbf{H}$ *can be realised as an effective coupling graph* if there exists a coupling function g for which $\Theta_{\mathrm{g}}(\mathbf{H})$ is not empty.

For particular structural network graphs $\mathbf{A}$ of (16) there are a large number of symmetries, i.e., the automorphism group $\Gamma(\mathbf{A})$ may be large. At the same time, $\Gamma(\mathbf{A})$ acts on the underlying phase space.

We now show how the symmetry of a point $\theta \in \mathbb{T}^N$ relates to the symmetries of the effective coupling graph at θ.

Lemma

Consider the system (16) *with structural network graph* $\mathbf{A}$ *and any coupling function* g. *For any* $\theta \in \mathbb{T}^N$, *we have* $\mathcal{G}_{\mathrm{g}}(\gamma\theta) = \gamma\mathcal{G}_{\mathrm{g}}(\theta)$ *for all* $\gamma \in \Gamma(\mathbf{A})$.

Corollary

Consider the system (16) *with structural network graph* $\mathbf{A}$ *and any coupling function* g. *For any* $\theta \in \mathbb{T}^N$, *we have*

$$\Sigma_\theta \subset \Sigma_{\mathcal{G}_{\mathrm{g}}(\theta)} \subset \Gamma(\mathbf{A}).$$

Proof.

To see this, note that if $\gamma \in \Sigma_\theta$ then $\gamma\theta = \theta$ and so $\mathcal{G}_{\mathrm{g}}(\theta) = \mathcal{G}_{\mathrm{g}}(\gamma\theta) = \gamma\mathcal{G}_{\mathrm{g}}(\theta)$ which implies that $\gamma \in \Sigma_{\mathcal{G}_{\mathrm{g}}(\theta)}$. $\square$

Note that the reverse containment of Corollary 1 does not necessarily hold, for example if there are no dead zones (i.e., if $\mathrm{DZ}(\mathrm{g})$ is empty) then clearly $\Sigma_\theta = \Gamma(\mathbf{A})$ for all θ.

We consider two special cases of **Q1**:

Q1a Given a point $\theta \in \mathbb{T}^N$ and a graph $\mathbf{H} \in \mathcal{H}_N$, is there a coupling function g such that $\mathcal{G}_{\mathrm{g}}(\theta) = \mathbf{H}$?

Q1b Is there a coupling function that realises all graphs for appropriate choice of θ? That is, is there a g such that $\mathcal{G}_{\mathrm{g}}(\mathbb{T}^N) = \mathcal{H}_N$?

For almost all θ, the answer to **Q1a**, while answer to **Q1b** is "yes".

One can also consider what possible effective coupling graphs will be realised for a coupling function g: this is important if we wish to understand the dynamics of (17) with a fixed coupling function.

Restrictions on the effective coupling graph

Given $\theta \in \mathbb{T}^N$, what do the properties of θ impose on the effective coupling graphs of (17)? The isotropy of $\theta \in \mathbb{T}^N$ has some important consequences on the possible effective coupling graphs realised at θ:

Proposition

Consider the system all-to-all coupled oscillator network (17) *with coupling function* g*. Assume that* $\theta \in \mathcal{C} \subset \mathbb{T}^N$*:*

1. (i) *If* θ *has isotropy* Σ_θ *then* $\mathcal{G}_\mathrm{g}(\theta)$ *must have at least the same isotropy.*
2. (ii) *For full synchrony* $\Theta^{\mathrm{sync}} = (a, \ldots, a)$ *we have* $\mathcal{G}_\mathrm{g}(\Theta^{\mathrm{sync}}) \in \{\emptyset_N, \mathbf{K}_N\}$*.*
3. (iii) *Suppose there exists* $0 < a < 2\pi/N$ *such that* $\theta_{k+1} - \theta_k = a$ *for any* $k \in \{1, \ldots, N-1\}$*. Then one of the following cases occurs:*
   1. *The directed path* $\mathbf{P}_{N,N-1,\ldots,1}$ *is a subgraph of* $\mathcal{G}_\mathrm{g}(\theta)$ *but* $\mathbf{P}_{1,2,\ldots,N}$ *is not.*
   2. *The directed path* $\mathbf{P}_{1,2,\ldots,N}$ *is a subgraph of* $\mathcal{G}_\mathrm{g}(\theta)$ *but* $\mathbf{P}_{N,N-1,\ldots,1}$ *is not.*
   3. *The undirected path* $\bar{\mathbf{P}}_{1,2,\ldots,N}$ *is a subgraph of* $\mathcal{G}_\mathrm{g}(\theta)$*.*
   4. $\mathcal{G}_\mathrm{g}(\theta)$ *is a n-partite graph (with* $n = [N/2]$ *if* N *is even or* $n = [N/2] + 1$ *if not).*

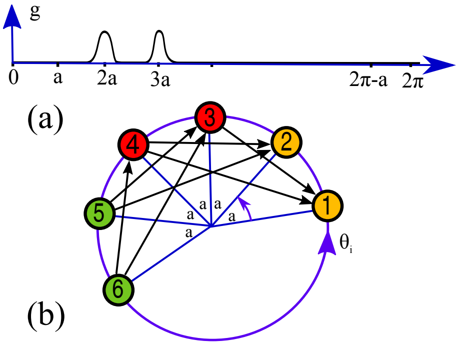


Figure: Illustration of the case (4)(iii) for $N = 6$ oscillators: the coupling function g shown in Panel (a) has two live zones centered at $2a$ and $3a$, the remainder consists of two dead zones. The diagram in Panel (b) shows the phases θ_k at one instant in time such that $\theta_j - \theta_i = a(j - i)$ for all $j > i$. This coupling graph is tripartite as indicated by the node colouring.

Proposition

For a generic choice of $\theta \in \mathbb{T}^N$*, and for any subgraph* $\mathbf{H} \in \mathcal{H}_N$*, there exists a coupling function* g *such that* $\mathcal{G}_{\mathrm{g}}(\theta) = \mathbf{H}$*.*

Corollary

There exists a coupling function g *for* (17) *such that for any subgraph* $\mathbf{H} \in \mathcal{H}_N$ *we have* $\mathcal{G}_{\mathrm{g}}(\theta) = \mathbf{H}$ *for some* θ*.*

Coupling functions for an interaction graph

- Given a coupling function g, which properties of g imply certain effective coupling graphs realised by g?
- Given $\theta \in \mathcal{C}$ and $\mathbf{H}$, how can one construct a coupling function g such that $\mathbf{H} = \mathcal{G}_{\mathrm{g}}(\theta)$?

Clearly, the number of dead zones plays a major role in these questions, since it constrains the resulting effective coupling graphs.

Definition

Let $n \in \mathbb{N}$. We denote by $\mathcal{F}(n)$ the set of coupling functions having n dead zones.

Note that if there are $n > 1$ dead zones there must also be n live zones, while for $n = 1$ there can be 0 or 1 live zones, and for $n = 0$ there is necessarily one live zone.

Proposition

Consider system (17) with coupling function g.

1. (i) *The coupling function* g *is dead zone symmetric if and only if all effective coupling graphs for* g *are undirected.*
2. (ii) *Assume that* $\mathrm{g} \in \mathcal{F}(1)$ *is dead zone symmetric with* $\mathrm{LZ}(\mathrm{g}) = [-a, a]$. *If* $a < 2\pi/N$, *then for any* $1 \leq k \leq N$ *and any sequence* $k, \ldots, k+p$ *in* $\{1, \ldots, N\}$ *we have that* $\emptyset_N, \mathbf{K}_N$ *and the embeddings of* $\bar{\mathbf{P}}_{k,\ldots,k+p}$ *and* $\mathbf{K}_{k,\ldots,k+p}$ *can be realised as effective coupling graphs for* g. *If* $a = 2\pi/N$, *then* $\mathbf{K}_N, \bar{\mathbf{P}}_{1,\ldots,N}, \bar{\mathbf{C}}_{1,\ldots,N}$, *and the embeddings of graphs* $\bar{\mathbf{P}}_{k,\ldots,k+p}$ *and* $\mathbf{K}_{k,\ldots,k+p}$ *can be realised as effective coupling graphs for* g.
3. (iii) *Assume that* $\mathrm{g} \in \mathcal{F}(1)$ *is dead zone symmetric with* $\mathrm{LZ}(\mathrm{g}) = [\pi - a, \pi + a]$ *and* $a \leq 2\pi/N$. *Then* $\emptyset_N$ *and* $\mathbf{K}_N$ *can be realised as effective coupling graphs for* g.

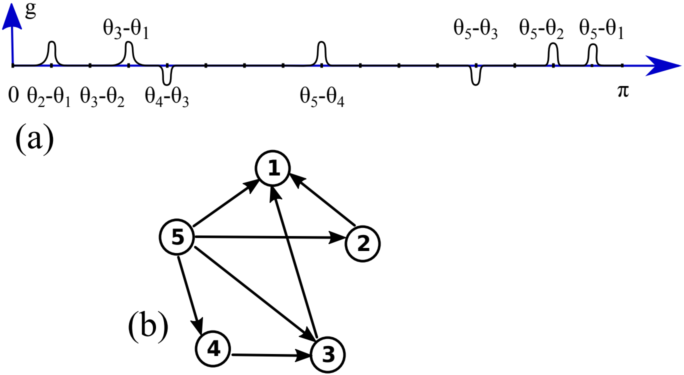


Figure: An example of directed graph (b) within $\mathbf{K}_5$ with 7 edges realised as an effective coupling graph with a coupling function g in $\mathcal{F}(7)$

Effective coupling and dynamic stability

We say a graph $\mathbf{H}$ can be *stably realised* if there is an asymptotically stable invariant open set $\mathcal{A}$ with $\mathcal{A} \subset \Theta_{\mathrm{g}}(\mathbf{H})$.

Proposition

For any $\mathbf{H} \in \mathcal{H}_N$ *admitting a spanning diverging tree, there is a coupling function* g *such that the oscillator network* (17) *has a locally asymptotically stable relative equilibrium* $(\Omega t + \theta_1^o, \ldots, \Omega t + \theta_N^o)$ *satisfying* $\mathcal{G}_{\mathrm{g}}(\theta^o) = \mathbf{H}$.

In other words, in the all-to-all coupled case, for any $\mathbf{H}$ there exists a coupling function g that stably realises $\mathbf{H}$.

Proof (Sketch).

- Choose a θ^o with trivial isotropy and set dead zones such that $\mathcal{G}_{\mathrm{g}}(\theta^0) = \mathbf{H}$ for all g with these dead zones.
- Choose values of $\mathrm{g}(\theta_i^o - \theta_j^o)$ in the live zones such that $(\Omega t + \theta_1^o, \ldots, \Omega t + \theta_N^o)$ is a relative equilibrium.
- Show that all eigenvalues of θ^o can be made negative by suitable choice of $\mathrm{g}'(\theta_i^o - \theta_j^o)$.

□

Uses:

Proposition (Agaev et al 2009)

Let $\mathbf{H}$ *be a graph admitting a spanning diverging tree. Consider the Laplacian matrix* $L^{\mathbf{H}}$ *with coefficients*

$$L^{\mathbf{H}}_{jk} = \begin{cases} -A^{\mathbf{H}}_{jk} & \text{if } j \neq k, \\ \sum_{\ell=1,\ell\neq k}^{N} A^{\mathbf{H}}_{\ell k} & \text{if } k = j. \end{cases}$$

Then the multiplicity of the eigenvalue 0 *in the spectrum of* $L^{\mathbf{H}}$ *is one.*

Corollary

Assume that $\mathbf{H} \in \mathcal{H}(\mathbf{A})$ *admits a spanning diverging tree. Then there is a coupling function* g *such that* (16) *has an asymptotically stable relative equilibrium* $(\Omega t + \theta_1^\circ, \ldots, \Omega t + \theta_N^\circ)$ *satisfying* $\mathcal{G}_{\mathrm{g}}(\theta^\circ) = \mathbf{H}$*.*

In other words, also in the non-symmetric case, for any $\mathbf{H}$ there exists a coupling function g that stably realises $\mathbf{H}$.

Effective coupling graphs for networks of two and three oscillators

Two oscillators

One can easily demonstrate that a single dead zone and a single live zone is sufficient to realise all effective coupling graphs for (17) with $N = 2$ oscillators. More precisely, choose any $\mathrm{g} \in \mathcal{F}(1)$ with $\mathrm{LZ}(\mathrm{g}) = [-a, 2a]$ for $a < \pi/2$, where all inequalities are understood in the interval $[-\pi, \pi]$. Then

$$\mathcal{G}_{\mathrm{g}}(0, c) = \begin{cases} \mathbf{K}_2 & \text{if } c \in (-a, a), \\ \mathbf{P}_{1,2} & \text{if } c \in (a, 2a), \\ \mathbf{P}_{2,1} & \text{if } c \in (-2a, -a), \\ \emptyset_2 & \text{if } c \in (-\pi, -2a) \cup (2a, \pi). \end{cases}$$

This shows that there is a single coupling function that realises all four subgraphs of $\mathbf{K}_2$. Note that if g is dead zone symmetric then only the undirected graph $\mathbf{K}_2$ and $\emptyset_2$ can be realised.

Three oscillators

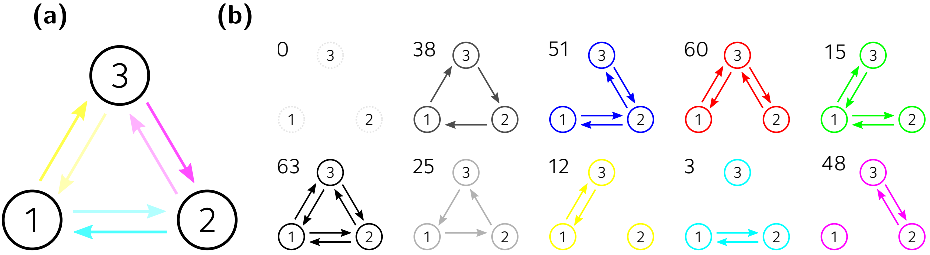


Figure: We use a colour scheme to identify the graphs in $\mathcal{H}_3$. Panel (a) shows the shades of cyan, magenta, and yellow identified with each directed edge of $\mathbf{K}_3$. If multiple edges are present, the colours are added. Examples of graphs $\mathbf{H} \in \mathcal{H}_3$ in their associated colours, as well as the corresponding graph numbers $\nu(\mathbf{H})$,are shown in Panel (b). The subgraphs where all edges to/from a given node are present (and no others) are associated with the colours red, green, and blue. Any symmetry that permutes the three nodes acts on the colour scheme by permuting the colour channels. Note that white corresponds to $\emptyset_3$, black to $\mathbf{K}_3$, and shades of gray for the directed cycles $\mathbf{C}_{1,2,3}, \mathbf{C}_{3,2,1}$.

Define the *graph number*

$$\nu(\mathbf{H}) = A^{\mathbf{H}}_{12} + 2A^{\mathbf{H}}_{21} + 4A^{\mathbf{H}}_{13} + 8A^{\mathbf{H}}_{31} + 16A^{\mathbf{H}}_{23} + 32A^{\mathbf{H}}_{32} \in \{0, \ldots, 63\}\,, \qquad (21)$$

which uniquely encodes the realised effective coupling graph as an integer. In particular, we have $\nu(\emptyset_3) = 0$ and $\nu(\mathbf{K}_3) = 63$.

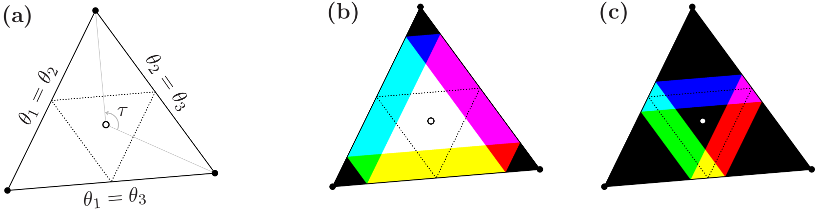


Figure: The sets Θ_{g} partition the canonical invariant region $\mathcal{C}$ for the fully symmetric system of $N = 3$ oscillators. The CIR is sketched in Panel (a): Its boundary is given by the sets $\theta_1 - \theta_2 = 0$, $\theta_2 - \theta_3 = 0$, and $\theta_3 - \theta_1 = 0$ (black lines) which intersect in Θ^{sync} (black dot, $\bullet$). The splay phase Θ^{splay} is the centroid (hollow dot, $\circ$) and is the fixed point of the residual $\mathbb{Z}_3 = \langle\tau\rangle$ symmetry which rotates the CIR (indicated by gray lines). Dashed lines indicate phase configurations where one phase difference is equal to π. For a dead zone symmetric coupling function $\mathrm{g} \in \mathcal{F}(1)$ only the undirected subgraphs of $\mathbf{K}_3$ can be realised. Panel (b) shows the partition of the CIR for $\mathrm{DZ}(\mathrm{g}) = \left(\frac{\pi}{3}, \frac{5\pi}{3}\right)$. Panel (c) shows the partition for a dead zone symmetric coupling function with $\mathrm{DZ}(\mathrm{g}) = \left(\frac{5\pi}{6}, \frac{7\pi}{6}\right)$.

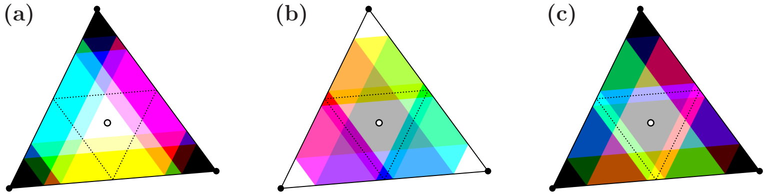


Figure: Many effective coupling graphs are possible for $N = 3$ oscillators and a general coupling function $\mathrm{g} \in \mathcal{F}(1)$ with one dead zone. We have $\mathrm{DZ}(\mathrm{g}) = \left(\frac{\pi}{3}, \frac{3\pi}{2}\right)$ in Panel (a), $\mathrm{DZ}(\mathrm{g}) = \left(-\frac{\pi}{3}, \frac{11\pi}{12}\right)$ in Panel (b) and $\mathrm{DZ}(\mathrm{g}) = \left(\frac{\pi}{3}, \frac{11\pi}{12}\right)$ in Panel (c).

We now look at examples of the system's dynamics and explore how the effective coupling graph changes along trajectories. To this end, we examine the dynamics of (17) with $N = 3$ and the coupling function

$$\mathrm{g}(\psi) = -\sin(\psi + \alpha)h(\psi) \quad \text{where } h(\psi) = \frac{1}{2}\left(\tanh(\varepsilon^{-1}(\cos b - \cos(a - \psi)) + 1\right) \tag{22}$$

for constants $a \in [0, 2\pi)$, $b \in [0, \pi)$, $\varepsilon > 0$ and $\alpha \in [0, 2\pi)$. This coupling function is a modulated Kuramoto–Sakaguchi coupling with phase-shift parameter α.

We call

$$\mathrm{DZ}^{\varepsilon}(\mathrm{g}) = \left\{ \theta \;\middle|\; \left|\theta - a\right| < b \right\} \tag{23}$$

the approximate dead zone of the coupling function (22) since in the limit $\varepsilon \to 0$ the coupling function (22) has a single dead zone $\mathrm{DZ}(\mathrm{g}) = \{ \theta \mid |\theta - a| < b \}$ centred at a of half-width b; here the inequality is to be understood modulo 2π. In the following we fix $\varepsilon = 10^{-2}$ and $\alpha = 1.3$.

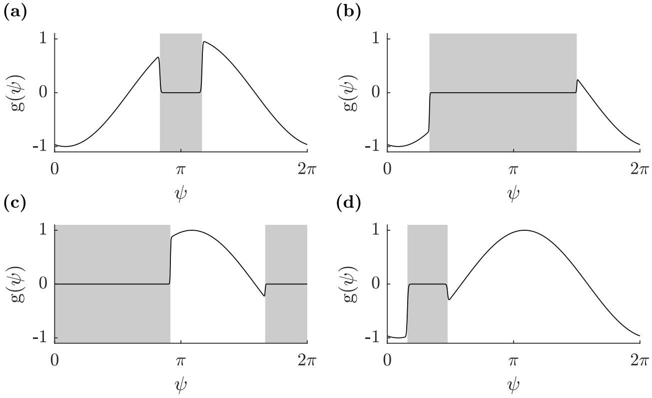
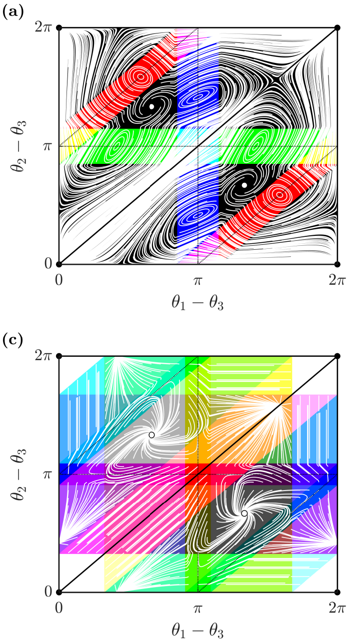


Figure: The coupling functions (22) provide examples of coupling functions $\mathrm{g} \in \mathcal{F}(1)$ with one dead zone; here $\varepsilon = 10^{-3}$ and $\alpha = 1.3$. The shaded area indicates the dead zone of the coupling function. In Panel (a) we have a dead zone symmetric coupling function with $\mathrm{DZ}(\mathrm{g}) \approx \left(\frac{5\pi}{6}, \frac{7\pi}{6}\right)$; In Panel (b) we have $\mathrm{DZ}^{\varepsilon}(\mathrm{g}) = \left(\frac{\pi}{3}, \frac{3\pi}{2}\right)$; In Panel (c) we have $\mathrm{DZ}^{\varepsilon}(\mathrm{g}) = \left(-\frac{\pi}{3}, \frac{11\pi}{12}\right)$In Panel (d) we have $\mathrm{DZ}^{\varepsilon}(\mathrm{g}) = \left(0.5, 1.5\right)$.


(a)
(b)
(c)
(d)
2π
π
0
$\theta_2 - \theta_3$
$\theta_1 - \theta_3$

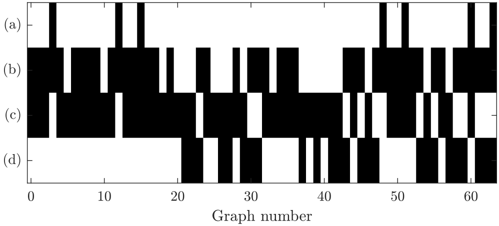


Figure: The possible effective coupling graphs realised using $N = 3$ and (22) for parameters as in Figure 10 and some θ. Black indicates $\Theta_{\mathrm{g}}(\mathbf{H}) \neq \emptyset$ for $\mathbf{H} \in \mathcal{H}_3$ with a given graph number, and white indicates $\Theta_{\mathrm{g}}(\mathbf{H}) = \emptyset$. Since (a) is a dead zone symmetric coupling function, only undirected subgraphs are realised. By contrast, the general coupling functions with one dead zone in (b) and (c) between them together realise all possible subgraphs $\mathbf{H} = \mathcal{G}_{\mathrm{g}}(\theta)$ for some choice of θ.

There several natural questions that relate to the number, location and lengths of the dead zones to the set of realisable effective coupling graphs. For example, the coupling functions (b,c) above together can realise all possible (embedded) subgraphs of $\mathbf{K}_3$. Two specific questions in this direction are:

- What is the minimum number of dead zones $n = n(N)$ such that there is a $\mathrm{g} \in \mathcal{F}(n)$ that realises all $\mathbf{H} \in \mathcal{H}_N$?
- For any $\ell < n(N)$, what is the minimum m such that there exists $\{\mathrm{g}_1, \ldots, \mathrm{g}_m\} \subset \mathcal{F}(\ell)$ between them realise any given $\mathbf{H} \in \mathcal{H}_N$?

Discussion

Summary:

- A definition for weak chimera states and identification of weak chimera states small networks of oscillators
- Six oscillators: Proof of existence of continuum of weak chimeras for $r = \beta = 0$. Some understanding of persistence and bifurcations nearby.
- "Dead zones" can modulate the effective coupling and dynamics in a nontrivial manner.

Further questions:

- Scaling of weak chimeras to chimeras in the continuum limit.
- Detailed dynamics of weak chimeras in medium-sized networks?
- Implications for effective coupling in applications?

Refs:

- PA, Oleksandr Burylko, Chaos, 2015
- Mary Thoubaan, PA, Chaos, 2018
- PA, Chris Bick, Camille Poignard, arXiv:1904.00626 2019