In between phase and amplitude oscillators

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Outline

Mean-field ensemble of Stuart-Landau oscillators

The oscillators bounded along a (smooth) curve

Phase diagram

Characterization of the collective dynamics

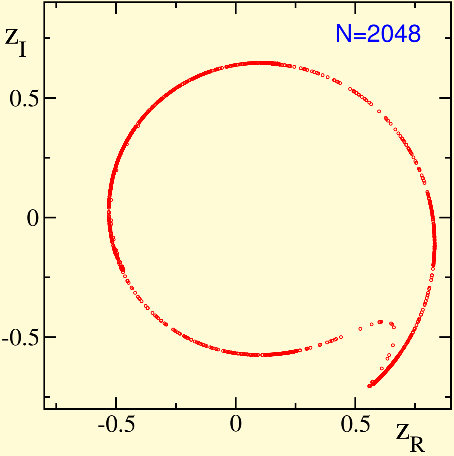
Validation of generality

Globally coupled identical Stuart-Landau oscillators

$$\dot{z}_j = z_j - (1 + ic_2)|z_j|^2 z_j + K(1 + ic_1)(\bar{z} - z_j)$$

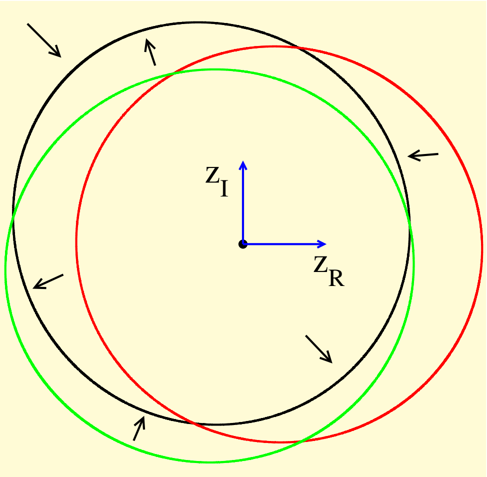
c_1=-2 ; c_2=3 ; K=0.47

$\bar{z} = \frac{1}{N} \sum_{m=1}^{N} z_m$

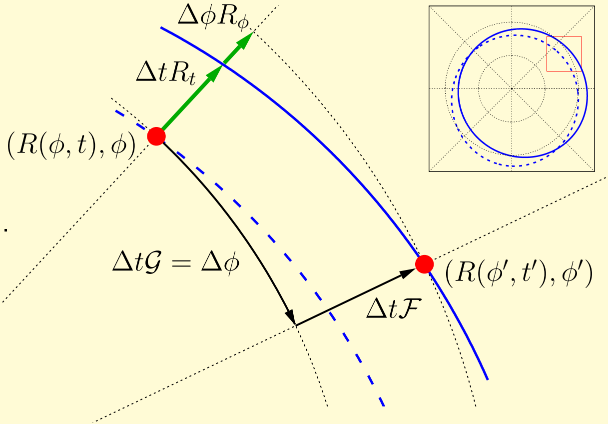


In between phase and amplitude behavior: Stuart-Landau oscillators again

$$\dot{z}_j = z_j - (1 + ic_2)|z_j|^2 z_j + K(1 + ic_1)(\overline{z} - z_j)$$



Phase-space representation



.

$$t' = t + \Delta t$$

$$Q(z,t) = Q(z,t)\delta(|z| - R(\phi,t)) =: P(\phi,t).$$

Polar coordinates

$$\dot{r}_j = \mathcal{F}[r_j, \phi_j, \overline{z}]$$
$$\dot{\phi}_j = \mathcal{G}[r_j, \phi_j, \overline{z}] \, ,$$

$$\mathcal{F}[r, \phi, \overline{z}] = (1 - K - r^2)r + K \,\mathrm{Re}[(1 + ic_1)\overline{z}\, \mathrm{e}^{-i\phi}]$$
$$\mathcal{G}[r, \phi, \overline{z}] = -c_1 K - c_2 r^2 + \frac{K}{r} \,\mathrm{Im}[(1 + ic_1)\overline{z}\, \mathrm{e}^{-i\phi}] \, .$$

General equations

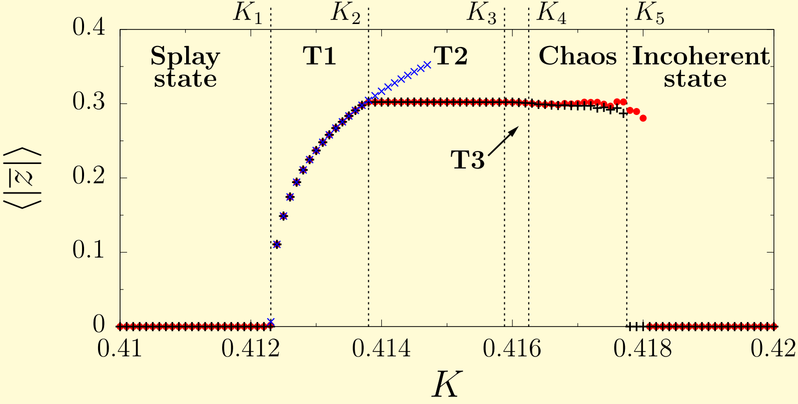
$$\frac{\partial R}{\partial t}(\phi, t) = \mathcal{F}[R, \phi, \overline{z}] - \mathcal{G}[R, \phi, \overline{z}] R_\phi \ .$$

$$\frac{\partial P}{\partial t}(\phi, t) = -\frac{\partial}{\partial \phi}\Big\{ P(\phi, t)\mathcal{G}[R, \phi, \overline{z}] \Big\}.$$

$$\overline{z}(t) = \int_0^{2\pi} P(\phi, t) R(\phi, t) e^{i\phi} d\phi.$$

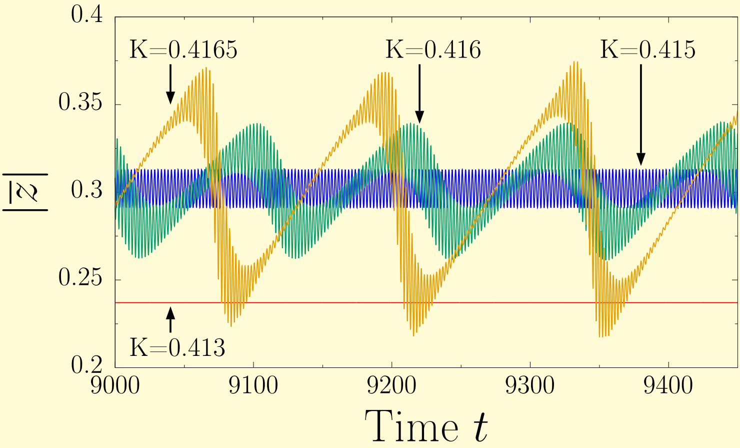
$$\frac{\partial P}{\partial t}(\phi, t) = -\frac{\partial}{\partial \phi}\Big\{ P(\phi, t)\mathcal{G}[R, \phi, \overline{z}] \Big\} + D\frac{\partial^2}{\partial \phi^2} P(\phi, t)$$

Phase diagram (see also Nakagawa & Kuramoto 1993)



$c_1 = -2 \qquad c_2 = 3$

Collective dynamics



Stability analysis of the splay state

$$R = \sqrt{1-K} + v(\phi) \quad ; \quad P = 1/(2\pi) + u(\phi) \quad ; \quad \overline{z} = \overline{w}$$

$$v_t = -2(1-K)v + Av_\phi + K\,\mathrm{Re}[(1+ic_1)\overline{w}\mathrm{e}^{-i\phi}]\,,$$

$$u_t = Au_\phi + \frac{c_2\sqrt{1-K}}{\pi}v_\phi + \frac{K\,\mathrm{Re}[(1+ic_1)\overline{w}\mathrm{e}^{-i\phi}]}{2\pi\sqrt{1-K}}\,,$$

$$A = c_1K + c_2(1-K)$$

$$\overline{w} := \sqrt{1-K}\int_0^{2\pi} d\phi\; u(\phi,t)\,\mathrm{e}^{i\phi} + \frac{1}{2\pi}\int_0^{2\pi} d\phi\; v(\phi,t)\,\mathrm{e}^{i\phi}$$

.... in Fourier space

$$\tilde{v}(k,t) = \int_0^{2\pi} d\phi v(\phi,t)\,\mathrm{e}^{ik\phi} \qquad \text{and}$$

$$\tilde{u}(k,t) = \int_0^{2\pi} d\phi u(\phi,t)\,\mathrm{e}^{ik\phi}\ ,$$

$$[\tilde{v}(k)]_t = [-2(1-K) - ikA]\,\tilde{v}(k)$$

$$[\tilde{u}(k)]_t = -ik\frac{c_2\sqrt{1-K}}{\pi}\tilde{v}(k) - ikA\tilde{u}(k)$$

Two bands of eigenvalues

$k > 1$

$$\lambda_k^{(v)} = -2(1-K) - ik(Kc_1 + c_2(1-K)) \quad \text{and}$$
$$\lambda_k^{(u)} = -ik\left(Kc_1 + c_2(1-K)\right) \ .$$

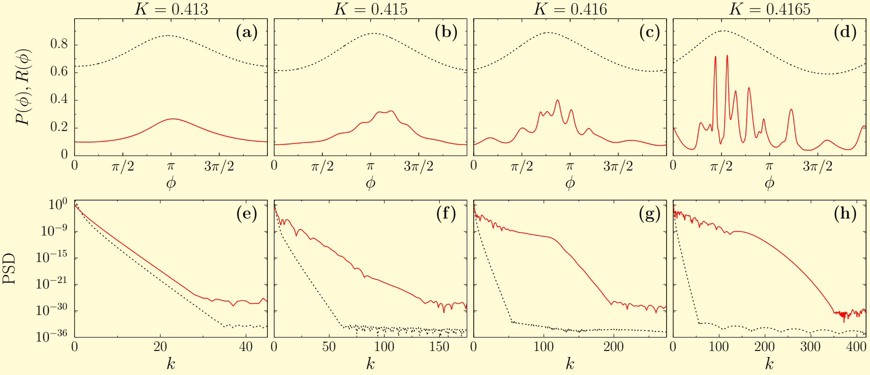
The curve dynamics is stable - probability eigenvalues are marginal

$k = 1$

v and u dynamics are mutually coupled.

$K_1 = (65 - \sqrt{1025})/80 = 0.412304\ldots$

Snapshots



Dashed black = R

Solid red = P

Self-consistent partial synchrony (SCPS)

Consider a rotating frame

$$R_t = \mathcal{F}[R,\theta,\overline{z}] + \{\omega - \mathcal{G}[R,\theta,\overline{z}]\}\, R_\theta$$

$$P_t = \frac{\partial}{\partial\theta}\Big(P(\theta,t)\big\{\omega - \mathcal{G}[R,\theta,\overline{z}]\big\}\Big)$$

$$\int_0^{2\pi} d\theta\; P(\theta,t) R(\theta,t) e^{i\theta} =: \overline{z}\ .$$

and determine the stationary solution

$$[R_0(\theta)]_\theta = -\frac{\mathcal{F}[R_0,\theta,\overline{z}]}{\omega - \mathcal{G}[R_0,\theta,\overline{z}]}$$

$$P_0(\theta) = -\frac{\eta}{\omega - \mathcal{G}[R_0,\theta,\overline{z}]}$$

Onset of SCPS in standard phase oscillators

Kuramoto-Daido setup

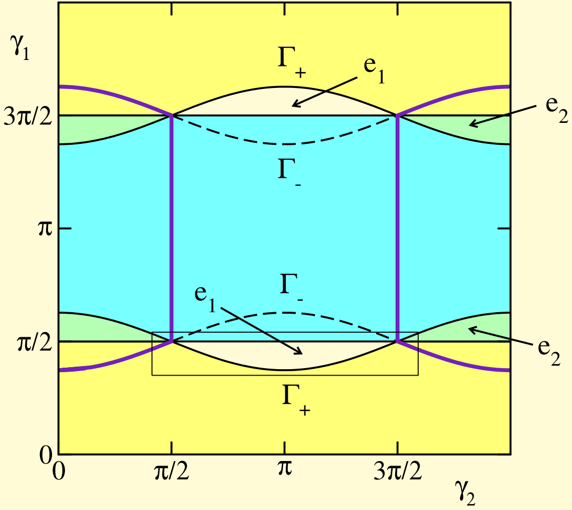
$$\dot{\phi}_i = \frac{1}{N} \sum_j G(\phi_j - \phi_i)$$

No partial synchrony in
Kuramoto-Sakaguchi: $G(\phi) = \sin(\phi + \alpha)$

Either full synchrony or splay state.

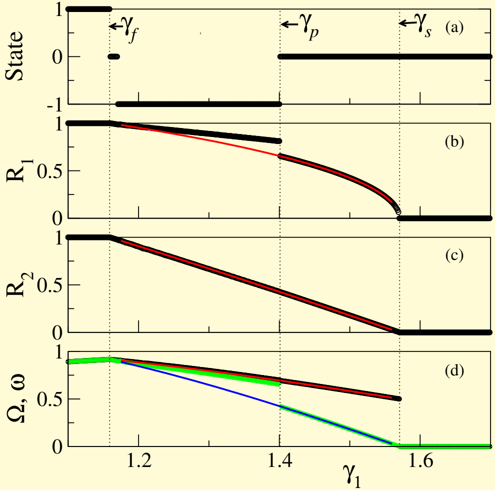
Minimal model of SCPS (Clusella, Rosenblum, AP, 2016)

$$G(\phi) = \sin(\phi + \gamma_1) + a\sin(2\phi + \gamma_2) \qquad a = 0.2$$



Phenomenology

$$Z_m = R_m e^{i\beta_m} = \frac{1}{N}\sum_j \mathrm{e}^{im\phi_j}$$



Back to Stuart-Landau: stability of SCPS

$$[v(\theta,t)]_t = v(\theta,t)F^{(v)}(\theta) + [v(\theta,t)]_\theta\, G^{(v)}(\theta) + w(t)X^{(v)}(\theta) + \hat{w}(t)Y^{(v)}(\theta)$$

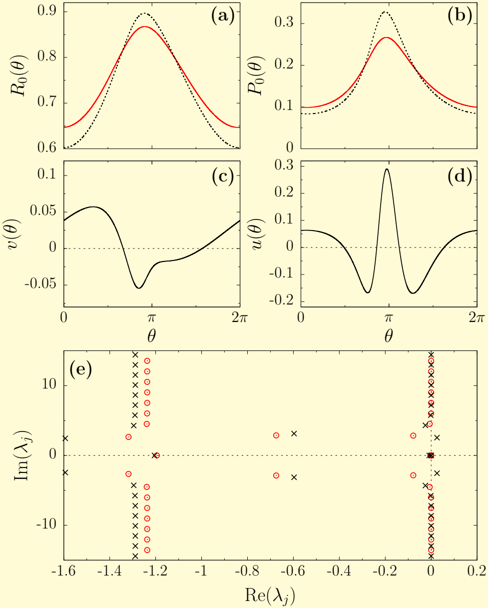
$$[u(\theta,t)]_t = \frac{d}{d\theta}\Big[v(\theta,t)F^{(u)}(\theta) + u(\theta,t)G^{(u)}(\theta) + w(t)X^{(u)}(\theta) + \hat{w}(t)Y^{(u)}(\theta)\Big]$$

$$w(t) = \int_0^{2\pi} d\theta e^{i\theta}\left(vP_0 + uR_0\right)$$

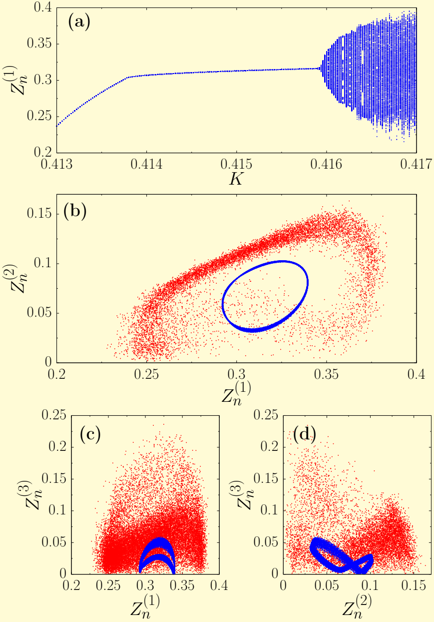
$$\hat{w}(t) = \int_0^{2\pi} d\theta e^{-i\theta}\left(vP_0 + uR_0\right)$$

$$K_2 = 0.413765$$

Graphical representation

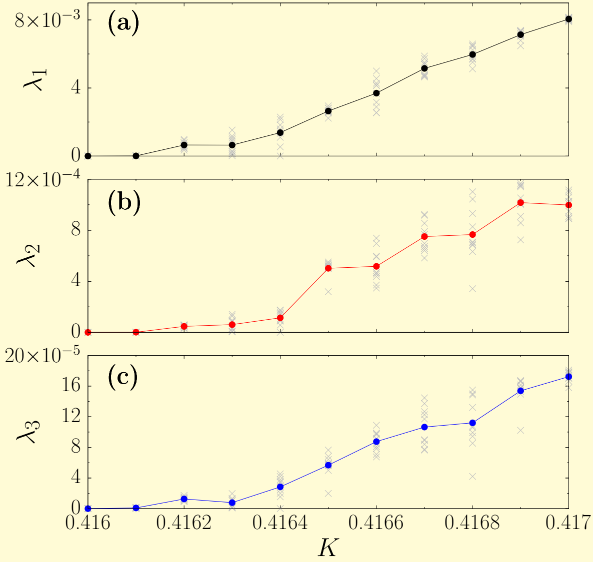


Larger coupling strength

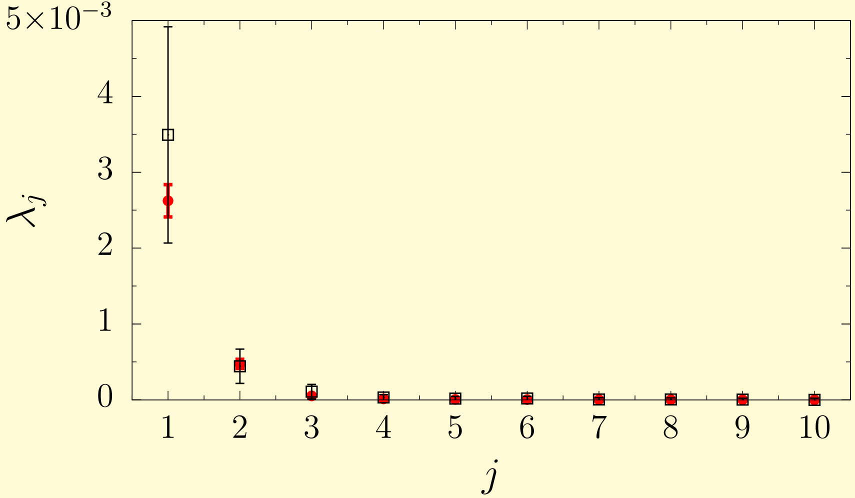


$$Z_n^{(k)} = \left| \int_0^{2\pi} d\psi R(\psi, t_n) P(\psi, t_n) \mathrm{e}^{ik\psi} \right|$$

The three largest microscopic Lyapunov exponents



Microscopic versus macroscopic Lyapunov exponents

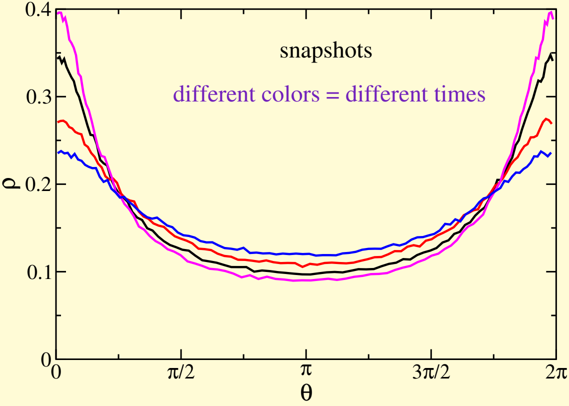


Red circles = microscopic exponents

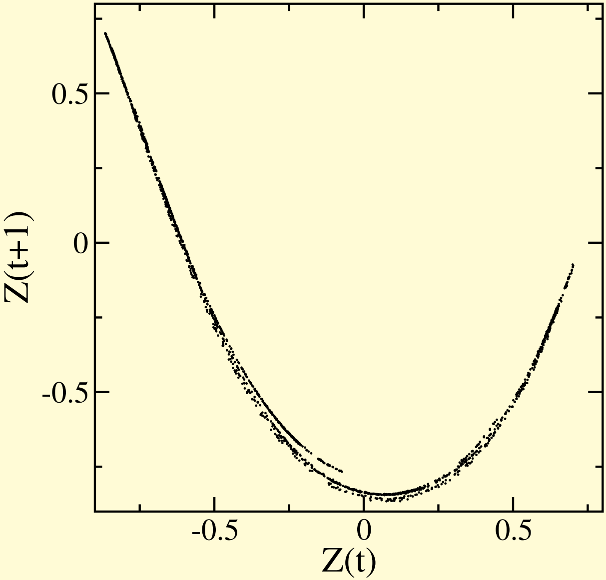
Open black squares = macroscopic exponents

An example of collective chaos: nonlinearly coupled coupled Bernoulli maps (AP, Pikovsky, Ullner, 2017)

$$\theta_j(t+1) = F(\theta_j, Z) = 2[\theta_j(t) + g(1 - 2a^2 Z^2)\sin\theta_j],$$
$$Z = \frac{1}{N}\sum \cos\theta_j(t)$$

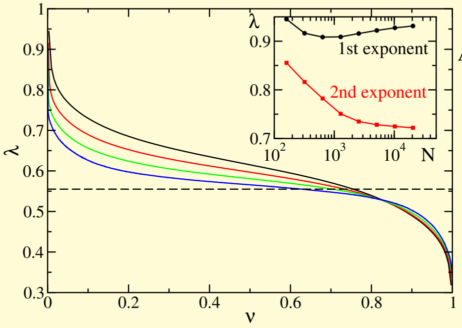


Representation of collective chaos

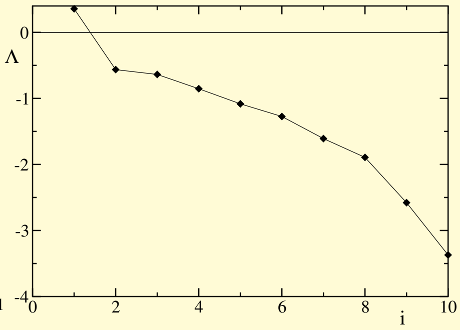


Micro and macro Lyapunov exponents

Microscopic spectrum



Macroscopic spectrum



Generalized Lyapunov exponents

$$\mathcal{L}(q) = \lim_{\tau\to\infty} \frac{1}{q\tau} \ln\langle |H(\tau)u|^q \rangle$$

$$S(\Lambda) = \lim_{\tau\to\infty} -\frac{\ln \mathcal{P}(\Lambda,\tau)}{\tau} \ .$$

$$q\mathcal{L}(q) = q\Lambda^* - S(\Lambda^*) \ ,$$

where $q = S'(\Lambda^*)$. In the Gaussian approximation

$$S(\Lambda) = \frac{(\Lambda - \lambda_T)^2}{2D}$$

$$D = \lim_{\tau\to\infty} \tau \left(\overline{\Lambda(\tau)^2} - \lambda_T^2 \right) \ .$$

Topological entropy

$$\mathcal{L}(1) = \lambda_T + \frac{D}{2} \ .$$

From numerical studies of Stuart-Landau oscillators

$$\mathcal{L}(1) \approx 0$$

Rayleigh oscillators

$$\ddot{x}_j - \zeta(1-\dot{x}_j^2)\dot{x}_j + x_j = K\,\mathrm{Re}[\mathrm{e}^{i\gamma}\overline{x} + i\overline{y}]$$

$$\overline{x} = \frac{1}{N}\sum_{m=1}^{N} x_m \quad \overline{y} = \frac{1}{N}\sum_{m=1}^{N} \dot{x}_m$$