Introduction to QFT (2)

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Yesterday

- Field theory is a generalization of classical mechanics
  - $t \to t_\mu \equiv x_\mu$
  - $x(t) \to \phi(t_\mu)$
  - $L(x, \dot{x}) \to \mathcal{L}(\phi, \partial_\mu \phi)$
  - $S = \int L dt \to S = \int \mathcal{L} d^4 t$
- We work around the minimum so we think of SHO
- Particles are excitation of fields, so just of SHOs
- How to deal with higher order terms?

Perturbation theory

Perturbation theory

$$H = H_0 + H_1 \qquad H_1 \ll H_0$$

- In many cases, perturbation theory is a mathematical tool
- There are cases, however, that PT is a better way to describe the physics
- Many times we prefer to work with EV of H (why?)
- Yet, at times it is better to work with EV of H_0 (why?)

PT for 2 SHOs

$$V(x,y) = \frac{kx^2}{2} + \frac{4ky^2}{2} + \alpha x^2 y$$

- We assume that α is small
- Classically α moves energy between the two modes
- How it goes in QM?

Drops

- Recall the Fermi golden rule

$$P \propto |\mathcal{A}|^2 \times \mathrm{P.S.} \qquad \mathcal{A} \sim \langle f|\alpha x^2 y|i\rangle$$

- The relevant thing to calculate is the transition amplitude, $\mathcal{A}$.

1st and 2nd order PT

$$H = H_0 + H_1 \qquad H_1 \ll H_0$$

- In first order we care only about the states with the same energy

$$\mathcal{A}(i \to f) \sim \langle f|H_1|i\rangle \qquad E_f = E_i$$

- 2nd order pertubation theory probe the whole spectrum

$$\mathcal{A}(i \to f) \sim \sum_n \frac{\langle f|H_1|n\rangle\langle n|H_1|i\rangle}{E_n - E_f} \qquad E_f = E_i \qquad E_n \neq E_i$$

Transitions

$$V(x,y) = \frac{kx^2}{2} + \frac{4ky^2}{2} + \alpha x^2 y$$

- Recall

$$x \sim a_x + a_x^\dagger \qquad y \sim a_y + a_y^\dagger$$

- For a given i, for what f we have $\mathcal{A} \neq 0$?

$$\mathcal{A} \sim \langle f | \alpha x^2 y | i \rangle$$

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- Since $H_1 \sim x^2 y$ we see that $\Delta n_y = \pm 1$ and $\Delta n_x = 0, \pm 2$
- What could you say if the perturbation was $x^2 y^3$?

Two SHOs with small α

$$V(x,y) = \frac{kx^2}{2} + \frac{4ky^2}{2} + \alpha x^2 y \qquad \omega_y = 2\omega_x$$

- Consider $|i\rangle = |0,1\rangle$
- Since $\omega_y = 2\omega_x$ only $f = |2,0\rangle$ is allowed by energy conservation and by the perturbation

$$\mathcal{A} \sim \langle 2,0|\alpha x^2 y|0,1\rangle \sim \alpha\langle 2,0|(a_x+a_x^\dagger)(a_x+a_x^\dagger)(a_y+a_y^\dagger)|0,1\rangle$$

- a_y in y annihilates the y "particle" and $(a_x^\dagger)^2$ in x^2 creates two x "particles"
- It is a decay of a particle y into two x particles with width $\Gamma \propto \alpha^2$ and $\tau = 1/\Gamma$

Even More PT

$$H_1 = \alpha x^2 z + \beta xyz \qquad \omega_z = 10,\ \omega_y = 3,\ \omega_x = 1$$

- Calculate $y \to 3x$ using 2nd order PT

$$\mathcal{A} \sim \langle 3,0,0|\mathcal{O}|0,1,0\rangle \qquad \mathcal{O} \sim \sum \frac{\langle 3,0,0|V'|n\rangle\langle n|V'|0,1,0\rangle}{E_n - E_{0,1,0}}$$

- Which intermediate states? $|1,0,1\rangle$ and $|2,1,1\rangle$
  - $\mathcal{A}_1 = |0,1,0\rangle \xrightarrow{\beta} |1,0,1\rangle \xrightarrow{\alpha} |3,0,0\rangle$
  - $\mathcal{A}_2 = |0,1,0\rangle \xrightarrow{\alpha} |2,1,1\rangle \xrightarrow{\beta} |3,0,0\rangle$
- The total amplitude is then

$$\mathcal{A} \propto \alpha\beta \left(\frac{\#}{\Delta E_1} + \frac{\#}{\Delta E_2} \right)$$

Closer look

$$V' = \alpha x^2 z + \beta xyz \qquad \omega_z = 10,\ \omega_y = 3,\ \omega_x = 1$$

- $\mathcal{A}_1 = |0,1,0\rangle \xrightarrow{\beta} |1,0,1\rangle \xrightarrow{\alpha} |3,0,0\rangle$
- $\mathcal{A}_2 = |0,1,0\rangle \xrightarrow{\alpha} |2,1,1\rangle \xrightarrow{\beta} |3,0,0\rangle$

Closer look

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- $\mathcal{A}_2 = |0,1,0\rangle \xrightarrow{\alpha} |2,1,1\rangle \xrightarrow{\beta} |3,0,0\rangle$

(HW) Show that the sum of the two diagrams give

$$\frac{1}{\omega_z^2 - q^2}$$

where q is the “energy flow” in the “z line”

General method for SHO PT

- Every x^n term with $n \geq 3$ is a vertex
- We write all the ways to get from “in” to “out”
- Each amplitude is the product of the couplings and the “off-shell” intermediate states

$$\frac{1}{\omega^2 - q^2}$$

- There are few more rules
- We add all amplitude square them and use the Fermi Golden rule

Feynman diagrams

Using PT for fields

- For one SHO we have $x \sim a + a^\dagger$
- For many SHOs we have $x_i \sim a_i + a_i^\dagger$
- For fields we then have $\phi \sim \int \left[a(k) + a^\dagger(k)\right]\, dk$

Perturbation theory for fields is a generalization of that of SHO

- $\omega \to p_\mu$
- $\omega^2 \to m^2$
- We can have any energy (but one mass)

Calculations

- We usually care about $1 \to n$ or $2 \to n$ processes
- We need to make sure we have energy conservation
- External (Internal) particles are called on(off)–shell
  - On-shell: $E^2 = p^2 + m^2$
  - Off-shell: $E^2 \neq p^2 + m^2$
- $\mathcal{A} =$ the product of all the vertices and internal lines
- Each internal line with q^μ gives suppression

$$\frac{1}{m^2 - q^2}$$

- There are many more rules to get all the factors right

Examples of amplitudes

$$\mathcal{L} = \lambda_1 XYZ + \lambda_2 X^2 Z$$

$$\boxed{\Gamma(Z \to XY)}$$

- Energy conservation condition
- Draw the diagram and estimate the amplitude

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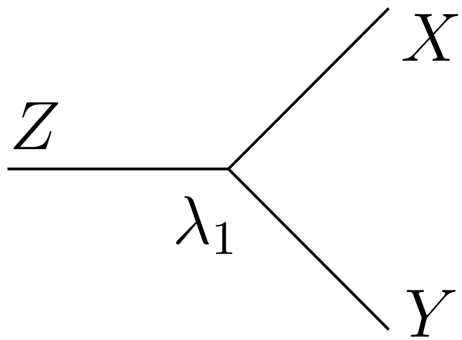
- Energy conservation condition $m_Z > m_X + m_Y$
- Draw the diagram and estimate the amplitude



$$\mathcal{A} \propto \lambda_1$$

Examples of amplitudes (2)

$$\mathcal{L} = \lambda_1 XYZ + \lambda_2 X^2 Z$$

$$\boxed{\Gamma(Y \to 3X)}$$

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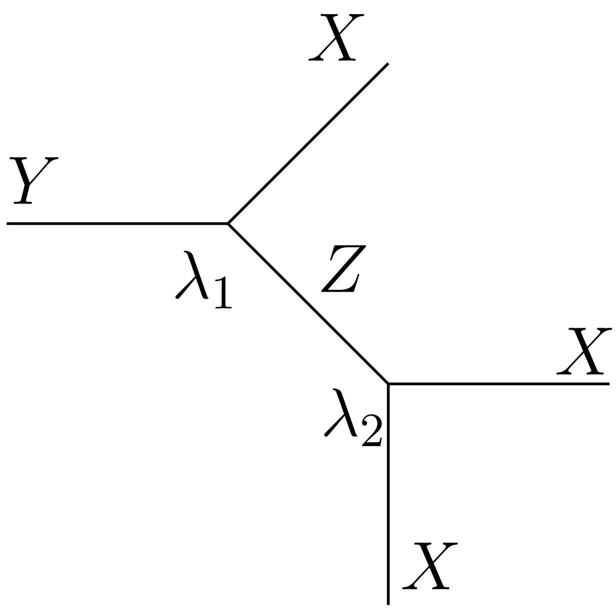
Examples of amplitudes (2)

$$\mathcal{L} = \lambda_1 XYZ + \lambda_2 X^2 Z \qquad \boxed{\Gamma(Y \to 3X)}$$

- Energy conservation condition $m_Y > 3m_X$
- Draw the diagram and estimate the amplitude



$$\mathcal{A} \propto \lambda_1 \lambda_2 \times \frac{1}{\Delta E_Z^2}$$

$$= \lambda_1 \lambda_2 \times \frac{1}{m_Z^2 - q^2}$$

Examples of amplitudes (HW)

$$\mathcal{L} = \lambda_1 XYZ + \lambda_2 X^2 Z$$

$$\boxed{\sigma(XX \to XY)}$$

- Energy conservation condition
- Draw the diagram and estimate the amplitude

Some summary

- Quadratic terms describe free fields. Free particles cannot be created nor decay
- We use perturbation theory were terms with 3 or more fields in $\mathcal{L}$ are considered small
- These terms can generate and destroy particles and give dynamics
- Feynman diagrams are a tool to calculate transition amplitudes
- Many more details are needed to get calculation done
- Once calculations and experiments to check them are done, we can test our theory

Symmetries

How to “built” Lagrangians

- $\mathcal{L}$ is:
  - The most general one that is invariant under some symmetries
  - We work up to some order (usually 4)
- We need the following input:
  - What are the symmtires we impose
  - What DOFs we have and how they transform under the symmtry
- The output is
  - A Lagrangian with N parameters
  - We need to measure its parameters and test it

Symmetries and representations

Example: 3d real space in classical mechanics

- We require that $\mathcal{L}$ is invariant under rotation
- All our DOFs are assigned into vector representations $(\vec{r}_1,\ \vec{r}_2, ...)$
- We construct invariants from these DOFs. They are called singlets or scalars

$$C_{ij} \equiv \vec{r}_i \cdot \vec{r}_j$$

- We then require that V is a function of the C_{ij}s

Generalizations

- In mechanics, $\vec{r}$ lives in 3d real space and is a vector
- Fields do not live in real space. They live in some mathematical space
- They also do not have to be vectors, but can be scalars or tensors (representation)
- The idea is similar to what we did in mechanics
  - We require $\mathcal{L}$ to be invariant under rotation in that mathematical space
  - Thus $\mathcal{L}$ depends only on combinations of fields that form singlets
- All this is related to a subject called Lie groups
- We usually care about $SO(N)$, $SU(N)$ and $U(1)$

Combining representations

- It is all about generating singlets
- We all know that we can combine vectors in real space to generate singlets
- We also know how to make a spin zero from 2 spin half spinors (spin zero is a singlet!)
- There is a generalization of this procedure to any mathematical space
- As of now, all we need to know are $SU(3)$, $SU(2)$ and $U(1)$

Invariant of complex numbers

- $U(1)$ is rotation in 1d complex space
- Each complex number comes with a q that tells us how much it rotates
- When we rotate the space by an angle θ, the number rotates as

$$X \to e^{iq\theta} X$$

- Consider $q_X = 1, \; q_Y = 2, \; q_Z = 3$ and write 3rd and 4th order invariants

$$XX^*YY^* \qquad X^2Y^*$$

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- Consider $q_X = 1,\ q_Y = 2,\ q_Z = 3$ and write 3rd and 4th order invariants

$$XX^*YY^* \qquad X^2Y^* \qquad XYZ^* \qquad X^3Z^* \qquad Y^2X^*Z^*$$

$SU(2)$

- $U(2)$ is rotation in 2d complex space. We have $U(2) = SU(2) \times U(1)$
- $SU(2)$ is localy the same as rotation in 3d real space
- Rotations in this space are non-Abelian (non-commutative)
- It depends on the representation: scalar, spinor, vector
- Spin in QM is described by SU(2) rotations, so we use the same language to describe it
- For the SM all we care is that $1/2 \times 1/2 \ni 0$ so we know how to generate singlets
- How can we generate invarints from spin $1/2$ and spin $3/2$?

$SU(3)$

- $U(3)$ is rotation in 3d complex space. We have $U(3) = SU(3) \times U(1)$
- The representations we care about are singlets, triplets and octets
- Unlike $SU(2)$, in $SU(3)$ we have complex representations, 3 and $\bar{3}$
- The three quarks form a triplet (the three colors)
- To form a singlet we need to know that

$$3 \times \bar{3} \ni 1 \qquad 3 \times 3 \times 3 \ni 1$$

- This is why we have baryons and mesons

A game

A game calls "building invariants"

- Symmetry is $SU(3) \times SU(2) \times U(1)$
  - $U(1)$: Add the numbers ($\bar{X}$ has charge $-q$)
  - $SU(2)$: $2 \times 2 \ni 1$ and recall that 1 is a singlet
  - $SU(3)$: we need $3 \times \bar{3} \ni 1$ and $3 \times 3 \times 3 \ni 1$
- Fields are

$$Q(3,2)_1 \qquad U(3,1)_4 \qquad D(3,1)_{-2} \qquad H(1,2)_3$$

- What 3rd and 4th order invariants can we built?

$$(HH^*)^2 \qquad H^3 \qquad UDD \qquad QUD \qquad HQU^*$$

- HW: Find more invariants