Problem Set: Kaluza-Klein theory as illustration of Swampland conjectures

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As discussed in my lectures, many of the features that the Swampland Distance Conjecture and (Tower/Sublattice) Weak Gravity Conjectures tell us are true in general are visible in the simple example of Kaluza-Klein theory, i.e. the gauge theory we obtain by dimensionally reducing a theory of pure gravity on a circle. In this problem set you will work out some of the details.

1 Weyl rescaling of a metric

Consider general relativity in $D = d + 1$ spacetime dimensions. Suppose that we carry out a field redefinition, defining a new metric

$$\widetilde{g}_{\mu\nu}(x) = \mathrm{e}^{2\omega(x)} g_{\mu\nu}(x), \tag{1}$$

for some scalar function $\omega(x)$. Check that the following quantities transform in the indicated manner, where we use $\mathcal{R}$ to denote the Ricci scalar:

(a) $\sqrt{|\widetilde{g}|} = \mathrm{e}^{D\omega(x)}\sqrt{|g|}$.

(b) $\widetilde{g}^{\mu\nu} = \mathrm{e}^{-2\omega(x)} g^{\mu\nu}(x)$.

(c) $\widetilde{\mathcal{R}}_{\mu\nu} = \mathcal{R}_{\mu\nu} - (D-2)\nabla_\mu\nabla_\nu\omega - g_{\mu\nu}g^{\rho\sigma}\nabla_\rho\nabla_\sigma\omega + (D-2)(\nabla_\mu\omega)(\nabla_\nu\omega) - (D-2)g_{\mu\nu}g^{\rho\sigma}(\nabla_\rho\omega)(\nabla_\sigma\omega)$.

(d) $\widetilde{\mathcal{R}} = \mathrm{e}^{-2\omega(x)}\left[\mathcal{R} - 2(D-1)g^{\mu\nu}\nabla_\mu\nabla_\nu\omega - (D-2)(D-1)g^{\mu\nu}(\nabla_\mu\omega)(\nabla_\nu\omega)\right]$.

2 Compactifying on a circle and going to Einstein frame

Suppose that we compactify our D-dimensional theory to a theory in d spacetime dimensions, on a spatial circle. We choose an ansatz for the D-dimensional metric,

$$ds_D^2 = g_{\mu\nu}(x)dx^\mu dx^\nu + L(x)^2 d\theta^2, \tag{2}$$

where the x^μ are d-dimensional coordinates and $g_{\mu\nu}$ is a d-dimensional metric, and θ is a dimensionless angular coordinate for the compact dimension, $\theta \cong \theta + 2\pi$. Then $L(x)$ has units of length and describes the size of the compact dimension.

Notice that we have taken both $g_{\mu\nu}(x)$ and $L(x)$ to be independent of θ. This is not the most general ansatz. We have also ignored terms of the form $A_\mu dx^\mu d\theta$, which will be important below. Our ansatz corresponds to keeping the "zero modes"—those which are uniform in the extra dimension—for the metric and the scalar *radion* field that parametrizes the size of the circle.

(a) Show that the higher-dimensional Einstein-Hilbert action, for this ansatz, reduces to the d-dimensional action

$$S_{\text{EH}} = \frac{1}{16\pi G_D} \int d^d x \sqrt{-g}\, 2\pi L(x)\, \mathcal{R}_d, \tag{3}$$

where $\mathcal{R}_d$ is the Ricci scalar associated to the d-dimensional metric $g_{\mu\nu}(x)$.

(b) Notice that the field $L(x)$ has no kinetic term, though it does have a *kinetic mixing* with the graviton because it appears multiplying $\mathcal{R}_d \sim \partial^2 h + \cdots$. Show that we can remove this kinetic mixing by a field redefinition,

$$\widetilde{g}_{\mu\nu}(x) = \left(\frac{L(x)}{\langle L \rangle}\right)^{\alpha} g_{\mu\nu}(x), \tag{4}$$

where α is a constant and $\langle L \rangle$ is a constant reference value of the radion field—for instance, we might assume that asymptotically $L(x) \mapsto \langle L \rangle$ in all directions. After this field redefinition, with an appropriate choice of α, you should obtain what is referred to as the "Einstein frame" action where the kinetic terms of gravity and the scalar are independent of each other,

$$S_{\text{Einstein}} = \frac{1}{16\pi G_d} \int d^d x \sqrt{-\widetilde{g}} \big(\widetilde{\mathcal{R}}_d - C_d \widetilde{g}^{\mu\nu} \partial_\mu (\log L) \partial_\nu (\log L)\big). \tag{5}$$

(You may find that this is only correct up to boundary terms.) What is the value of the exponent α that we should use when rescaling to achieve this separation of kinetic terms? How are the constants G_d and C_d related to the parameters of the D-dimensional theory and our ansatz?

3 Scalar coupling to the gauge kinetic term

Now that you have seen how to rescale the metric to eliminate the kinetic mixing of the radion and the graviton, we can just build it into our ansatz to begin with. This time, however, let us consider a more general ansatz that also includes the "graviphoton"—the spin-1 field that comes from taking metric modes with one leg along the compact direction, roughly $A_\mu \sim g_{\mu\theta}$. To be more precise, take the ansatz:

$$ds^2 = \left(\frac{L(x)}{\langle L \rangle}\right)^{-\alpha} \widetilde{g}_{\mu\nu}(x) dx^\mu dx^\nu + L(x)^2 (d\theta + A_1)^2, \tag{6}$$

where the 1-form $A_1 = A_\mu dx^\mu$ will be our d-dimensional gauge field.

(a) Show that, after dimensional reduction, the gauge field in d dimensions has the kinetic term

$$-\int d^d x \sqrt{-\widetilde{g}}\, \frac{1}{4e_d^2} \left(\frac{L(x)}{\langle L \rangle}\right)^{-\beta} \widetilde{g}^{\mu\rho} \widetilde{g}^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}, \tag{7}$$

where $F = dA_1 = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$ is the photon field strength. What are the constants e_d and β?

(b) Now define a *canonically normalized* scalar field $\varphi(x)$ that is related to $L(x)$, and rewrite the gauge field kinetic term in terms of $\varphi(x)$ rather than $L(x)$. You should find that the prefactor depends *exponentially* on the canonically normalized field.

4 Tower of massive particles

So far we have taken all of the terms in our ansatz to be zero modes, independent of the extra-dimensional coordinate. Now consider instead a more general ansatz that allows the metric modes to depend on the extra dimension,

$$ds^2 = \left(\frac{L(x)}{\langle L\rangle}\right)^{-\alpha} \widetilde{g}_{\mu\nu}(x,\theta)dx^\mu dx^\nu + L(x)^2(d\theta + A_1)^2, \tag{8}$$

and decompose the metric into "Kaluza-Klein modes" with definite momentum around the extra dimension,

$$\widetilde{g}_{\mu\nu}(x,\theta) = \sum_{n=0}^{\infty} \widetilde{g}^{(n)}_{\mu\nu}(x)\mathrm{e}^{\mathrm{i}n\theta}. \tag{9}$$

(a) Argue that the d-dimensional fields $\widetilde{g}^{(n)}_{\mu\nu}(x)$ with different values of n are orthogonal to each other (i.e., they have independent kinetic terms).

(b) Argue that the d-dimensional field $\widetilde{g}^{(n)}_{\mu\nu}(x)$ has a mass

$$m_n = \frac{n}{\langle L\rangle}, \tag{10}$$

and that it has charge n under the gauge field A_1. Rewrite m_n in terms of the d-dimensional Planck scale and the gauge coupling e_d. It turns out that these Kaluza-Klein modes each saturate the Weak Gravity Conjecture inequality relating m_n^2 with $n^2 e_d^2 M_d^{d-2}$.[1]

(c) Argue that the masses m_n are *exponentially* small in terms of the canonically normalized radion field. This is the prediction of the "Swampland Distance Conjecture": when going a large distance in some scalar field space, an infinite tower of modes becomes light in a manner that is exponential in the distance.

[1] We define the Planck scale M_d with the convention that $\frac{1}{8\pi G_d} = M_d^{d-2}$, in any dimension d.