Questions on Cosmology and Dark Matter Physics

1. Cosmology review

(a) Assume the universe is either dominated by radiation, matter, or dark energy, with negligible contributions to the energy density from the other components. In each case, solve the Friedmann equation to determine the relationship between redshift $1+z$ and elapsed coordinate time t. For each choice, does the expansion of the universe speed up or slow down with increasing time?

Solution:

Note $H(z) = (1/a)da/dt = (1+z)d/dt(1+z)^{-1} = -\frac{1}{1+z}\frac{dz}{dt}$.

Radiation domination: $H^2 = \Omega_{\rm rad} H_0^2 (1+z)^4 \Rightarrow \left(\frac{dz}{dt}\right)^2 = H_0^2 \Omega_{\rm rad}(1+z)^6$, i.e. $-(1+z)^{-3}dz = H_0\sqrt{\Omega_{\rm rad}}dt \Rightarrow (1/2)/(1+z)^2 = \sqrt{\Omega_{\rm rad}}t \Rightarrow t = \frac{1}{H_0}\frac{1}{2(1+z)^2\sqrt{\Omega_{\rm rad}}} \Rightarrow 1+z = \sqrt{\frac{1}{H_0 t \times 2\sqrt{\Omega_{\rm rad}}}}$.

Matter domination: $H^2 = \Omega_m H_0^2 (1+z)^3 \Rightarrow \left(\frac{dz}{dt}\right)^2 = H_0^2\Omega_m(1+z)^5$, i.e. $-(1+z)^{-2.5}dz = H_0\sqrt{\Omega_m}dt \Rightarrow (2/3)/(1+z)^{1.5} = \sqrt{\Omega_m}t \Rightarrow t = \frac{1}{H_0}\frac{2}{3(1+z)^{3/2}\sqrt{\Omega_m}} \Rightarrow 1+z = \left(\frac{1}{H_0 t}\frac{2}{3\sqrt{\Omega_m}}\right)^{2/3}$.

Dark energy domination: $H^2 = \Omega_\Lambda H_0^2 \Rightarrow \left(\frac{dz}{dt}\right)^2 = H_0^2\Omega_\Lambda(1+z)^2$, i.e. $-(1+z)^{-1}dz = H_0\sqrt{\Omega_\Lambda}dt \Rightarrow C - \ln(1+z) = \sqrt{\Omega_\Lambda}t \Rightarrow t = (t_0 H_0\sqrt{\Omega_\Lambda} - \ln(1+z))\frac{1}{H_0\sqrt{\Omega_\Lambda}} \Rightarrow 1+z = e^{-H_0\sqrt{\Omega_\Lambda}(t-t_0)}$.

(b) Using the Planck 2018 cosmological parameters, estimate (1) the redshift of matter-radiation equality, when the contributions to the energy density from matter and radiation are equal, (2) the redshift of matter-dark energy equality, when the contributions to the energy density from matter and dark energy are equal.

Solution:

Matter-radiation equality: $(1+z)\Omega_{\rm rad} = \Omega_m \Rightarrow 1+z = \frac{0.32}{10^{-4}} \sim 3000$.

Matter-dark energy equality: $(1+z)^3\Omega_m = \Omega_\Lambda \Rightarrow (1+z)^3 = \frac{0.68}{0.32} \Rightarrow 1+z \approx 1.28$.

(c) Using your results from (a) and (b), estimate the age of the universe at Big Bang nucleosynthesis ($T \sim 1$ MeV), matter-radiation equality, recombination ($z \sim 1000$), and matter-dark energy equality. You may assume the present-day temperature of the

CMB is 2.725 K ($\sim 2 \times 10^{-4}$ eV), and neglect changes in the number of relativistic degrees of freedom (although if you wish to include them, that is also fine).

Solution: The first two processes occur during radiation domination (or at its end), and occur respectively at $z \sim 5 \times 10^9$ (since temperature redshifts roughly as $(1+z)$) and $z \sim 1000$. So using our result above for radiation domination, $t = \frac{1}{H_0}\frac{1}{2(1+z)^2\sqrt{\Omega_{\rm rad}}}$, we obtain for BBN $t \approx 2 \times 10^{-18}/H_0 \approx 1s$, and for recombination $t \approx 6 \times 10^{-6}/H_0 \approx 70,000$ years.

Recombination and matter-DE equality occur during matter domination, so substituting $z = 1000$ and $z = 1.28$ into $t = \frac{1}{H_0}\frac{2}{3(1+z)^{3/2}\sqrt{\Omega_m}}$, we obtain: $t = 4 \times 10^{-5}/H_0 \approx 480,000$ years for recombination, $0.8/H_0 \approx 11$ billion years.

(d) Using the Friedmann equation, we can relate H to the energy density of the universe; during radiation domination, the energy density is controlled by the temperature of the radiation bath (as well as the number of degrees of freedom). As a consequence, during radiation domination we can estimate $H \sim T^a m_{\rm Pl}^b$ for some coefficients a and b, where we have neglected numerical prefactors (from e.g. the number of relativistic degrees of freedom), and $m_{\rm Pl}$ is the Planck mass. Determine a and b.

We know that $H^2 \sim \rho G \sim T^4/m_{\rm Pl}^2$, as $m_{\rm Pl} \sim 1/\sqrt{G}$. Thus $H \sim T^2/m_{\rm Pl}$, $a = 2$ and $b = -1$.

2. Dark matter interactions and decoupling

Suppose the dark matter possesses a non-gravitational interaction, via a new light $U(1)$ gauge boson ϕ (much lighter than the dark matter) which couples both to the dark matter (with coupling g_D) and to electrons (with coupling g_V). At temperatures much less than the mass of the ϕ, but much greater than the electron mass, the dark matter scatters off the thermal bath of relativistic electrons with a scattering cross section of order,

$$\langle \sigma v \rangle \sim \alpha_D \alpha_V T^2/m_\phi^4.$$

For the momentum of the dark matter to be appreciably changed by repeated such scatterings (i.e. for the total momentum transfer to be comparable to the initial momentum of the dark matter), roughly $m_{\rm DM}/T$ interactions are required. The dark matter will kinetically decouple from the thermal bath – i.e. the dark matter temperature will no longer be fixed to the photon temperature – when the expansion timescale H^{-1} becomes comparable to the timescale for sizable momentum transfer.

(a) Estimate the temperature of the universe when the dark matter kinetically decouples as a function of m_ϕ, α_D and α_V. Give the numerical value of the decoupling temperature when the dark matter mass is 1 TeV and $\alpha_D = 1/30$, for the two cases where (i) $\alpha_V = 1/30$, $m_\phi = m_Z$, and (ii) $\alpha_V = 10^{-6} \times 1/30$, $m_\phi = 30$ MeV (the first case corresponds to the standard WIMP calculation, the second is an example of dark matter in a hidden sector with a light force carrier). You may neglect $\mathcal{O}(1)$ factors in your calculation, and you may assume kinetic decoupling occurs during radiation domination and while the electrons and positrons are relativistic.

(Note that this calculation only holds when $T \gtrsim m_e$, which implies radiation domination – once the electrons become non-relativistic, unless the dark matter has a large coupling to the neutrinos, the rate becomes very small due to a lack of targets and decoupling is usually immediate.)

Solution:

We require $\langle \sigma v \rangle T^3 T/m_{\rm DM} \sim H \Rightarrow \alpha_D \alpha_V T^6/(m_D M m_\phi^4) \sim T^2/m_{\rm Pl}$, from above. Thus we obtain $T \sim \left(\frac{m_\phi^4 m_{\rm DM}}{m_{\rm Pl}\alpha_D \alpha_V}\right)^{1/4}$. Substituting in the numbers for the two cases gives:

(i) 50 MeV

(ii) 500 keV

(Note in both cases we have taken $m_{\rm Pl} \sim 10^{19}$ GeV, slightly different answers may be obtained depending on choices of O(1) factors.)

We see that in the second case, the DM would stay coupled right down to the electron decoupling scale, where this calculation breaks down. However this typically requires quite a light DM mass.

(b) The horizon size cH^{-1} at kinetic decoupling roughly sets the scale of the smallest dark matter structures. Matter fluctuations that enter the horizon at earlier times are suppressed by the coupling of the dark matter to the thermal bath. As we discussed today, the presence of small-scale structure can be probed by measurements of the Lyman-α forest, and such structures can also be erased by a high free-streaming length for the dark matter.

Dark matter structures too small to probe directly (e.g. with the Lyman-α observations) can be very important in searches for dark matter annihilation (where dark matter particles collide and produce visible particles), for example in studies of the extragalactic gamma-ray background where most of the predicted power comes from annihilations occurring in large numbers of very small dark matter halos. As you will see, unfortunately, this small-scale cutoff is very model-dependent.

For kinetic decoupling temperatures of 1 MeV and 1 GeV, compute M_{cutoff} as the dark matter mass enclosed inside the horizon at the time of decoupling. (It is a little more complicated than this, as there are competing damping scales - see `arXiv::0903.0189` for a discussion.) You may assume a spatially flat universe, and the Planck measurement of the dark matter density.

Solution: These temperatures are during radiation domination, so we can estimate $H \sim T^2/m_{\text{Pl}}$. The enclosed mass is $\rho_{\text{DM}}(4\pi/3)H^{-3} \sim \rho_{\text{DM,today}}(T/T_0)^3 H^{-3} \sim \rho_{\text{DM,today}}(m_{\text{Pl}}/(TT_0))^3$. The DM density today is $\rho_{\text{DM,today}} \approx 10^{-6}$ GeV/cm^3. Thus for temperatures of 1 MeV and 1 GeV we obtain enclosed masses of $\sim 10^{57}$ GeV and $\sim 10^{48}$ GeV respectively, or ~ 1 solar mass and $\sim 10^{-9}$ solar masses.

(c) The presence of the light force carrier also mediates a scattering interaction between dark matter particles, which could become important in the dense cores of dark matter halos at late times. Consider a galaxy cluster with typical DM velocity dispersion ~ 1000 km/s. At the virial radius where the dark matter density is roughly 200 times the cosmological value, what must the DM-DM scattering cross section σ be such that the scattering time for a DM particle is less than 10^{10} years? (Your answer will depend on the DM mass.)

Solution: The scattering rate (inverse of time) is $n\sigma v = \rho v \sigma / m_{\text{DM}}$. Thus we must have $\sigma/m_{\text{DM}} \gtrsim \frac{1}{10^{10}\text{years} \times \rho v}$; taking $\rho = 200 \times 10^{-6}$ GeV/cm^3 and $v = 1000$ km/s, we obtain $\sigma/m_{\text{DM}} \gtrsim 80$ cm^2/g.

(d) For a 100 GeV WIMP, compare this cross section to the DM-baryon cross section constrained by XENON1T, which can be as small as $\sigma \approx 4 \times 10^{-47}$ cm^2 for 30 GeV DM. If we take this cross section to be given parametrically by $\sim 1/m^2$, what is the characteristic mass scale m (keeping the DM mass at 100 GeV)?

Solution: This cross section corresponds to $\sigma/m_{\rm DM} \gtrsim 10^{-22}$ cm^2/GeV, so for 100 GeV DM, this would correspond to $\sigma \gtrsim 10^{-20}$ cm^2. This is a completely enormous cross section by direct-detection standards! We can also express this as $\sigma \gtrsim 1/(10^{-4} GeV)^2$, i.e. the relevant mass scale is 100 keV. Cross sections of this size typically require a very light mediator.