Questions on Dark Matter Production and Direct Detection

1. Asymmetric dark matter

Suppose that the dark matter is a Dirac fermion, and there is an asymmetry in the dark matter sector analogous to that in the baryon sector, so the abundance of dark matter and anti-dark-matter is different. In this case, either the depletion of the less abundant component (as for the baryons) *or* the expansion of the universe (as for conventional WIMP dark matter, as we have discussed in lectures) can cut off the annihilation processes that deplete the dark matter density. This idea has gained considerable interest over the last few years, in part because of the apparent coincidence between the amounts of dark and baryonic matter in the universe – if their abundances are set by entirely different processes, why are they only different by a factor of five or so?

In this problem we will work out the freeze-out of annihilations for this scenario.

(a) Let us denote the DM by χ^+ and the anti-DM by χ^-. Without loss of generality, we will assume χ^+ is more abundant than χ^-. Let the average $\chi^+\chi^-$ annihilation cross section be given by $\langle\sigma v\rangle$. Justify (qualitatively, in words) the coupled Boltzmann equations for this system:

$$\frac{dn^\pm}{dt} + 3Hn^\pm = -\langle\sigma v\rangle(n^+n^- - n^+_{\rm eq}n^-_{\rm eq}).$$

Here n describes the number density of the two species and the subscript "eq" refers to an equilibrium value. (We are assuming here that the annihilation reaction is the only number-changing reaction for either DM or anti-DM.)

Solution: When $\langle\sigma v\rangle \to 0$, the abundance of the DM should redshift as a^{-3}, i.e. $d(n_\pm a^3)/dt = 0$; this is captured by the terms on the LHS of the equation. Once $\langle\sigma v\rangle$ is turned on, we need to take into account the DM depletion via annihilation, and production via the reverse process. The rate of annihilations is $n^+n^-\langle\sigma v\rangle$, as each annihilation requires both a + and - particle; each annihilation removes a single χ^+ and a single χ^-, so there is no numerical prefactor. The production term cannot depend on the DM abundances, only on the temperature of the SM bath, and must approximately cancel the depletion term in the limit of large $\langle\sigma v\rangle$, when the DM abundances must also approach equilibrium. This tells us that the production term is $\langle\sigma v\rangle n^+_{\rm eq} n^-_{\rm eq}$, as given above.

(b) Let us define $Y^\pm = n^\pm/s$, where s is the entropy density of the universe. You may assume that entropy is conserved, so $s \propto a^{-3}$. Let $\eta = Y^+ - Y^-$. Show that η is conserved by the Boltzmann equations from (a), and explain the physical reason for this behavior.

Solution: The LHS of the equation can be rewritten as $a^{-3}\frac{d}{dt}(n^\pm a^3) = s\frac{d}{dt}(n^\pm/s)$. Thus we obtain:

$$s\frac{dY^\pm}{dt} = -s^2\langle\sigma v\rangle(Y^+Y^- - Y^+_{\rm eq}Y^-_{\rm eq}).$$

Taking the difference of the equations for Y^+ and Y^-, we obtain:

$$\frac{d\eta}{dt} = -s\langle\sigma v\rangle(Y^+Y^- - Y^+_{\rm eq}Y^-_{\rm eq}) + s\langle\sigma v\rangle(Y^+Y^- - Y^+_{\rm eq}Y^-_{\rm eq}) = 0.$$

Physically, this occurs because $Y^\pm$ is proportional to the total number of DM particles in a comoving volume (since s scales with the inverse of the physical size of a comoving volume), the expansion of the universe does not change the number of particles in a comoving volume, and the annihilation interactions deplete the total number of χ^+ and χ^- particles equally, so the difference in particle number within a comoving volume remains constant.

(c) Define the fractional asymmetry $r = n^-/n^+$. Note by definition $0 \le r \le 1$. What is the annihilation rate in terms of the overall DM density $n = n^+ + n^-$ and r? Comment on the limits $r = 1$ and $r = 0$.

Solution: We can write the total density $n = n^+(1+r)$, and consequently $n^+ = n/(1+r)$, $n^- = nr/(1+r)$. The annihilation rate $n^+n^-\langle\sigma v\rangle = n^2 r/(1+r)^2$. In the limit of $r = 0$ the annihilation rate approaches zero, as expected (we need some χ^- component for annihilations to occur), whereas in the limit of $r = 1$ the annihilation rate approaches $n^2/4$, a factor of 2 smaller than in the case of identical initial particles. This factor of 2 occurs because in the symmetric limit, only 1/2 the DM particles can annihilate with any given DM particle (i.e. only the χ^- can annihilate with any given χ^+, and vice versa).

(d) Assume the universe is radiation dominated so $H(T) \propto T^2$ and $T \propto 1/a$; you may assume no significant change in g_* over the time period of greatest interest. Define $r_{\rm eq} = n^-_{\rm eq}/n^+_{\rm eq}$, and $x = m/T$. Show that the Boltzmann equations yield a dynamical equation for r of the form:

$$\frac{dr}{dx} \propto -\eta\langle\sigma v\rangle x^{-2}\left[r - r_{\rm eq}\left(\frac{1-r}{1-r_{\rm eq}}\right)^2\right] \propto -\eta\langle\sigma v\rangle x^{-2}\left[r - \frac{Y^+_{\rm eq}Y^-_{\rm eq}}{\eta^2}(1-r)^2\right].$$

Solution: Let us start from the equation in the solution to part (b) above:

$$\frac{dY^\pm}{dt} = -s\langle\sigma v\rangle(Y^+Y^- - Y^+_{\rm eq}Y^-_{\rm eq}).$$

Note we can write $\frac{d}{dt} = \frac{1}{a}\frac{da}{dt}\frac{d}{d\ln a} = H(T)\frac{d}{d\ln(1/T)} = H(T)\frac{d}{d\ln x}$, via the assumptions above. Thus we can write:

$$H(T)x\frac{dY^\pm}{dx} = -s\langle\sigma v\rangle(Y^+Y^- - Y^+_{\rm eq}Y^-_{\rm eq}).$$

Now we have:

$$\frac{dr}{dx} = \frac{d}{dx}\left(\frac{Y^-}{Y^+}\right) = \frac{1}{Y^+}\left(\frac{dY^-}{dx} - \frac{Y^-}{Y^+}\frac{dY^+}{dx}\right) = \frac{1}{Y^+H(T)x}\left(-s\langle\sigma v\rangle(Y^+Y^- - Y^+_{\rm eq}Y^-_{\rm eq})\right)(1-r)$$

Since $\eta = Y^+ - Y^-$, we can write $\eta = (1-r)Y^+$, so $Y^+ = \eta/(1-r)$, $Y^- = \eta r/(1-r)$. Thus we obtain:

$$\frac{dr}{dx} = \frac{-s\langle\sigma v\rangle(1-r)^2}{\eta H(T)x}\left(\eta^2 r/(1-r)^2 - Y^+_{\rm eq}Y^-_{\rm eq}\right) = \frac{-s\langle\sigma v\rangle\eta}{H(T)x}\left(r - (1-r)^2Y^+_{\rm eq}Y^-_{\rm eq}/\eta^2\right)$$

The $s/H(T)$ factor is $\propto T^3/(T^2/m_{\rm Pl}) \propto Tm_{\rm Pl} \propto 1/x$. Using $Y^+_{\rm eq} = \eta/(1-r_{\rm eq})$, $Y^-_{\rm eq} = \eta r/(1-r_{\rm eq})$, we can rewrite the equation in the two required forms:

$$\frac{dr}{dx} \propto \frac{-\langle\sigma v\rangle\eta}{x^2}\left(r - (1-r)^2Y^+_{\rm eq}Y^-_{\rm eq}/\eta^2\right) = \frac{-\langle\sigma v\rangle\eta}{x^2}\left(r - \frac{(1-r)^2}{(1-r_{\rm eq})^2}r_{\rm eq}\right)$$

(e) As the DM freezes out, the $Y_{\rm eq}$ terms in the above equation will be exponentially suppressed (by the usual Boltzmann suppression). In the limit where this term approaches zero, solve the resulting differential equation for the late-time value of r. How does it depend on the annihilation cross section? Comment.

Solution: Dropping the equilibrium terms, we obtain:

$$\frac{dr}{dx} = \frac{-A\langle\sigma v\rangle r}{x^2},$$

where A is a (positive) constant. Solving this differential equation gives:

$$\int \frac{1}{r} dr = -A\langle\sigma v\rangle \int \frac{1}{x^2} dx \Rightarrow \ln r = C + A\langle\sigma v\rangle/x,$$

where C is a constant of integration, i.e. $r = e^C e^{A\langle\sigma v\rangle/x}$. If x_f denotes the value of x at which the abundances start to significantly diverge from their equilibrium values, then $r(x_f) \approx 1$ (assuming equal masses and degrees of freedom for the χ^+ and χ^- components). Thus we have $r \approx e^{A\langle\sigma v\rangle(1/x - 1/x_f)}$. Thus as $x \to \infty$, the ratio r converges to a constant value given approximately by $e^{-A\langle\sigma v\rangle/x_f}$. The late-time overall abundance is given by $Y^+ + Y^- = \eta(1+r)/(1-r)$, so if r becomes exponentially small, $Y^+ + Y^- \approx \eta$, which is fixed throughout the freezeout process and is totally determined by the primordial asymmetry. The residual relic abundance of the subdominant component – and the late-time annihilation rate – is thus *exponentially* sensitive to the annihilation cross section.

This exponential dependence contrasts with the standard symmetric freezeout scenario (corresponding to $A = 0$, as $A \propto \eta$) where the relic density of both components is inversely proportional to the annihilation cross section. The key difference is that in the case of symmetric freezeout, both the species undergoing depletion and its annihilation partner are simultaneously being exponentially depleted, leading to a rapid cutoff in the annihilation rate. In the asymmetric case, the subdominant species is annihilating against a partner whose abundance is *not* exponentially depleted (being fixed by η at late times), and thus its annihilations remain much more efficient.

2. Astrophysics-independent comparisons in direct detection

As mentioned in lecture, the usual calculation for direct detection makes several assumptions about the relevant particle physics, nuclear physics and astrophysics. In this problem, we will look at one way to factor out the unknown astrophysics, given certain assumptions on the particle physics model.

For elastic scattering, we will write the differential rate with respect to recoil energy in the form:

$$\frac{dR}{dE_R} = \frac{N_T M_T \rho}{2 m_\chi \mu^2} \sigma(E_R) \int_{v_{\min}}^{\infty} dv \frac{f(v,t)}{v}. \tag{1}$$

For the purposes of this problem, ignore both the motion of the Sun and the motion of the Earth around the Sun, and assume the DM velocity distribution in the Galactic frame is isotropic.

(a) Suppose a candidate WIMP event is detected in a silicon detector with recoil energy 12.3 keV (you may round all atomic masses to the nearest integer), and interpreted as the scattering of a 10 GeV WIMP. What is the smallest possible speed of the WIMP in question, in the lab frame?

Solution: The minimum relative velocity of the two particles, as derived in lectures, is $v_{\min} = \sqrt{m_T E_R / 2\mu^2}$. The atomic mass of silicon is 28 (to the nearest integer). One atomic mass unit corresponds to 0.93 GeV, for a total mass of 26.0 GeV. The reduced mass is thus $26 \times 10/(26 + 10) \approx 7.2$ GeV. This gives a minimum relative velocity of $v_{\min} = 1.8 \times 10^{-3} c = 530$ km/s, which is also the lab-frame speed of the WIMP (since in that frame the target is at rest).

(b) Suppose the scattering occurs at that minimum velocity. What is the maximum recoil energy of an identical WIMP (with the same velocity) scattering in a detector with target material: (i) Xenon, (ii) Sodium, (iii) Iodine, (iv) Germanium.

Solution: The atomic masses of the relevant species are 131 (xenon), 23 (sodium), 127 (iodine), and 73 (germanium). Taking $(E_R)_{\max} = 2\mu^2 v_{\rm rel}^2 / m_T$, we obtain the recoil energies (i) 4.3 keV (ii) 13.3 keV (iii) 4.4 keV (iv) 6.9 keV.

(c) Suppose a differential rate $dR/dE_R = K(E_R)$ is measured at one experiment, with target mass M_T^1, target number N_T^1 and cross section $\sigma_1(E_R)$. By writing $\int_{v_{\min}}^{\infty} dv \frac{f(v,t)}{v}$ in terms of $K(E_R)$, one can immediately predict the differential rate at a different experiment at the energy corresponding to the same $v_{\min}$. Use this idea to write down the predicted differential rate as a function of recoil energy at an experiment with target mass M_T^2, target number N_T^2 and cross section $\sigma_2(E_R)$, eliminating all dependence on the function $f(v)$.

This approach has been used to perform "astrophysics-independent" comparisons between possible signals (and constraints) at different experiments, although it does rely on knowing the kinematics of the interaction.

Solution: A recoil energy of E_R at experiment 2 corresponds to a $v_{\min}$ of $\sqrt{m_T^2 E_R/2(\mu_2)^2}$, where μ_2 is the reduced mass for a target of mass M_T^2. In turn, at experiment 1 this is the $v_{\min}$ for a recoil energy of $E_R' = 2\mu_1^2 v_{\min}^2/m_T^1 = E_R \frac{m_T^2}{m_T^1}(\mu_1/\mu_2)^2$, where μ_1 is the reduced mass calculated for a target of mass M_T^1.

Now for this same fixed $v_{\min}$ (corresponding to a recoil energy of E_R at experiment 2, and E_R' at experiment 1), we can write:

$$K(E_R') = \frac{N_T^1 M_T^1 \rho}{2m_\chi \mu_1^2}\sigma(E_R')\int_{v_{\min}}^{\infty} dv f(v,t)/v,$$

and the desired rate at experiment 2 is given by:

$$\begin{aligned}\frac{dR}{dE_R} &= \frac{N_T^2 M_T^2 \rho}{2m_\chi \mu_2^2}\sigma(E_R)\int_{v_{\min}}^{\infty} dv f(v,t)/v \\ &= \frac{N_T^2 M_T^2 \rho}{2m_\chi \mu_2^2}\frac{\sigma(E_R)}{\sigma(E_R')}K(E_R')\frac{2m_\chi \mu_1^2}{N_T^1 M_T^1 \rho} \\ &= \frac{N_T^2 M_T^2 \mu_1^2}{N_T^1 M_T^1 \mu_2^2}\frac{\sigma(E_R)}{\sigma(E_R')}K(E_R'), \end{aligned} \tag{2}$$

or writing out the dependences of E_R' explicitly,

$$\frac{dR}{dE_R} = \frac{N_T^2 M_T^2 \mu_1^2}{N_T^1 M_T^1 \mu_2^2}\frac{\sigma(E_R)}{\sigma\left(E_R \frac{m_T^2}{m_T^1}(\mu_1/\mu_2)^2\right)}K\left(E_R \frac{m_T^2}{m_T^1}(\mu_1/\mu_2)^2\right).$$