Lecture Notes for Trieste Summer School : Black holes, Binaries and GW observations

0 Syllabus

(75 minute lectures)

1. Kerr Black Hole
   - 1.1 History & metric
   - 1.2 Properties (horizon, ergo-sphere, singularity)
   - 1.3 geodesics
2. EMRIs
   - 2.1 Formation Scenarios and Modeling
   - 2.2 Semi-relativistic Approximation
   - 2.3 Kludge waveforms
   - 2.4 Self-force
3. Comparable-Mass Binaries
   - 3.1 Landau-Lifschitz formulation
   - 3.2 Relaxed Einstein Equations
   - 3.3 DIRE Approach
   - 3.4 End result (waveform and EoM)
4. Black Hole Perturbation Theory
   - 4.1 Scalar Field Example
   - 4.2 Schwarzschild Perturbations

N.B. These notes were prepared for lectures @ ICTP for grad students.
The notes have not been checked in detail, so use @ your own risk!

① Kerr Black Holes

References: Poisson, A Relativistic Toolkit, Cambridge Press
MTW, Gravitation

1.1 History and Metric

1916. Schw
1922 – 1924. Friedmann
1930s. Oppenheimer
1939 - 1945. War
1960s. New telescopes
New diff geom results
more scientists (end of nuclear race)

→ Golden Age

- Princeton (Wheeler)
- Cambridge (Penrose & Sciama, Carter) 1964 ↳ Physical Interpretation → 1964 Boyer-Lindquist
- Moscow (Zeldovich)
- Hamburg (Jordan & Ehlers)
- Texas (Schild, Kerr, Boyer) ↳ 1963 Solution → 1964 Boyer-Lindquist

Boyer Lindquist Coordinates

$$\boxed{ds^2 = -\left(1-\frac{2Mr}{\rho^2}\right)dt^2 - 4\frac{Mar}{\rho^2}\sin^2\theta\, dt\, d\varphi + \frac{\Sigma}{\rho^2}\sin^2\theta\, d\varphi^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2}$$

defs $\rho^2 \equiv r^2 + a^2\cos^2\theta \quad , \quad \Delta \equiv r^2 - 2Mr + a^2 \quad , \quad \Sigma = (r^2+a^2)^2 - a^2\Delta\sin^2\theta$

Ingoing Kerr Coordinates $\quad v = t + r^* \quad , \quad \psi = \varphi + r^\# \quad , \quad r^* \equiv \int \frac{r^2+a^2}{\Delta}dr, \quad r^\# \equiv \int \frac{a}{\Delta}dr$

$$\boxed{ds^2 = -\left(1-\frac{2Mr}{\rho^2}\right)dv^2 + 2\,dv\,dr - 2a\sin^2\theta\, dr\, d\psi - \frac{4Ma}{\rho^2}r\sin^2\theta\, dv\, d\psi + \frac{\Sigma}{\rho^2}\sin^2\theta\, d\psi^2 + \rho^2 d\theta^2}$$

Kerr-Schild Coordinates $\quad x + iy = (r + ia)\sin\theta\, e^{i\psi} \qquad z = r\cos\theta \qquad t' = v - r$

$$\boxed{ds^2 = \eta_{\alpha\beta}\,dx^\alpha dx^\beta + H\, l_\alpha l_\beta\, dx^\alpha dx^\beta}$$

defs $H \equiv \frac{2Mr}{\rho^2} \qquad l^\alpha\partial_\alpha = \frac{r^2+a^2}{\Delta}\partial_t - \partial_r + \frac{a}{\Delta}\partial_\varphi$

1.2 Properties

- Asymptotically flat
- Stationary but not static (t-independent but not invariant under time-reversal)
- M is the ADM mass and the Komar mass and the Hawking mass
  $a = J^{ADM}/M^{ADM}$ is the Kerr spin parameter
- $t^\alpha = \frac{\partial x^\alpha}{\partial t}$ and $\phi^\alpha = \frac{\partial x^\alpha}{\partial \phi}$ are Killing vectors → stationarity, → axisymmetry

- Special Observers (a) ZAMO = zero-angular momentum observer i.e. (1) observers that "rotate with the BH".

because if u^α 4-velocity / $\boxed{\tilde{L} = u_\alpha \phi^\alpha = 0}$ by (1),

then $g_{t\phi} \dot{t} + g_{\phi\phi} \dot{\phi} = 0 \longrightarrow \frac{\dot{\phi}}{\dot{t}} = \frac{d\phi}{dt} \equiv \omega = -\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{4Mar\sin^2\theta}{\rho^2} \frac{\rho^2}{\Sigma \sin^2\theta}$

this increases as ZAMO gets closer to BH and ZAMOs rotate in the same direction as BH and ZAMOs rotate @ same speed as horizon if ZAMOs located there ← $\boxed{\omega = \frac{2Mar}{\Sigma}} \sim \frac{2Ma}{r^3}$ as $r \gg M$

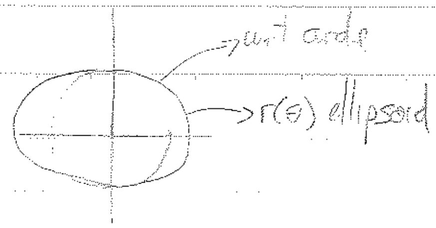
(b) Static Observers those with u^α / $u^\alpha = \gamma t^\alpha$

γ is a normalization constant so $u^\alpha u_\alpha = -1 \Rightarrow \gamma = (g_{tt})^{-1/2}$

[Note: static observers are non-geodesic since obs is "held in place" by external agent]

But static observers do not exist everywhere. If $g_{tt} = 0$ ∴ $\gamma \nexists$ ∴ no static observer

$$\boxed{r_{ergo}(\theta) = M + M\left(1 - \frac{a^2}{M^2}\cos^2\theta\right)^{1/2}}$$ ERGOSPHERE



+ Shape: If $a = M$, then $r_{ergo}(\theta) = M(1+\sin\theta)$

+ Interpretation: In $r < r_{ergo}(\theta)$ ∴ observers cannot remain static (indep of external force) and must rotate with BH ⟶ DRAGGING OF INERTIAL FRAMES

(c) Stationary observer = static + moving in ϕ direction with arbitrary velocity

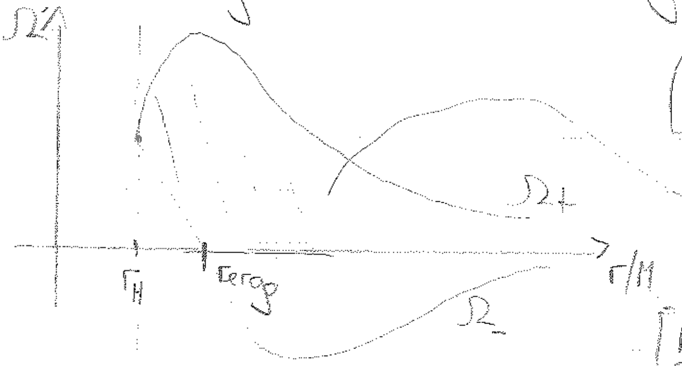
$$u^\alpha = \gamma(t^\alpha + \Omega\phi^\alpha) \quad \text{where } \gamma = g_{\phi\phi}^{-1/2}\left(2\omega\Omega - \Omega^2 - g_{tt}/g_{\phi\phi}\right)^{-1/2}$$

with $\Omega = \frac{d\phi}{dt}$ and $\omega \equiv -g_{t\phi}/g_{\phi\phi} = \frac{2Mar}{\Sigma}$

[Note. Stationary obs [does not perceive variations in the grav. field; not necessarily distinct from ZAMO; $t^\alpha + \Omega\phi^\alpha$ also a Killing vector.]]

But stationary observers do not exist anywhere. If $\gamma^{-2} = 0$ ∴ γ ∄ ∴ no stationary obs.

$$\boxed{\Omega_- < \Omega < \Omega_+ \quad \text{with } \Omega_\pm = \omega \pm \frac{\Delta^{1/2}\rho^2}{\Sigma\sin\theta}}$$



→ Stationary obs exist here but not in $r < r_H$

[Note: @ $r = r_H$, stationary obs = ZAMO]

What is r_H? It is when $\Omega_+ = \Omega_-$ ∴ $\frac{\Delta^{1/2}\rho^2}{\Sigma\sin\theta} = 0 \Longrightarrow \Delta = 0 \longrightarrow \boxed{r_\pm = M \pm M\left(1 - \frac{a^2}{M^2}\right)^{1/2}}$

$\boxed{r_H = r_+}$ → $\boxed{\text{EVENT HORIZON}}$

[Note: r_- is an apparent horizon that plays an important role in the maximal extension of Kerr]

[Note: @ $r = r_H$, $\omega_H \equiv \omega(r_H) = \Omega_H = \frac{a}{r_+^2 + a^2}$]

But how do you know it's a horizon?

- If $F = r - r_H = 0$ is a sfc, then its normal is null, i.e. $(\partial_\alpha F)(\partial^\alpha F) = 0$ (or $g^{rr} = 0 \rightarrow \frac{\Delta}{\rho^2} = 0$)
- Also an apparent horizon because the null expansion = 0, a Killing horizon because $t^\alpha + \Omega \phi^\alpha$ is null @ r_H.

But how do you know it's a BH?

Kretschmann scalar: $R_{\alpha\nu\gamma\beta} R^{\alpha\nu\gamma\beta} = \frac{48 M^2}{\rho^{12}} (r^2 - a^2\cos^2\theta)(\rho^4 - 16 a^2 r^2 \cos^2\theta)$

diverges @ $\rho^2 = 0 = r^2 + a^2\cos^2\theta \longrightarrow$ @ $r = 0$ and $\theta = \frac{\pi}{2}$

in Kerr Schild coords $x^2 + y^2 = (r^2 + a^2)\sin^2\theta$ \& $z = r\cos\theta$ So $x^2 + y^2 = a^2$ \& $z = 0$ circular ring on equator

[Note: $\rho = 0$ is always inside event horizon unless $a > M$]

1.3 Geodesics

$\frac{d^2 x^\alpha}{d\tau^2} = \Gamma^\alpha_{\mu\nu} u^\mu u^\nu$ | → 2nd Order System

Conserved Quantities
- → $\tilde{E} = -u_\alpha t^\alpha$ Energy
- → $\tilde{L} = u_\alpha \phi^\alpha$ Angular Momentum
- → $Q = u_\alpha u_\beta \xi^{\alpha\beta}$ Carter Constant → Rank 2 Killing Vector

First Order Reduction

Use E, L, Q and $u_\mu u^\mu = -1$ to solve for $(\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi})$

Defs
$$P \equiv \tilde{E}(r^2 + a^2) - a\tilde{L}$$
$$R \equiv P^2 - \Delta\left[(\tilde{L} - a\tilde{E})^2 + Q\right]$$
$$\Theta \equiv Q + \cos^2\theta\left(a^2\tilde{E}^2 - \tilde{L}^2/\sin^2\theta\right)$$

$$\rho^2 \dot{t} = -a(a\tilde{E}\sin^2\theta - \tilde{L}) + (r^2 + a^2) P/\Delta$$
$$\rho^2 \dot{r} = \pm R^{1/2} \qquad \rho^2 \dot{\theta} = \pm \Theta^{1/2}$$
$$\rho^2 \dot{\phi} = -(a\tilde{E} - \tilde{L}/\sin^2\theta) + aP/\Delta$$

Conservation of Quantities

$$\frac{d}{d\tau} = u^\alpha \nabla_\alpha \qquad \frac{d}{d\tau} E = u^\alpha \nabla_\alpha (-u_\beta \xi^\beta) = -u^\alpha u^\beta \nabla_\alpha \xi_\beta - \underbrace{u^\alpha \xi^\beta \nabla_\alpha u_\beta}_{=0 \text{ (geodesic)}}$$

$$= -u^\alpha u^\beta \nabla_{(\alpha} \xi_{\beta)} = \underbrace{0}_{\text{Killing Eq}}$$

$$\frac{d}{d\tau} Q = u^\alpha \nabla_\alpha (K_{\beta\gamma} u^\beta u^\gamma) =$$

$$= \underbrace{u^\alpha u^\beta u^\gamma \nabla_\alpha K_{\beta\gamma}}_{=0} + 2K_{\beta\gamma} u^\alpha \underbrace{\xi^\beta \nabla_\alpha u^\gamma}_{=0 \text{ geodesic Eq}}$$

By Killing Tensor Eq

$$\nabla_{(\gamma} K_{\alpha\beta)} = 0$$

② EMRIs ≡ Extreme Mass-Ratio Inspiral

[Refs Amaro-Seoane et al CQG 24, ..
Barack, CQG 26, '09
Poisson, et al, LRR 14, '11]

2.1 Formation Scenarios & Modelling

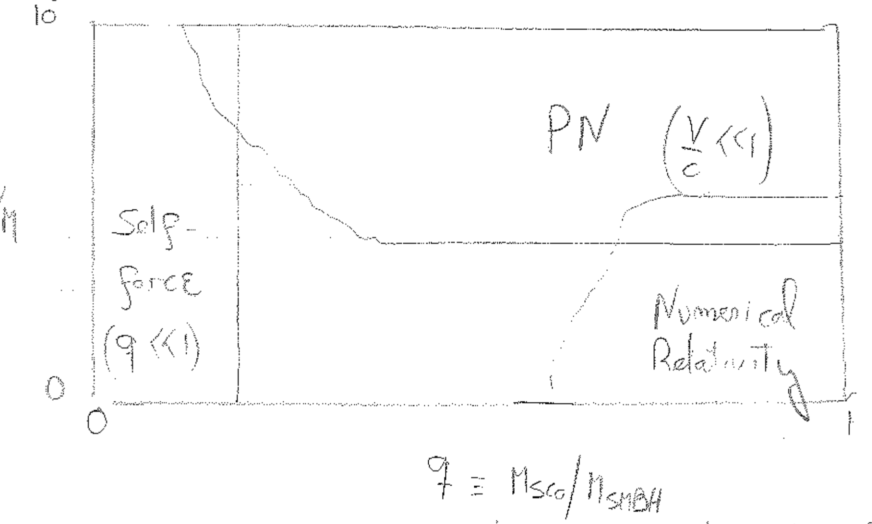
Capture : 3-Body processes swing SCO near SMBH
(more likely) SCO emits GW burst and it's captured

Disk Formation : Stars forms in outskirts of disk by fragmentation
(less likely) As it migrates inwards, it goes SN and forms a small BH
BH eventually migrates to inner disk

→ EMRI Stellar mass compact object in generic inspiral into SMBH

Modelling Must solve for geometry and motion $\boxed{G_{\mu\nu} = 8\pi T_{\mu\nu} \text{ and } \nabla^\mu G_{\mu\nu} = 8\pi \nabla^\mu T_{\mu\nu} = 0}$

Cannot be solved exactly in closed form. Coupled system of non-linear PDEs



NR for EMRIs too expensive → resolve SCO and GWs in the same grid; evolve for longer

PN for EMRIs valid only when $\frac{v}{c} \ll 1$ ⇒ Expand to extreme order and resum.

2.2 <u>Semi Relativistic Approximation</u> [Refs: Ruffini & Sasaki, PTP 66, 181]

Let $g_{\mu\nu} = g_{\mu\nu}^{SBH} + \varepsilon h_{\mu\nu}$ and $u^\mu = u^\mu_{geod} + \varepsilon \delta u^\mu$

and work to leading order in ε.

@ 0^{th} order $g_{\mu\nu} = g_{\mu\nu}^{SMBH}$ and $u^\mu = u^\mu_{gd}$

@ 1^{st} order Solve Einstein Eqs $G_{\mu\nu} = 8\pi T_{\mu\nu} \rightarrow \Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}$; $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h$
but use a Lorentz bg $g_{\mu\nu}^{SMBH} = \eta_{\mu\nu}$ and $T_{\mu\nu}$ of point particle

$$T^{pp}_{\mu\nu} = m \int_{-\infty}^{\infty} u^\mu u^\nu \delta^4(x - z(\tau))\, d\tau$$

↳ trajectory of SCO found by solving geod. eq

$$\therefore \boxed{\bar{h}_{\mu\nu} = -\frac{4m}{R} \frac{u_\mu u_\nu}{l_\alpha u^\alpha}\Big|_{ret}}$$

where $l^\alpha = (1, \vec{n})$ vector that points from field pt to the location of SCO
and $u^\alpha = \gamma(1, \vec{v})$
↳ Lorentz factor

can be proved thru Green's function

↳ Analogy to Lienard - Wiechert potential of E&M

2.3 <u>Kludge Waveforms</u> : Approximate solution that includes some relativistic aspects of the motion (more than the semi rel approx)

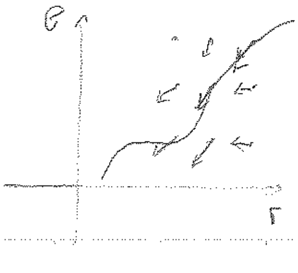
<u>Adiabatic Approximation</u> ≡ assume orbit evolves "slowly", ie its a sequence of <u>osculating</u> orbits
<u>Balance Law</u> ≡ relate the rate of change of "conserved quantities" to fluxes @ $\mathcal{J}^+$ and across horizon

$$\begin{aligned} \dot{t} &= \dot{t}(r,\theta; E, L, Q) \\ \dot{r} &= \dot{r}(r,\theta; E, L, Q) \\ \dot{\theta} &= \dot{\theta}(r,\theta; E, L, Q) \\ \dot{\varphi} &= \dot{\varphi}(r,\theta; E, L, Q) \end{aligned} \quad + \quad \begin{aligned} \dot{E} &= \dots \\ \dot{L} &= \dots \\ \dot{Q} &= \dots \end{aligned}$$

from GW emission

<u>PN methods</u> $\dot{E} \sim \dddot{I}_{ij} \dddot{I}^{ij} + \dots$

<u>BH PT methods</u> $\dot{E} \sim \dot{\psi}_4^2 + \dots$



and then insert into GW $h_{+,\times} = h_{+,\times}(t(\tau), r(\tau), \theta(\tau), \varphi(\tau))$

[See eg Hughes, PRD 64 '00]

Harmonic Gauge Condition

$$\boxed{\Box X^{\mu} = 0} \quad \text{Harmonic Coords}$$

$$g^{\alpha\beta} \nabla_\alpha \nabla_\beta X^{\mu} = 0$$

$$\frac{1}{\sqrt{-g}} \partial_\alpha \left(\sqrt{-g}\, g^{\alpha\beta} \partial_\beta X^{(\mu)} \right) = 0$$

Pick $X^\mu = x^\mu$ cartesian coords

$$\partial_\beta \left(\sqrt{-g}\, g^{\alpha\beta} \right) = 0 = g^{\mu\nu} \Gamma^{\alpha}_{\mu\nu}$$

Let $\sqrt{-g}\, g^{\alpha\beta} = \mathfrak{g}^{\alpha\beta}$ gothic metric

and expand $\mathfrak{g}^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta}$ ...

$$\boxed{\partial_\beta h^{\alpha\beta} = 0} \quad \text{Harmonic Gauge Condition}$$

Note: $(-g) = 1 + h$, $h = h_{\mu\nu}\eta^{\mu\nu}$ so $g_{\mu\nu} = \frac{1}{\sqrt{-g}}(\eta_{\mu\nu} + h_{\mu\nu}) = \left(1 - \frac{h}{2}\right)(\eta_{\mu\nu} + h_{\mu\nu})$

$$g_{\mu\nu} = \eta_{\mu\nu} + \left(h_{\mu\nu} - \frac{1}{2} h\, \eta_{\mu\nu} \right) \rightarrow \bar{h}_{\mu\nu}$$

Lienard - Wiechert derivation

$$\varphi(\vec{r}, t) = \frac{1}{4\pi} \int \frac{\rho(\vec{r}\,', t_r')}{|\vec{r} - \vec{r}\,'|} d^3 r' \quad \text{and } \rho(\vec{r}\,', t') = q\, \delta^3(\vec{r}\,' - \vec{r}_s(t_r'))$$

$$= \frac{1}{4\pi} \iint q \frac{\delta^3(\vec{r}\,' - \vec{r}_s(t'))}{|\vec{r} - \vec{r}\,'|} \delta(t' - t_r')\, dt'\, d^3\vec{r}\,'$$

But $\delta(f(t')) = \sum_i \frac{\delta(t' - t_i)}{|f'(t_i)|}$ $\forall\, t_i$ that are zeroes of f then

$$\delta(t' - t_r') = \frac{\delta(t' - t_r)}{\frac{\partial}{\partial t'}(t' - t_r')\big|_{t' = t_r}} = \frac{\delta(t' - t_r)}{\partial_{t'}\left(t' - (t - |\vec{r} - \vec{r}_s(t')|)\right)\big|_{t'=t_r}} = \frac{\delta(t' - t_r)}{1 - \vec{v}_s \cdot (\vec{r} - \vec{r}_s)/|\vec{r} - \vec{r}_s|} \quad \text{So}$$

$$\varphi(\vec{r}, t) = \frac{1}{4\pi} \left(\frac{q}{(1 - \vec{n} \cdot \vec{v}_s)\, |\vec{r} - \vec{r}_s|} \right) \Bigg|_{t_r} \quad \Box$$

2.4 Self-force : Do a formal expansion in $q : \frac{M^{sco}}{M^{SMBH}} \equiv \frac{m}{M} \ll 1$ -

Motion (I) $\frac{d^2 z'^\alpha}{d\tau'^2} + \Gamma'^\alpha_{\mu\nu} \frac{dz'^\mu}{d\tau'} \frac{dz'^\nu}{d\tau'} = 0$ prime denotes trajectory z'^μ in $g_{\mu\nu} = g^{SMBH}_{\mu\nu} + \varepsilon h_{\mu\nu}$

and re-interpret this as forced geodesic motion (II) $m\left(\frac{d^2 z^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu} \frac{dz^\mu}{d\tau} \frac{dz^\nu}{d\tau}\right) = F^\alpha_{\text{self-force}}$ (II)

But what is the self-force? $\frac{d}{d\tau} = \left(\frac{d\tau'}{d\tau}\right) \frac{d}{d\tau'}$

Convert from prime to unprime $\frac{d^2 z^\alpha}{d\tau^2} = \frac{d^2 z^\alpha}{d\tau'^2}\left(\frac{d\tau'}{d\tau}\right)^2 + \left(\frac{d^2\tau'}{d\tau^2}\right)\frac{dz^\alpha}{d\tau'}$

using (I) in (II), we get

$$F_{\text{self-force}} = m\left(\frac{d\tau'}{d\tau}\right)^2\left(-\Gamma'^\alpha_{\mu\nu}\frac{dz^\mu}{d\tau'}\frac{dz^\nu}{d\tau'}\right) + m\left(\frac{d^2\tau'}{d\tau^2}\right)\left(\frac{dz^\alpha}{d\tau'}\right) + m\,\Gamma^\alpha_{\mu\nu}\frac{dz^\mu}{d\tau}\frac{dz^\nu}{d\tau}$$

$$F_{sf} = m\left(\frac{d^2 z^\alpha}{d\tau'^2}\left(\frac{d\tau'}{d\tau}\right)^2 + \frac{d^2\tau'}{d\tau^2}\frac{dz^\alpha}{d\tau'}\right) + m\,\Gamma^\alpha_{\mu\nu}\frac{dz^\mu}{d\tau}\frac{dz^\nu}{d\tau}$$

$$= -m\underbrace{\left(\Gamma'^\alpha_{\mu\nu} - \Gamma^\alpha_{\mu\nu}\right)}_{\equiv\, \Delta\Gamma^\alpha_{\mu\nu} \text{ is a tensor}}\frac{dz^\mu}{d\tau}\frac{dz^\nu}{d\tau} + \underbrace{m\left(\frac{d^2\tau'}{d\tau^2}\right)\left(\frac{d\tau}{d\tau'}\right)\left(\frac{dz^\alpha}{d\tau}\right)}_{\text{Parallel to } u^\alpha}$$

but F_{sf} is $\perp$ to u^α

Proof: $F^\alpha_{sf} = m u^\mu \nabla_\mu u^\alpha$; $F^\alpha_{sf} u_\alpha = m u^\mu u^\alpha \nabla_\mu u_\alpha = 0$

∴ must contain a piece that is parallel to u^α that cancels the term

Since we only care about $\perp$ part, project!

$$\boxed{F^\alpha_{sf} = (g^{\alpha\lambda} + u^\alpha u^\lambda) F_\lambda = -\frac{1}{2} m (g^{\alpha\lambda} + u^\alpha u^\lambda)(\nabla_\nu h_{\lambda\mu} + \nabla_\mu h_{\lambda\nu} - \nabla_\lambda h_{\mu\nu}) u^\mu u^\nu \equiv m \nabla^{\alpha\beta\gamma} h_{\beta\gamma}}$$

"Small" Problems (I) test particle has divergent self-field @ its location

(II) delta functions are not strong-field representations of pt. ptcls → Hoop Conjecture (if you squeeze, you collapse) (Thorne '72)

Resolution: Treat SCO as a BH and use asymptotic matching

$$\boxed{F^\alpha_{sf} = m \nabla^{\alpha\beta\gamma} h^{(R)}_{\beta\gamma} \quad \text{where} \quad h^{(R)}_{\beta\gamma} = h_{\beta\gamma} - h^{(S)}_{\beta\gamma}} \rightarrow \text{MiSaTaQuWa Eq}$$

∴ Given from $\Box \bar{h}_{\alpha\beta} + 2 R^\mu{}_\alpha{}^\nu{}_\beta \bar{h}_{\mu\nu} = -16\pi T_{\alpha\beta}$ via Green function

→ must be solved simultaneously

3. Comparable-Mass Binaries

[Refs: Poisson & Will, Gravity, Cambridge Press]

3.1 Landau-Lifshitz Formulation

Gothic inv. metric $\mathfrak{g}^{\alpha\beta} \equiv \sqrt{-g}\, g^{\alpha\beta}$ tensor density

[Note: $\det(\mathfrak{g}^{\alpha\beta}) = g = (\sqrt{-g})^4 \frac{1}{g} = g$
so $g^{\alpha\beta} = \frac{1}{\sqrt{-g}} \mathfrak{g}^{\alpha\beta}$]

H tensor $H^{\alpha\mu\beta\nu} \equiv \mathfrak{g}^{\alpha\beta}\mathfrak{g}^{\nu\mu} \rightarrow$ same symmetries as $R^{\alpha\mu\beta\nu}$

Notably $\boxed{\partial_{\mu\nu} H^{\alpha\mu\beta\nu} = 2(-g)G^{\alpha\beta} + 16\pi(-g)\, t_{LL}^{\alpha\beta}} \rightarrow$ Landau-Lifshitz pseudo-tensor

$t_{LL}^{\alpha\beta} = \frac{1}{16\pi}(-g)^{-1}\left\{ \partial_\lambda \mathfrak{g}^{\alpha\beta}\partial_\mu \mathfrak{g}^{\lambda\mu} + \dots \right\}$

$\Downarrow$

Field Eqs become $\boxed{\partial_{\mu\nu} H^{\alpha\mu\beta\nu} = 16\pi(-g)\left(T^{\alpha\beta} + t_{LL}^{\alpha\beta}\right)}$

$\downarrow$ leads to conservation Eq

(good to find conserved quantities, like P^μ, $J^{\mu\nu}$) $\leftarrow \boxed{\partial_\beta\left[(-g)\left(T^{\alpha\beta} + t_{LL}^{\alpha\beta}\right)\right] = 0} \rightarrow$ because $\partial_{\beta\mu\nu} H^{\alpha\mu\beta\nu} = 0$ by symmetry

3.2 Relaxed Einstein Equations

(a) Harmonic Coordinates $\Box x^\alpha = 0 \longrightarrow \boxed{\partial_\beta\left(\sqrt{-g}\, g^{\alpha\beta}\right) = \partial_\beta \mathfrak{g}^{\alpha\beta} = 0}$

(b) Metric Perturbation $\boxed{h^{\alpha\beta} \equiv \eta^{\alpha\beta} - \mathfrak{g}^{\alpha\beta}} \longrightarrow g^{\alpha\beta} = \frac{1}{\sqrt{-g}}\left(\eta^{\alpha\beta} - h^{\alpha\beta}\right)$

[Note: (a)+(b) imply $\partial_\beta h^{\alpha\beta} = 0$, which is similar to Lorentz gauge, though that's only to $\mathcal{O}(h)$]

(c) Einstein Equations $\partial_{\mu\nu} H^{\alpha\mu\beta\nu} = -\Box_\eta h^{\alpha\beta} + \underbrace{h^{\mu\nu}\partial_{\mu\nu} h^{\alpha\beta} - \left(\partial_\mu h^{\alpha\nu}\right)\left(\partial_\nu h^{\beta\mu}\right)}_{\equiv 16\pi g\, t_H^{\alpha\beta}}$ (Exact)

$\therefore \boxed{\Box_\eta h^{\alpha\beta} = -16\pi\,\tau^{\alpha\beta} = -16\pi(-g)\left(T^{\alpha\beta} + t_{LL}^{\alpha\beta} + t_H^{\alpha\beta}\right)} \rightarrow$ Relaxed Einstein Eqs

$\rightarrow$ Exact!

Note: Since partials commute, Relaxed Einstein Eqs imply $\boxed{\partial_\alpha \tau^{\alpha\beta} = 0}$ → similar to SET conservation

Formal Solution $\boxed{h^{\alpha\beta} = 4\int G(x,x')\,\tau^{\alpha\beta}(x')\,d^4x'}$

Retarded Green's Function for $\Box_\eta$, i.e. $\Box_\eta G(x,x') = -4\pi\,\delta^4(x-x')$

$$\boxed{h^{\alpha\beta} = 4\int \frac{\tau^{\alpha\beta}(t-|\vec{x}-\vec{x}'|)}{|\vec{x}-\vec{x}'|}\,d^3x'} \qquad G(x,x') = \frac{\delta(t-t'-|\vec{x}-\vec{x}'|)}{|\vec{x}-\vec{x}'|}$$

General Procedure: Expand in powers of G: $h^{\alpha\beta} = G\,k_1^{\alpha\beta} + G^2\,k_2^{\alpha\beta} + \ldots$ → $\boxed{\text{POST-MINKOWSKIAN EXPANSION}}$

0^{th} Order: $h_0^{\alpha\beta} = 0 \longrightarrow g_{\alpha\beta} = \eta_{\alpha\beta} \longrightarrow \tau_0^{\alpha\beta} = T_m^{\alpha\beta}(\eta^{\mu\nu})$ since $t_{LL_0}^{\alpha\beta} = 0 = t_{H_0}^{\alpha\beta}$

1^{st} Order: $\Box h_1^{\alpha\beta} = -16\pi\,\tau_0^{\alpha\beta} = -16\pi\,T_m^{\alpha\beta}(\eta^{\alpha\beta}) \xrightarrow{\text{solve to}}$ get $h_1^{\alpha\beta} \longrightarrow$ construct $\tau_1^{\alpha\beta}$

will depend on $T_m^{\alpha\beta}[g_1^{\mu\nu}]$, $t_{LL}^{\alpha\beta}[h_1^{\mu\nu}]$ & $t_H^{\alpha\beta}[h_1^{\mu\nu}]$

⋮

Once you stop, you'll have $g_{\mu\nu}^{(n)}$ in terms of the trajectory $z^\alpha(\tau)$ which you determine by requiring $\underline{\partial_\beta \tau_{n-1}^{\alpha\beta} = 0}$ → gives you Eq of motion

construct $h^{\alpha\beta}(t)$ by composition. ⟵ Solve them in slow-motion approx ⟵ $\dfrac{d^2\vec{z}_A}{dt^2} = \mathcal{O}(G) + \mathcal{O}(G^2) + \mathcal{O}(G^3) + \ldots$

3.3 DIRE Approach : Direct Integration of the Relaxed Einstein Equations.

Split Regions of Integration: Near Zone (N) $(r \ll \lambda_c)$ & Wave Zone (W) $(r \gg \lambda_c)$

$\lambda_c \equiv$ characteristic wavelength of radiation $\left(\equiv \frac{b}{v}\right.$ for a binary$\left.\right)$

Note: Retardation is $\begin{pmatrix}\text{important} \\ \text{unimportant}\end{pmatrix}$ in the $\begin{pmatrix} NW \\ W \end{pmatrix}$ because $T_R \equiv t - r$ ↗ $T_R \ll \lambda_c$ in W ; ↘ $T_R \gg \lambda_c$ in NW

This implies that in W $\quad r\frac{\partial T}{\partial t} \approx r\frac{T}{\lambda_c} \ll T \quad$ so $\boxed{\frac{|\partial T/\partial t|}{|\partial T/\partial \vec{x}|} \ll 1 \text{ in } W}$

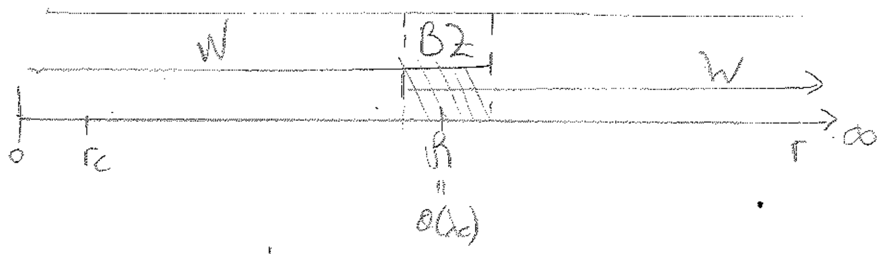
Slow-motion condition in W

require source has compact support $r_c \ll \lambda_c$ (in W)

require source moves slowly $v_c \equiv \frac{r_c}{t_c} \approx \frac{r_c}{\lambda_c} \ll 1$ (in W)

↳ speed with which changes in src propagate through regions occupied by src

Eg, v_c = sound speed for fluid
v_c = orbital speed for binary



$\therefore \; h_{\mu\nu} = h^{W}_{\mu\nu} + h^{W}_{\mu\nu}$ and you calculate each piece separately using different approximations valid in each region.

3.4 End Result (for a binary in a quasi-circular orbit) (See Blanchet's LRR)

$$h_+ = \frac{1}{2}\left(e^j_x e^k_y - e^j_y e^k_y\right) h_{jk} = 2\eta \frac{m}{\Delta} X \left[\underset{0}{H_+} + \underset{1/2}{H_+} X^{1/2} + \underset{1}{H_+} X + \dots \right] \qquad X \equiv (m\omega)^{2/3} \approx v^2$$

$$h_\times = \frac{1}{2}\left(e^j_x e^k_y + e^j_y e^k_x\right) h_{jk} = 2\eta \frac{m}{\Delta} X \left[\underset{0}{H_\times} + \underset{1/2}{H_\times} X^{1/2} + \underset{1}{H_\times} X + \dots \right] \qquad \eta = m_1 m_2 / m^2, \quad m = m_1 + m_2$$

Defs $\quad \underset{0}{H_+} = -(1 + C_i^2)\cos(2\phi) - \frac{1}{96} S_i^2 (17 + C_i^2)$

$\underset{0}{H_\times} = -2 C_i \sin(2\phi)$

$\vdots$

So how do you deal with BHs? → BHPT

4 Black Hole Perturbation Theory

[Refs: Berti et al, QNM of BHs & Black Branes
Teukolsky, ApJ 185, 173
Kokkotas & Schmidt, LRR
Nollert, CQG 16, '98]

4.1 Scalar Field Example

- Consider a Complex Scalar Field Φ

$$\mathcal{L} = -(\partial_\mu \Phi)^\dagger (\partial^\mu \Phi) - \frac{d-2}{4(d-1)} \gamma R \Phi^\dagger \Phi - m^2 \Phi^\dagger \Phi$$

on a d-dimensional Schw-AdS BH $ds^2 = -f\,dt^2 + f^{-1} dr^2 + r^2 d\Omega^2_{d-2}$; $f = 1 + \frac{r^2}{L^2} - \frac{r_0^{d-3}}{r^{d-3}}$

$L \equiv$ ADM radius, $r_0 \equiv$ mass in 4D ; $\gamma \equiv$ coupling constant $\quad M = \frac{d-2}{16\pi} A_{d-2}\, r_0^{d-3}$

$$A_{d-2} = (2\pi)^{(d-1)/2} / \Gamma[(d-1)/2]$$

- Equations of Motion

$$\Box \Phi = \frac{d-2}{4(d-1)} \gamma R \Phi \quad (\text{setting } m=0) \quad \text{and} \quad G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T^{\text{scalar field}}_{\mu\nu}$$

- Perturb $\quad g_{\mu\nu} = g^B_{\mu\nu} + h_{\mu\nu}$ and $\Phi = \Phi_B + \varphi$

If $g^B_{\mu\nu}$ = Schw-AdS $\therefore \Phi_B = 0 \therefore$ $\boxed{\Box^B \varphi = \frac{d(d-2)}{4L^2} \gamma \varphi}$ (1)

- Ansatz $\quad \varphi = \sum_{lm} e^{i\omega t} \frac{\Psi(r)}{r^{(d-2)/2}} Y_{lm}(\theta)$ ↳ d^2-dimensional scalar sph. harmonics

- Master Eq (1) becomes $f^2 \Psi'' + f f' \Psi' + (\omega^2 - V)\Psi = 0$

$$\text{with } V = f\left[\frac{l(l+d-3)}{r^2} + \frac{d-2}{4}\left(\frac{(d-4)}{r^2} f + \frac{2f'}{r} + \frac{d\gamma}{L^2}\right)\right]$$

or using tortoise coords, $\frac{dr^*}{dr} = 1/f$ $\quad \boxed{\frac{d^2\Psi}{dr_*^2} + (\omega^2 - V)\Psi = 0} \longrightarrow$ Schrödinger-like Eq

4.2 Schwarzschild Perturbation Theory

- Let $g_{\mu\nu} = g^{schw}_{\mu\nu} + h_{\mu\nu}$ and decompose $h_{\mu\nu} = \sum_{lm} {}^{A}h^{lm}_{\mu\nu} + {}^{P}h^{lm}_{\mu\nu}$

where

$$ {}^{A}h^{lm}_{\mu\nu} = \left(\begin{array}{cc|c} 0 & 0 & h^{lm}_A S^{lm}_a \\ 0 & 0 & \\ \hline & / & H^{lm} S^{lm}_{ab} \end{array}\right) \quad \text{and} \quad {}^{P}h^{lm}_{\mu\nu} = \left(\begin{array}{c|c} h_{AB} Y^{lm} & P^{lm}_A Y^{lm}_a \\ \hline / & r^2 (K^{lm} Y^{lm}_{ab} + G^{lm} Z^{lm}_{ab}) \end{array}\right) $$

$(A, B, ..) = (t, r)$ and $(a, b, ..) = (\theta, \varphi)$ Eg, ${}^{A}h^{lm}_{t\theta} = h^{lm}_t S^{lm}_\theta$ ${}^{P}h^{lm}_{t\nu} = P^{lm}_t Y^{lm}_\theta$

- Tensor Spherical harmonics are eigen-functions of ang. sector of Linearized EEs in Sch(=)

Scalar: Y^{lm} **Vector**: $Y^{lm}_a = Y^{lm}_{:a}$ (axial) **Tensor** $Y^{lm}_{ab} = Y^{lm}\,\Omega_{ab}$ (polar)

$S^{lm}_a = \varepsilon_a{}^b Y^{lm}_b$ (polar) $Z^{lm}_{ab} = Y^{lm}_{:ab} + \frac{l(l+1)}{2} Y^{lm}\Omega_{ab}$ (polar)

$S^{lm}_{ab} = S_{(a:b)}$ (axial)

Defs: $:$ = cov. deriv wrt Ω_{ab}

Ω_{ab} = metric of 2-sphere

Polar = $\hat{P}[h^{lm}] = (-1)^l h^{lm}$

Axial = $\hat{P}[h^{lm}] = (-1)^{l+1} h^{lm}$

Note: Tensor Sph H satisfy orthogonality conditions. eg $\int Y^{lm} Y^{l'm'} d\Omega = \delta \delta$

- Since Tensor Sph. Harmonics satisfy the Linearized Einstein Eqs, the latter become ODEs for radial functions $(h^{lm}_A, h^{lm}_{AB}, H^{lm}, K^{lm}, G^{lm})$

Axial Sector

- But ODEs are coupled! You can decouple by combining eqs intelligently.

First, evaluate Linearized Einstein Eqs $G_{AB} = \sum_{lm} g^{lm}_{AB} Y^{lm}$, $G_{Aa} = \sum_{lm} g^{lm}_A Y^{lm}_a + \Omega^{lm}_A S^{lm}_a$

$$G_{ab} = \sum_{lm} g^{lm} Y^{lm}_{ab} + \Omega^{lm} Z^{lm}_{ab} + I^{lm} S^{lm}_{ab}$$

Because of orthogonality every differential prefactor vanishes independently, ie, eg if $G_{Aa} = 0 \therefore g^{lm}_A = 0$, $\Omega^{lm}_A = 0$

Now combine components $\frac{f}{r}\left[\frac{2}{r}\left(1 - \frac{3M}{r}\right) I^{lm} + 2f\,\Omega^{lm}_r - f\,\partial_r I^{lm}\right] = 0$

where $f \equiv 1 - 2M/r$

RW gauge $H^{lm} = 0$

$G^{lm} = P^{lm}_t = P^{lm}_r = 0$

finally simplify to find $\boxed{\left(-\partial_t^2 + \partial_{r*}^2 - V^{lm}_{RW}\right)\psi^{lm}_{RW} = 0}$ Regge-Wheeler Eq.

where $V^{lm}_{RW} = \frac{f}{r^2}\left[l(l+1) - 6\frac{M}{r}\right]$, $\partial_{r*} = f\partial_r$, $\psi^{lm}_{RW} = \frac{f}{r}\left(h^{lm}_r - \frac{H_{,r}}{2} + \frac{H}{r}^{lm}\right)$

↳ Regge-Wheeler function

Polar Sector

- Combine Eqs

$$\frac{4f^2}{\Lambda_l} g^{lm}_r - \frac{2f}{r} H^{lm} + \frac{2f}{2\Lambda_l(\lambda_l+1)}\Big\{-2r^2 f\partial_r M^{lm} + \frac{12M}{\Lambda_l} f^2 g^{lm}_{rr}$$
$$+\frac{4f}{r} g^{lm} + \frac{r}{2\lambda_l}\left[4\lambda_l(\lambda_l-1) + 12(2\lambda_l - 3)\frac{M}{r} + 84\frac{M^2}{r^2}\right]M^{lm}\Big\}$$

where $M^{lm} = -\frac{g^{lm}_{tt}}{f} + f g^{lm}_{rr}$, $\lambda_l = \frac{(l+2)(l-1)}{2}$, $\Lambda_l = \lambda_l + \frac{3M}{r}$

Simplify $\boxed{\left(-\partial_t^2 + \partial_{r*}^2 - V^l_{ZM}\right)\psi^{lm}_{ZM} = 0}$ Zerilli-Moncrief Eq

where $\psi^{lm}_{ZM} = \frac{r}{1+\lambda_l}\left\{K^{lm} + (1+\lambda_l)G^{lm} + \frac{f}{\Lambda_l}\left[f h^{lm}_{rr} - rK^{lm}_{,r} - \frac{2}{r}(1+\lambda_l)h^{lm}_r\right]\right\}$

Zerilli-Moncrief function

$$V^l_{ZM} = \frac{f}{r^2\Lambda_l^2}\left[2\lambda_l^2\left(1+\lambda_l+\frac{3M}{r}\right) + 18\frac{M^2}{r^2}\left(\lambda_l + \frac{M}{r}\right)\right]$$

[Note: ψ^{lm}_{ZM} and ψ^{lm}_{RW} are gauge invariant "master functions"]

spin-weighted spherical harmonics

Metric Reconstruction $\boxed{h_+ - i h_\times = \frac{1}{2r}\sum_{lm}\sqrt{\frac{(l+2)!}{(l-2)!}}\left(\psi^{lm}_{ZM} + i\,\psi^{lm}_{CPM}\right){}_{-2}Y^{lm}(\theta,\phi)}$

where $\psi^{lm}_{CPM} = \frac{r}{\lambda_l}\left(h^{lm}_{r,t} - h^{lm}_{t,r} + \frac{2}{r}h^{lm}_t\right)$ → similar master function to ψ^{lm}_{RW} but it requires different combination of components to decouple

$$\boxed{0 = 2\frac{rf}{\lambda_l}\left[H_{r,t} - H_{t,r} - \frac{f_{,r}}{f}H_t\right]}$$

(same potential V^l_{RW} though)

Fluxes @ $\mathscr{I}^+$

$$\dot{E}_{GW} = \frac{1}{64\pi}\sum_{lm}\frac{(l+2)!}{(l-2)!}\left[\left|\dot{\Psi}^{lm}_{CPM}\right|^2 + \left|\dot{\Psi}^{lm}_{ZM}\right|^2\right]$$

$$\dot{L}_{GW} = \frac{1}{64\pi}\sum_{lm} im\,\frac{(l+2)!}{(l-2)!}\left[\bar{\Psi}^{lm}_{CPM}\,\dot{\Psi}^{lm}_{CPM} + \bar{\Psi}^{lm}_{ZM}\,\dot{\Psi}^{lm}_{ZM}\right]$$

[Note: All of this assumed vaccum perturbations (like for ring down), but similar eqs are valid for EMRIs if you put a pt ptc SET on RHS of EEs]