

Lectures 1 & 2

Plan of the lectures: 1 & 2) Basic techniques to deal with open quantum systems. Few examples and an exactly solvable model.

3) Many-body open systems } Dissipative phase transitions & state preparation.

Let's start from the basics → recap some "trivial" observations that apply to classical systems.

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 the description of the dynamics should be given in terms of the variables of the system only, the presence of an external environment will enter through additional terms in the equations of motion that account for dissipation.

(γ, χ) → its form tells how dissipation affects the eq. of mot.

macroscopic parameter that tells the "strength" of the coupling to the external world

We would like to "export" the same strategy to the quantum world.

↳ modify the dynamical equations for the quantum state so that additional contributions appear taking care of the action of the bath on the system.

↳ also in the quantum case coupling constants capturing the "strength" of bath will appear.

There are various ways to tackle this problem in quantum systems.

formal, phenomenological, microscopic, ... (comment on some salient features of all of them.)

Since the dynamics is not unitary the state of the system is described by a density matrix ρ

ρ is { an hermitian operator
 non-negative "
 $\text{Tr} \rho = 1, \text{Tr} \rho^2 \leq 1$

Important concept that will be used several times is that of the reduced density matrix

$$\rho_A = \text{Tr}_B \rho$$

B can be the environment and A the system.

brief comment on the "formal" approach

What is the most general operation ^(transformations) that we can do on density matrices?

$M \in \mathcal{D}_H^2$

$$\rho' = \sum_{k=0}^{M^2-1} E_k \rho E_k^\dagger \quad \sum_k E_k^\dagger E_k = \mathbb{1}$$

Quantum operation:
it maps density matrices into density matrices.

Kraus operators

- If a map is
- 1) linear
 - 2) preserves hermiticity
 - 3) trace preserving
 - 4) completely positive?

→ has a Kraus representation.
(introduce an extra system R then the map that acts only on the system $\mathcal{H}_R \otimes \mathcal{H}_A$ is positive)

The Kraus representation includes description of the unitary evolution, measurement, ...
→ There is no notion of time/evolution.

Examples:

Bit-flip channels	$E_0 = \sqrt{1-p} \mathbb{1}$	$E_1 = \sqrt{p} \sigma_x$
Depolarizing "	$E_0 = \sqrt{1-p} \mathbb{1}$	$E_\alpha = \sqrt{\frac{p}{3}} \sigma_\alpha$

Exercise: try to apply these transformations many times and see what is the steady state of the system.

↓ "One step further" towards a dynamical equation for open system.

From Kraus representation to Lindblad equation

→ we would like to get a continuous (in time) transformation.

→ Start from an n -hermal Kraus transformation.

obviously

$$\rho(t+dt) = \sum_k E_k(dt) \rho(t) E_k^\dagger(dt)$$

$E_k(0) = \delta_{k,0}$ → for infinitesimal dt $E_0 = \mathbb{1} - \mathcal{L}dt$ → consider a t -independent process.

Specifically

$$E_0 = \mathbb{1} + (K - iH)dt$$

$$E_k = L_k \sqrt{dt}$$

$$K = \sum_{k=1}^M L_k^\dagger L_k$$

↳ from the normalization condition given above.

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it follows that

$$\dot{\rho} = -i[H, \rho] + \sum_{k=1}^{M-1} (L_k \rho L_k^\dagger - \frac{1}{2} \{L_k^\dagger L_k, \rho\})$$

It is a Markovian (memory-less) equation. Note that (in principle) the number of Lindblad operators scales exponentially (as the dimension of the Hilbert space) with the degrees of freedom.

■ Some considerations:

1) It may be employed in a "phenomenological" manner by choosing properly the L_k and giving proper coupling constants. [Note that it is not a phenomenological eq.]

2) Nature of the steady state. (number of possible fixed points, thermalization, limit cycles, ...) dep-d on the structure of the Lindblad operator.

↳ eigenvalues have a non-negative real part. (right)

3) The steady state is determined by a competition of the Lindblad operators and/or the competition between H and L .

4) In some cases the steady state is pure
for example $L = \sigma^-$ and $H = 0$

In some cases it is mixed $L_0 = \sigma^-$, $L_1 = \sigma^+$

5) An analogous "non-Markovian" eq. does not exist.

6) If coherences are neglected \rightarrow Lindblad \equiv master eq.

7) The steady-state density from Lindblad does not need to be thermal.

F_{co} - abstract \rightarrow to a "more physical" situation.

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\downarrow This tracing out the environmental degrees of freedom cannot be done exactly (except from few cases).

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V describes the interaction between the system and the environment.

$$V = \sum_i \hat{O}_i \hat{F}_i \quad \text{example } \sum_{k,r} g_{kr} (\hat{\sigma}^+ b_{kr} + \text{h.c.})$$

$\rho(t) \rightarrow$ describes the state of the universe $\dot{\rho} = -i[H, \rho]$

We would like an equation for $\rho(t) = \text{Tr}_E \rho(t)$

\downarrow I will describe a way to obtain it in ^{perturbative} an expansion in the coupling between the system and the environment. \rightarrow construct the mean of this perturbative exp.

"Rough derivation"

- Go ~~to~~ the interaction representation

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The eq. can be integrated with the result

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↳ This is a good starting point to trace out the environment.

$$\dot{\rho}_I(t) = -i \text{Tr}_E [V_I(t), \rho_I(t_0)] - \int_{t_0}^t dt' \text{Tr}_E [V_I(t), [V_I(t'), \rho_I(t')]]$$

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2) In the weak coupling approx the trace over the bath depends only on correlators of the form

$$\text{Tr}_E [V_I(t_1) V_I(t_2) \rho_E] \quad \text{stationary bath.}$$

the correlator above depends on $t_1 - t_2$

3) If the correlator is sufficiently short range → Markov approximation $\rho_I(t') \sim \rho_I(t) = \rho_I(t) \rho_E$

The equation reads

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Remember that $V_I = O_I F_I \Rightarrow$ One has to perform two types of traces.

$\text{Tr}_I F_I G_E =$ we assume to be zero. This is not really relevant since this term can be always eliminated by a shift in the Hamiltonian.

Example $\sigma^- \sigma^+$ for a thermal bath.

Another consideration: For a Markovian system the initial condition is irrelevant.

More important are the correlators $\text{Tr}_E [F_I(t_1) F_I(t_2) G_E]$

Note the "Lindblad structures"

$$O_I(t_1) \rho_I(t) O_I(t_2), \quad O_I(t_1) O_I(t_2) \rho_I(t), \quad \rho_I(t) O_I(t_1) O_I(t_2)$$

■ The effect of the environment manifests in two ways

(i) the couplings $\rightarrow \text{Tr}[F F G]$

(ii) the transition elements $\hat{O} \rho \hat{O}, \hat{O} \hat{O} \rho, \rho \hat{O} \hat{O}$

$$t_0 \rightarrow \infty$$

change of variables $\int_{-\infty}^t dt' \rightarrow \int_0^{\infty}$

$$t'' = t - t'$$

The equation becomes.

$$\dot{\rho}_I = - \int_0^{\infty} dt' \left\{ [O_I(t), O_I(t-t')] \rho_I(t) \langle F(t') F(0) \rangle + [O_I(t), \rho_I(t) O_I(t-t')] \langle F(0) F(t') \rangle \right\}$$

At this point ~~we~~ let's project the eqn. onto a given basis.

A convenient one is the system energy eigenbasis.

↳ The $\hat{O} \rightarrow \langle n | \hat{O} | m \rangle$

$$\dot{c}_{m|n} = \sum_{n'} c_{n'} R_{mn',nn'} e^{i(\bar{E}_{n'} - \bar{E}_m - \bar{E}_{n'} + \bar{E}_m)t}$$

the relevant correlation functions are of the form

$$\int_0^\infty dt' e^{-i(\bar{E}_c - \bar{E}_n)t'} \left. \begin{array}{l} \langle F(t') F(0) \rangle \\ \langle F(0) F(t') \rangle \end{array} \right\}$$

Secular approximation \rightarrow kill the oscillatory terms.

From the Master equation to the Fermi Golden rule.

$$\Gamma_{nm} = 2\pi \sum_{N, N'} |\langle m | V | N \rangle \langle N | V | n \rangle|^2 e^{-\beta \bar{E}_N} \delta(\bar{E}_n - \bar{E}_{N'} - \omega_{nm})$$

$$\frac{\Gamma_{nm}}{\Gamma_{mn}} = e^{-\beta(\bar{E}_n - \bar{E}_m)} \rightarrow \text{detailed balance.}$$

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General considerations.

- 1) Weak coupling here means essentially that the interaction with the environment occurs through single-photon transitions.
- 2) Short-time approximation vs perturbative expansion is the self-energy.
- 3) Thermalization emerges because we made appropriate and consistent approximations.
- 4) At this level of approximation (weak coupling) the only information about the environment is on the transitions that it is able to induce.
↳ Most of the properties derive from the density of states of the mode of the environment.

We know that different environment may behave completely differently



Let's see some cases that behave completely differently.

Decoherence of a single qubit (Pala, Suominen, Ekert '97)

$$H = \frac{\omega_0}{2} \sigma_z + \sum_k \omega_k b_k^\dagger b_k + \sum_k \sigma_z \left(g_k b_k^\dagger + g_k^* b_k \right)$$

Since $[\sigma_z, H] = 0$ the populations are not affected by the environment.

Only off-diagonal components of the density matrix will evolve in time.

This is the case of decoherence without dissipation.

For the next I consider bosonic bath but there is not always the case.

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One has to perform the trace over the thermal distribution.

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+ Feynman method for
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it follows that

$$\rho_{10}(t) = e^{-\Gamma(t)} \rho_{10}(t)$$

$$\Gamma(t) \sim \int d\omega G(\omega) |g(\omega)|^2 (1 + 2N_B(\omega)) \frac{1 - \cos\omega t}{\omega^2}$$

↳ Density of states $\omega^n e^{-\omega/c}$

Two 1-particle energy scales

$$\omega_c$$

$$k_B T$$

$$t < \omega_c^{-1}$$

$$\Gamma(t) \sim t^2$$

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This initial state evolves as

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$$|\mathcal{E}_0(t)\rangle = |\mathcal{E}_1(-t)\rangle = \sum_{n=0}^{2^N-1} c_n e^{-iB_n t/2} |n\rangle$$

$$B_n = \sum_{k=1}^N (-1)^{n_k} g_k$$

The coherence factor $\langle \mathcal{E}_0 | \mathcal{E}_1 \rangle = e^{-\Gamma(t)}$

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