A RESURGENT TRANSSERIES FOR N=4 SUSY YANG-MILLS

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Non-Perturbative Methods in Quantum Field Theory
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$$\sum_{n=0}^{\infty} E_n\, g^n \quad e^{-A/g}$$

PERTURBATION THEORY

Perturbation theory, fundamental in computation of observables, often leads to divergent asymptotic expansions

Surprisingly, this asymptotic behaviour carries crucial information about exponentially small, ***non-perturbative (NP) phenomena governing the global analytic properties*** of physical observables

In this talk:

Study the late-time behaviour of the energy density of a strongly coupled plasma, with the goal of obtaining its global analytic properties

OUTLINE

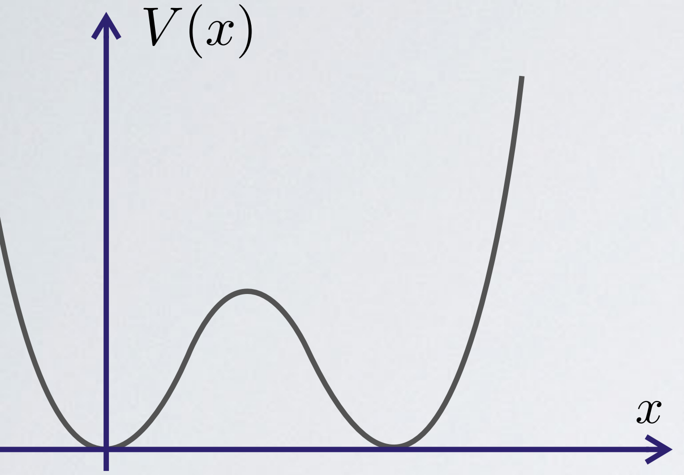
1. Introduction to resurgent transseries
2. Late-time behaviour for strongly coupled plasma
    - Microscopic description and dual gravity solution
    - Asymptotic analysis and QNMs
3. Müller-Israel-Stuart hydrodynamics
    - The attractor solution from asymptotic late-times?
4. Future directions

1.
INTRODUCTION TO RESURGENT TRANSSERIES

[IA,Basar,Schiappa'18]

PERTURBATION THEORY IN QM

g very small → Perturbation theory → $E_{g.s.}(g) \simeq \sum_{n=0}^{\infty} E_n \, g^n$



$$V(x) = \frac{1}{2} x^2 \left(1 - \sqrt{g}\, x\right)^2$$

- ***Series is asymptotic***: For large enough n

$$E_n \sim n! \, A^{-n}$$

Why asymptotic? Existence of instantons

Corrections to $E_{g.s.} \sim \mathrm{e}^{-A/g} \sum_{n=0}^{\infty} E_n^{(1)} \, g^n$ Suppressed!

BEYOND PERTURBATION THEORY

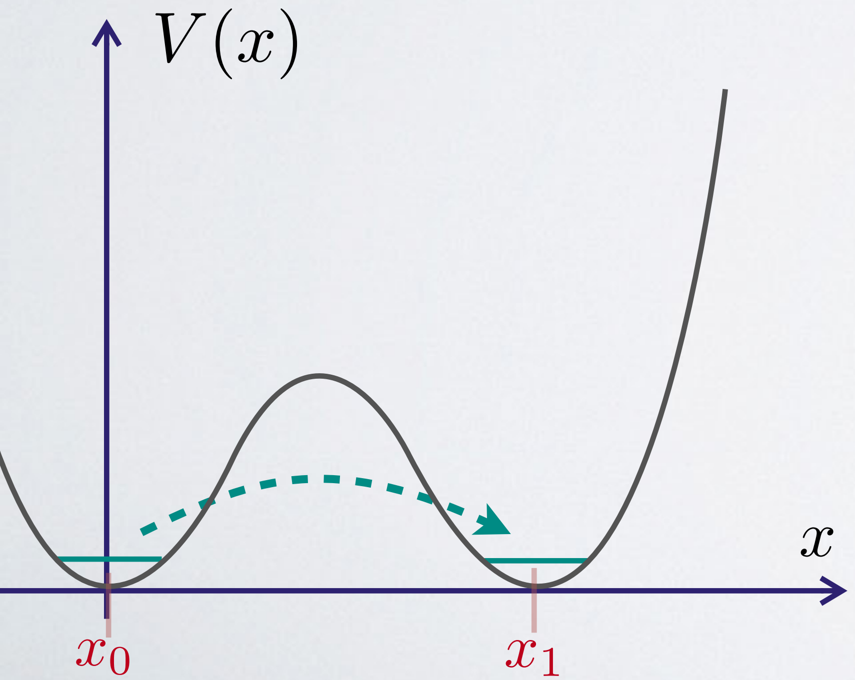
g very small → $E_{g.s.}(g) \simeq \sum_{n=0}^{\infty} E_n^{(0)} g^n$

+

Instanton corrections to $E_{g.s.}$ → $\sim e^{-A/g} \sum_{n=0}^{\infty} E_n^{(1)} g^n$

+

Higher instanton corrections → $\mathcal{O}\left(e^{-2A/g}\right)$



[Vanstein'64;Bender,Wu'73;Bogomolny,Zinn-Justin'80]

TRANSSERIES SOLUTION

$$E_{g.s.}(g,\sigma) \simeq \sum_{k=0}^{\infty} \sigma^k \mathrm{e}^{-kA/g} E^{(k)}(g) \qquad E^{(k)}(g) \simeq \sum_{n=0}^{\infty} E_n^{(k)} g^n$$

k-instanton contribution, each is asymptotic

$$E_n^{(k)} \sim n!\,(kA)^{-n}$$

Formal expansion in transmonomials

- the small parameter g
- non-perturbative term $\mathrm{e}^{-A/g}$
- σ encodes boundary/initial conditions

$E_{g.s.}(g,\sigma)$ requires all instantons to be well defined

[Edgar'08]

RESURGENCE

$$E^{(k)} \sim \sum_{n=0}^{\infty} E_n^{(k)} \, g^n$$

Coefficients between different sectors are related through large-order relations

Look at perturbative coefficients for large enough n

$$E_n^{(0)} \sim \frac{n!}{A^n} \left(E_1^{(1)} + \frac{A}{n-1} E_2^{(1)} + \cdots \right) + \frac{n!}{(2A)^n} \left(E_1^{(2)} + \frac{2A}{n-1} E_2^{(2)} + \cdots \right) + \cdots$$

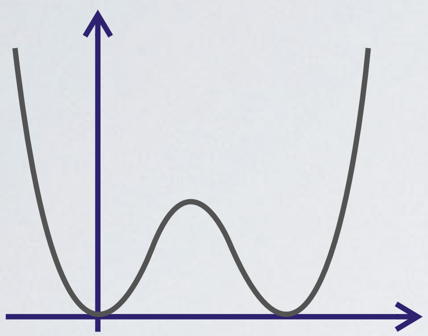
Same is true for all instanton coefficients

Using Resurgence

large order relations encode NP information in the perturbative series

BOREL TRANSFORMS

Determine NP phenomena from an asymptotic series



$$E_{g.s.}(g) \simeq \sum_{n=0}^{\infty} E_n^{(0)}\, g^n$$

$$E_n^{(0)} \sim \frac{n!}{A^n}$$ for large enough n

Remove the factorial growth to get a convergent series: inverse Laplace transform

$$B_E(s) = \sum_{n=0}^{\infty} \frac{E_n^{(0)}}{n!} s^n$$

- Non-perturbative phenomena: singularities in Borel plane
- Singularities usually will be branch cuts
- Singular directions: ***Stokes lines***

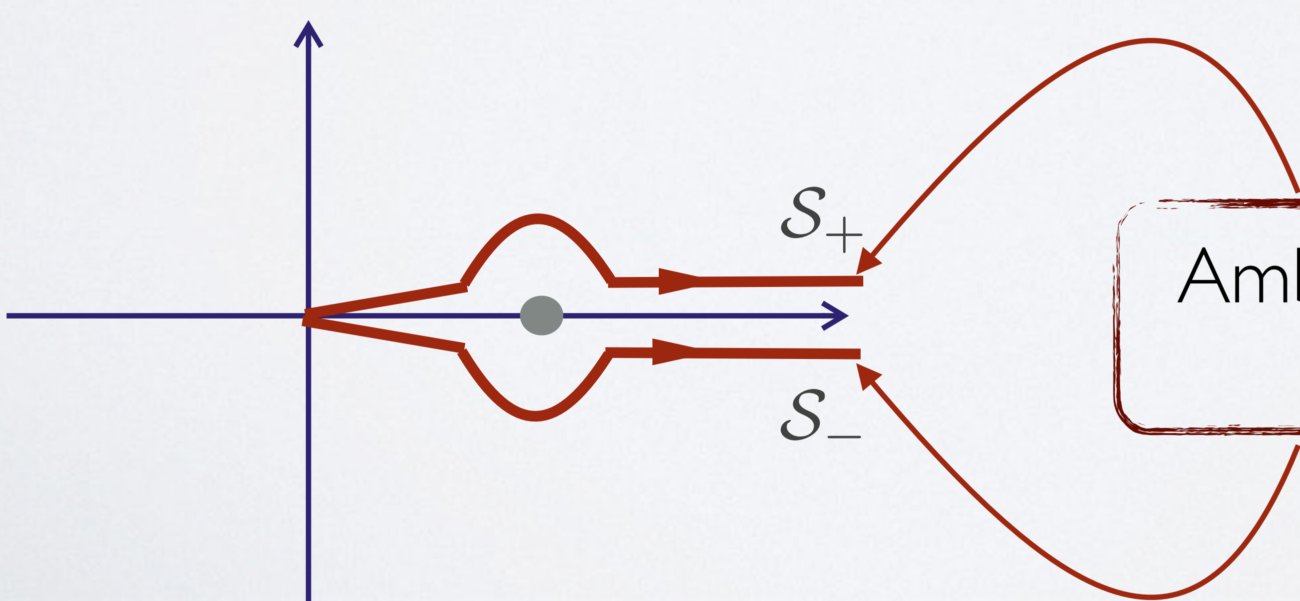
BOREL RESUMMATION

How to associate a function to the original asymptotic series?

Via ***Borel resummation***: Laplace transform

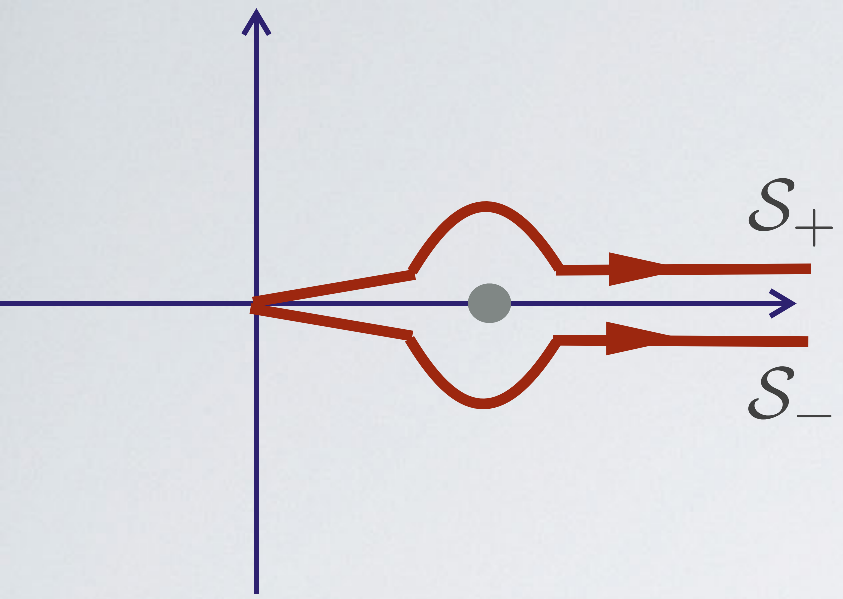
$$\mathcal{S}E_{g.s.}(g) = \int_0^\infty \mathrm{d}s B_E(s) \mathrm{e}^{-s/g}$$

- Borel resummation straightforward in the directions without singularities
- Re-summation along Stokes directions: ambiguities



Ambiguity in choice of contour

BOREL RESUMMATION



Ambiguities in the transseries

- all sectors have ambiguities
- Use resurgence to fix σ s.t.

$$(\mathcal{S}_+ - \mathcal{S}_-)\, E_{g.s.}(g, \sigma_0) = 0$$

[Delabaere'99][IA,Schiappa'13]

$$\mathcal{S}_+ E_{g.s.}(g, \sigma) = \mathcal{S}_- E_{g.s.}(g, \sigma + S)$$

Stokes constant (imaginary)

The full transseries is unambiguous, and we can construct an analytic solution in **any** direction

2.
LATE-TIME ASYMPTOTIC FOR STRONGLY COUPLED PLASMA IN $\mathcal{N} = 4$ SYM

RELATIVISTIC HYDRODYNAMICS

It provides a reliable description of strongly coupled systems

- real life: strongly coupled quark-gluon plasma in particle accelerators;
- To determine the kinetic parameters of hydrodynamic equations (e.g. shear viscosity): study the associated microscopic theory

The associated microscopic theory can be a QFT, such as strongly coupled $\mathcal{N} = 4$ Super Yang-Mills (SYM)

$N \to \infty$ ***gauge/gravity duality:*** determine hydrodynamic parameters, time dependent processes of the SYM plasma from dual geometry

[Policastro *et al* '01-'04; Nastase '05]

STRONGLY COUPLED SYSTEMS

Kinematic regime: ***expanding plasma*** in the so-called central rapidity region, where one assumes ***longitudinal boost invariance*** (Bjorken flow)

[Bjorken '83]

In hydrodynamic theories the energy-momentum tensor is given by

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + \mathcal{P}(\mathcal{E})(\eta^{\mu\nu} + u^\mu u^\nu) + \Pi^{\mu\nu}$$

Energy density

Pressure, in 4d conformal theories given by:

$$\mathcal{P}(\mathcal{E}) = \mathcal{E}/3$$

flow velocity

Shear stress tensor: dissipative effects

Symmetries: conformal invariance, transversely homogeneous, invariance under longitudinal Lorentz boosts

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Strongly coupled SYM boost invariant plasma:
all physics encoded in $\mathcal{E}(\tau)$.
Obtaining this function is in general too difficult:
perform a ***large proper time expansion*** $\tau \gg 1$.

LATE TIME BEHAVIOUR

Starting from highly non-equilibrium initial conditions, the microscopic theory will reveal the transition to hydrodynamic behaviour at late times

Conformal theories: late-time behaviour of energy density highly constrained

$$\mathcal{E}(\tau) = \frac{\Lambda}{(\Lambda\tau)^{1/3}}\left(1+\sum_{k=1}^{+\infty}\frac{\epsilon_k}{(\Lambda\tau)^{2k/3}}\right), \; \tau \gg 1$$

- Λ is a dimensionful parameter encoding initial non-eq. conditions
- Leading behaviour predicted by boost-invariant perfect fluid
- Subleading terms: dissipative hydrodynamic effects

Next: use dual geometry to analyse the expansion of boost invariant SYM plasma

SYM PLASMA FROM ADS/CFT

Equilibrium states of the microscopic theory (CFT) ⟷ black hole solutions [Witten '98]

flat space at boundary: planar horizons ⟷ black branes

Perturbative non-equilibrium phenomena ⟷ linearised perturbations of black brane solution

Non-hydrodynamic d.o.f. ⟷ exp. decaying black branes' quasi-normal modes

[Janik, Peschanski '05][Janik '05]

SYM PLASMA FROM ADS/CFT

Dual geometry given by boost invariant 5D metric

$$ds^2 = \frac{1}{z^2}\left(dz^2 - \mathrm{e}^{-A} d\tau^2 + \tau^2 \mathrm{e}^B dy^2 + \mathrm{e}^C d\boldsymbol{x}_\perp^2\right) = \frac{1}{z^2}\left(G_{\mu\nu} dx^\mu dx^\nu + dz^2\right)$$

Solve Einstein equations with negative cosmological constant
(asymptotic behaviour is AdS)

$$R_{\mu\nu} - \frac{1}{2} G_{\mu\nu} R - 6 G_{\mu\nu} = 0$$

- metric components depend on $z,\ \tau$
- boundary condition at $z = 0$:

$$G_{\mu\nu} = \eta_{\mu\nu} + z^4 g^{(4)}_{\mu\nu} + \cdots$$

Energy density $$\mathcal{E}(\tau) = -\lim_{z\to 0} \frac{A(z,\tau)}{z^4}$$

[Hare *et al* '00][Skenderis '02][Fefferman,Graham '85]

SYM PLASMA FROM ADS/CFT

Metric ansatz: multi-parameter transseries with exponential decaying sectors and perturbative expansions in proper time

The most general solution for the ***energy density*** of the SYM plasma is:

$$\mathcal{E}\left(u \equiv \tau^{2/3}, \boldsymbol{\sigma}\right) = \sum_{\boldsymbol{n}\in\mathbb{N}_0^{\infty}} \boldsymbol{\sigma}^{\boldsymbol{n}} \mathrm{e}^{-\boldsymbol{n}\cdot\boldsymbol{A}\,u} \Phi_{\boldsymbol{n}}(u) \ , \quad \Phi_{\boldsymbol{n}}(u) = u^{-\beta_{\boldsymbol{n}}} \sum_{k=0}^{+\infty} \varepsilon_k^{(\boldsymbol{n})} u^{-k}$$

exponentially decaying coupled QNMs $\omega_k = -\frac{2\mathrm{i}}{3} A_k$

perturbative late-time expansions

- Infinite number of QNMs $\boldsymbol{A} = (A_1, \bar{A}_1, A_2, \bar{A}_2, \cdots)$
- Parameters encoding non-hydro initial conditions $\boldsymbol{\sigma} = (\sigma_{A_1}, \sigma_{\bar{A}_1}, \sigma_{A_2}, \sigma_{\bar{A}_2}, \cdots)$

All expansions in the energy density are ***asymptotic***!

[Heller,Janik,Witaszczyk'15; IA *et al*'18]

ASYMPTOTIC ENERGY DENSITY

Hydrodynamic expansion:

$$\Phi_0(u) = u^{-2}\sum_{k=0}^{+\infty} \varepsilon_k^{(0)} u^{-k} \qquad \varepsilon_k^{(0)} \sim \frac{k!}{|A_1|}$$

Singularities in Borel plane:

ω_1; $2\omega_1$; $3\omega_1$

ω_2

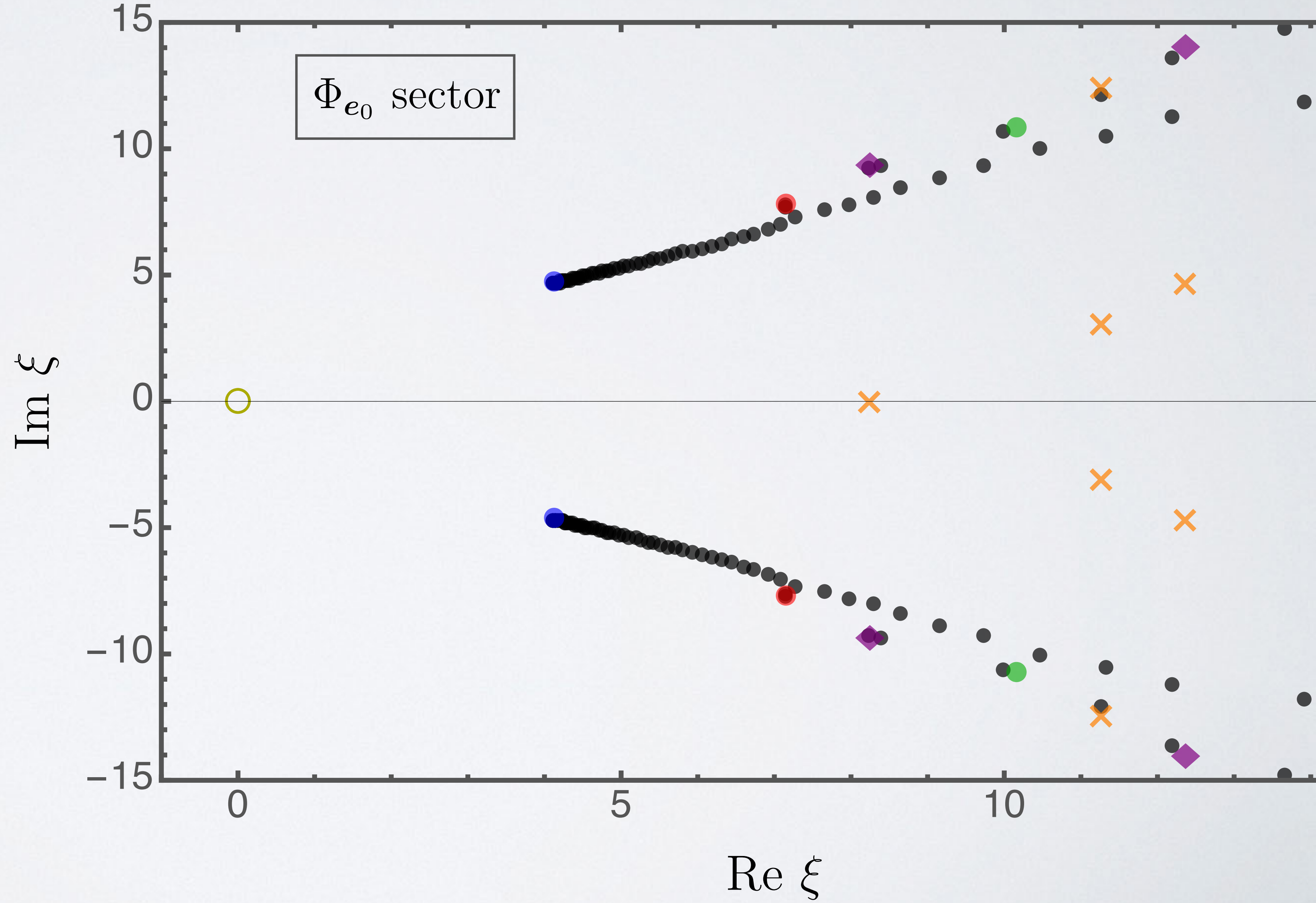
ω_3

$\overline{\omega}_i$

$$\omega_1 = \frac{3}{2}(2.746676 + 3.119452\mathrm{i})$$

$$\omega_2 = \frac{3}{2}(4.763570 + 5.169521\mathrm{i})$$

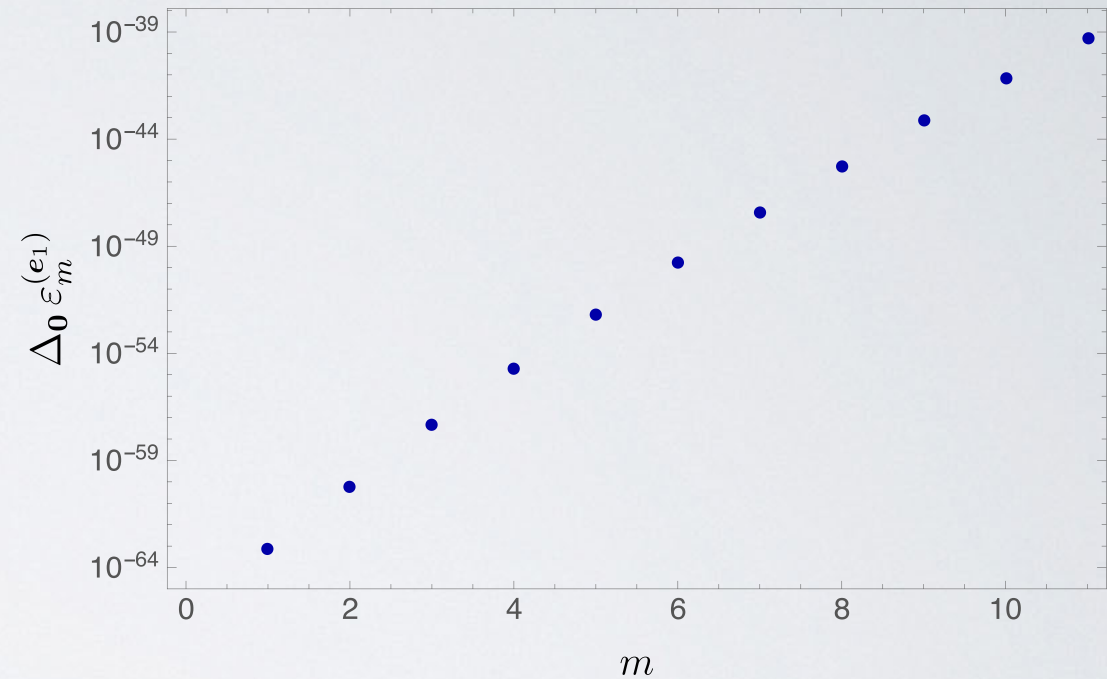
$$\omega_3 = \frac{3}{2}(6.769565 + 7.187931\mathrm{i})$$



ASYMPTOTIC ENERGY DENSITY

$$\mathcal{E}\left(u \equiv \tau^{2/3}, \boldsymbol{\sigma}\right) = \sum_{\boldsymbol{n}\in\mathbb{N}_0^\infty} \boldsymbol{\sigma}^{\boldsymbol{n}}\, \mathrm{e}^{-\boldsymbol{n}\cdot\boldsymbol{A}\,u}\, \Phi_{\boldsymbol{n}}(u)\ , \quad \Phi_{\boldsymbol{n}}(u) = u^{-\beta_{\boldsymbol{n}}} \sum_{k=0}^{+\infty} \varepsilon_k^{(\boldsymbol{n})}\, u^{-k}$$

- **NP description of the late time behaviour** of the SYM plasma
- Asymptotic analysis predicted **coupled QMN solutions** in gravity
- Agreement between gravity calculations and resurgence large-order predictions



[IA *et al*'18]

- Can we recover the non-equilibrium behaviour of early times?
- Dependence of the transseries parameters on initial conditions?

Study a simpler relativistic hydrodynamic system

3.
MÜLLER-ISRAEL-STUART HYDRODYNAMICS

MIS CAUSAL HYDRODYNAMICS

Solve evolution equations of the Energy momentum tensor

$$\nabla_\mu T^{\mu\nu} = 0$$

- Assume ***boost invariant flow, conformal invariance***
- ***Hydrodynamic gradient expansion***: approximate shear stress tensor by corrections to ideal fluid

Müller-Israel-Stuart (MIS) equations

$$z\, C_{\tau\Pi} f\, f' + 4C_{\tau\Pi} f^2 + \left(z - \frac{16C_{\tau\Pi}}{3}\right) f - \frac{4C_\eta}{9} + \frac{16C_{\tau\Pi}}{9} - \frac{2z}{3} = 0$$

- Non-linear ODE describing the energy density
- $C_{\tau\Pi},\, C_\eta$ are phenomenological parameters

MIS CAUSAL HYDRODYNAMICS

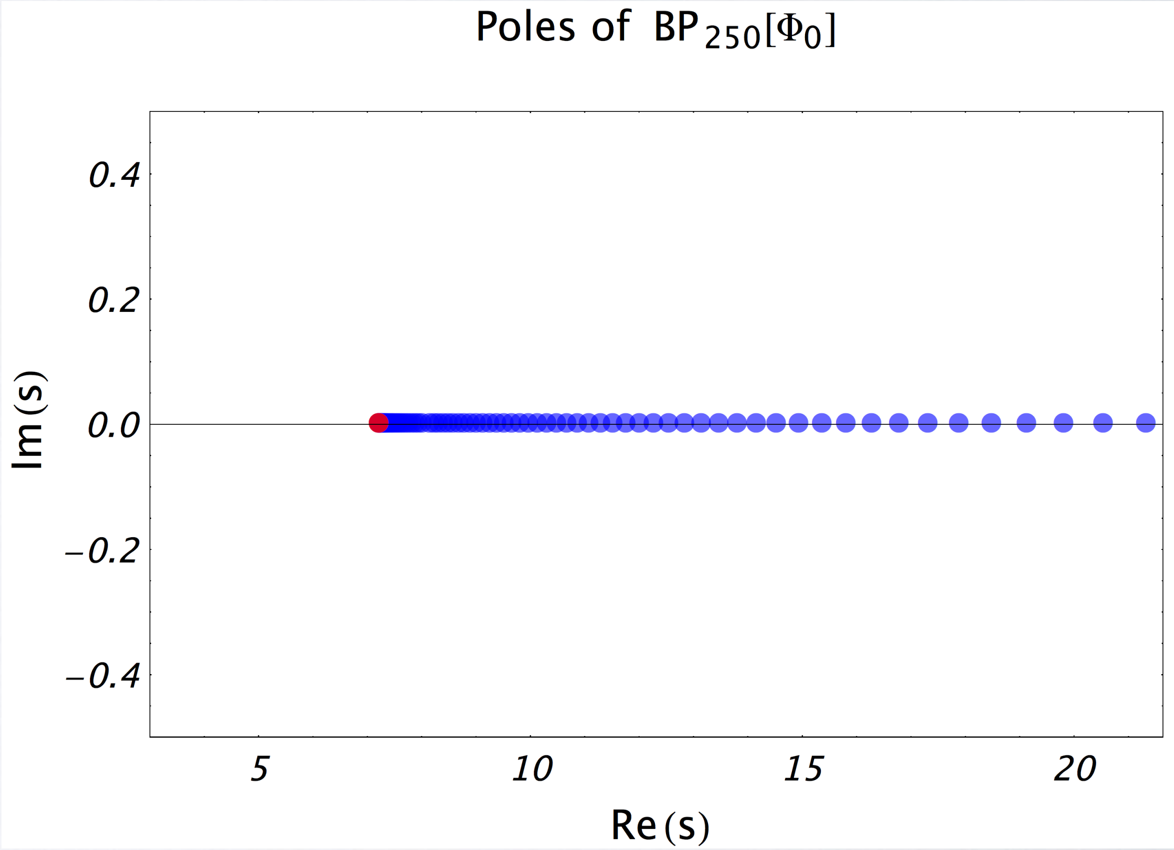
- We are interested in the late time regime $z \gg 1$
- It has a single, purely decaying non-hydrodynamic mode

Write the general solution as a transseries, sectors asymptotic.
Study resurgent properties

$$\mathcal{F}(z,\sigma) = \sum_{n=0}^{+\infty} \sigma^n \, \mathrm{e}^{-nAz} \, \Phi_n(z)$$

$$\Phi_n(z) = z^{-n\beta} \sum_{k=0}^{+\infty} a_k^{(n)} z^{-k}$$

$$A = \frac{3}{2C_{\tau\Pi}} \qquad \beta = -\frac{C_\eta}{C_{\tau\Pi}}$$



[Heller,Spalinski'15; Basar,Dunne'15; IA,Spalinski'15]

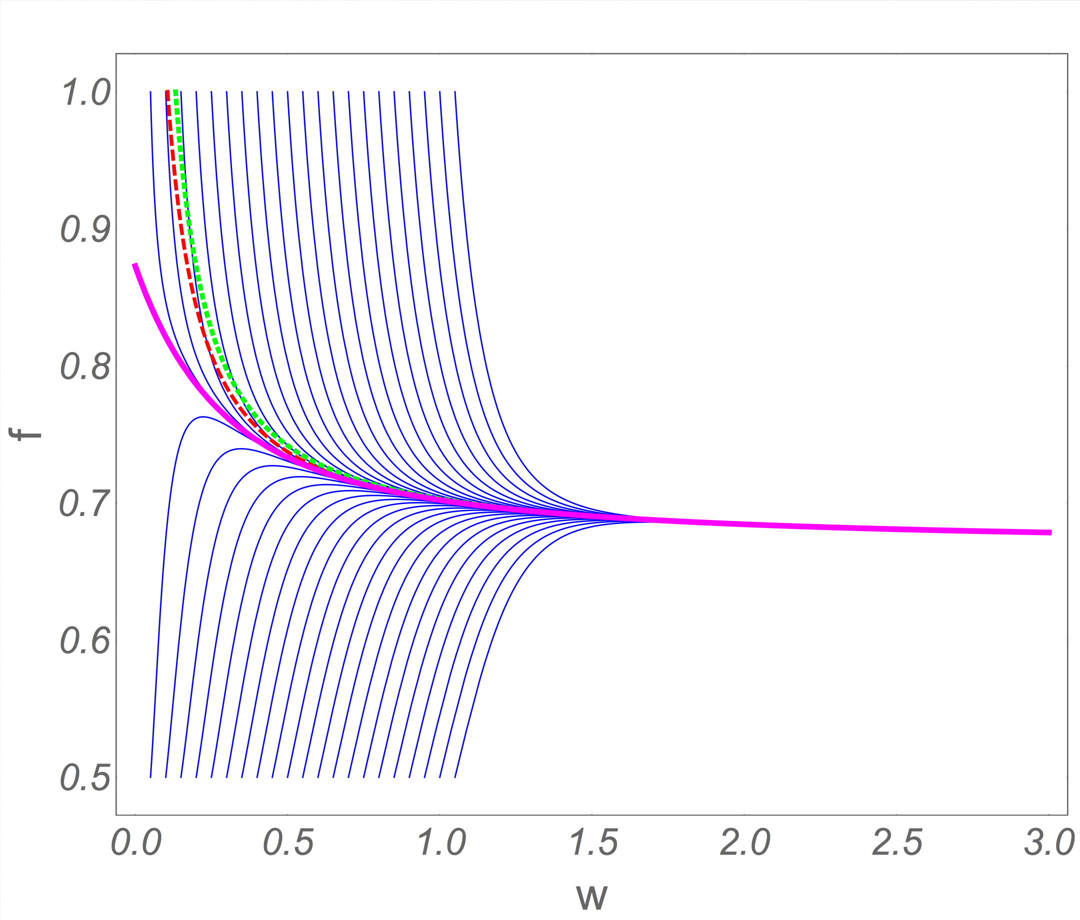
SOLUTION AT EARLY TIMES

Attractor solution: Stable solution, converging to a finite value at early times

Generic solution: divergent at early times, but will decay rapidly towards the attractor solution

$$\mathcal{F}_{\mathrm{Att}}(z) = \frac{2}{3} + \frac{1}{3}\sqrt{\frac{C_\eta}{C_{\tau\Pi}}} + \mathcal{O}(z)$$

Calculate attractor solution: Taylor expansion



[Heller,Spalinski '15]

SOLUTION AT EARLY TIMES

$$\mathcal{F}(z,\sigma) = \sum_{n=0}^{+\infty} \sigma^n \, \mathrm{e}^{-nAz} \, \Phi_n(z) \qquad\qquad \Phi_n(z) = z^{-n\beta} \sum_{k=0}^{+\infty} a_k^{(n)} z^{-k}$$

Can we recover the attractor solution
from the transseries expansion?

Need to fix the value of $\sigma = \sigma_R + \mathrm{i}\sigma_I$

- Ambiguity cancelation fixes its imaginary part
- Comparison with attractor fixes its real part

[Heller,Spalinski '15]

ANALYTIC TRANSSERIES SUM

The order of transmonomials in the transseries can be rearranged:

$$\mathcal{F}(z,\sigma) = \sum_{k=0}^{+\infty} z^{-k} \sum_{n=0}^{+\infty} \left(\sigma\, z^{-\beta} \mathrm{e}^{-Az}\right)^n a_k^n$$

Define a new variable: $\tau = \sigma\, z^{-\beta} \mathrm{e}^{-Az}$

We want to sum the transseries in a new regime: $z^{-1} \ll \tau \ll 1$

The sum over powers of τ can be done exactly!

$$\mathcal{F}(z,\tau) = \sum_{k=0}^{+\infty} z^{-k} F_k(\tau) \qquad F_k(\tau) = \sum_{n=0}^{+\infty} \tau^n a_k^n$$

[Costin et al'01-13; IA,Schiappa,Vonk 'to appear]

ANALYTIC TRANSSERIES SUM

$$\mathcal{F}(z,\tau) = \sum_{k=0}^{+\infty} z^{-k}\, F_k(\tau) \qquad F_k(\tau) = \sum_{n=0}^{+\infty} \tau^n\, a_k^n$$

Recursive calculation:

$$F_0(\tau) = \frac{2}{3}\left(1 + W\left(\frac{3}{2}\tau\right)\right)$$

Lambert-W function

$$W(x)\,\mathrm{e}^{W(x)} = x$$

$$F_1(\tau) = \frac{1}{F_0(\tau)} \sum_{r=0}^{3} f_1^{(r)}(C_\eta, C_{\tau\Pi}) F_0(\tau)^r$$

$$\vdots$$

$$F_k(\tau) = \frac{P_k\left(F_0(\tau)\right)}{Q_k\left(F_0(\tau)\right)}$$

Polynomials

CONNECTION TO ATTRACTOR

$$\mathcal{F}(z,\tau) = \sum_{k=0}^{+\infty} z^{-k}\, F_k(\tau) \qquad\qquad z^{-1} \ll \tau \ll 1$$

Choose z large enough to be in above regime, but small enough to compare to attractor solution $\mathcal{F}_{\rm Att}(z)$ at early times

- Choose z off the real axis $\quad z = z_R + \mathrm{i} z_I$
- Analytically continue attractor solution to complex plane

Solve $\quad \mathcal{F}(z,\tau) = \mathcal{F}_{\rm Att}(z) \quad$ to obtain $\quad \tau(z) = \sum_{r\geq 0} \tau_r\, z^{-r}$

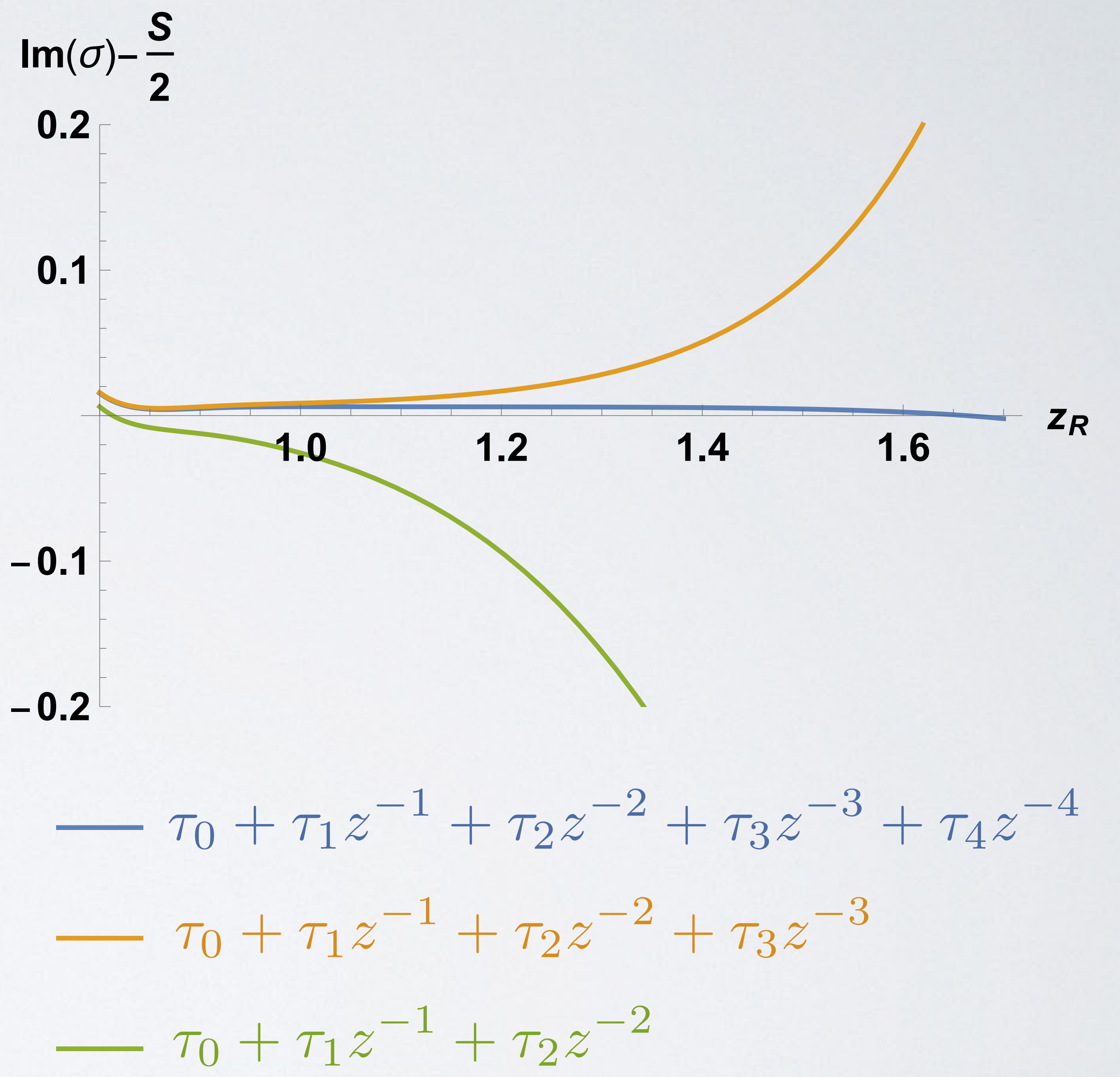
Transseries parameter: $\quad \sigma = z^{\beta} \mathrm{e}^{Az} \sum \tau_r\, z^{-r}$

CONNECTION TO ATTRACTOR

We obtain:

$$\sigma \sim -0.245 - 0.0128\mathrm{i}$$

Imaginary part approximates the value from ambiguity cancelation



4.
FUTURE DIRECTIONS

OPEN QUESTIONS

Transseries in gauge theories

- asymptotics with multi-parameters
- interpretation of non-perturbative contributions

Transseries summation and multiple scales

- different summations give rise to fast and slow scales
- boundary layers, initial value problems

Analytic properties of asymptotic observables

- phase transitions
- role of initial conditions
- probe of dualities

THANK YOU!

$$\sum_{n=0}^{\infty} E_n\, g^n \quad \mathrm{e}^{-A/g}$$