Renormalons in Quantum Mechanics

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based on 1906.07198
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Introduction

- Perturbative Series are generically divergent
  $f(\lambda) = \sum f_n \lambda^n \qquad (f_n \sim A^{-n} n!)$
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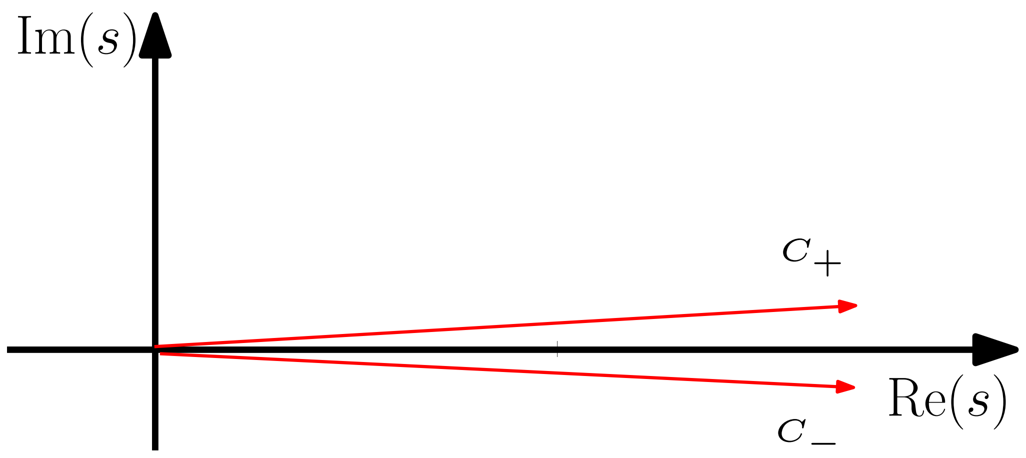
$$n! = \int_0^\infty \mathrm{d}s\, s^n e^{-s} \qquad \Longrightarrow \qquad f(\lambda) = \int_0^\infty \mathrm{d}s\, e^{-s} \hat{f}(s\lambda)$$

$$\hat{f}(s\lambda) = \sum_{n=0}^{\infty} \frac{f_n}{n!} (s\lambda)^n = \frac{A}{A - s\lambda}$$

$$\text{Im}\, f(\lambda) = Res(e^{-s} \hat{f}(s\lambda))|_{s=A/\lambda}$$



Choosing C_+ or C_- results in complex conjugate results! $\longrightarrow$ Borel Ambiguity

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- Types of Divergences:

  ↳ Proliferation of Diagrams

  - Proliferation of number diagrams $\longrightarrow$ Tunneling
  - Borel Ambiguity $\xleftrightarrow[\text{cancellation}]{\text{B-ZJ}}$ Instanton/WKB Action

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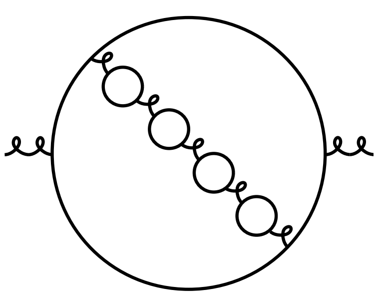
Loop Expansion

- Remnant of renormalization process of loop
- Borel Ambiguity $\longrightarrow$?



IR-Region: $\int \mathrm{d}k^2 k^2 \left(\ln \frac{\mu}{k^2}\right)^n \sim \int \mathrm{d}t e^{-t} t^n \sim n!$

UV Region: $\int \frac{\mathrm{d}k^2}{k^2} \left(\ln \frac{k^2}{\mu}\right)^n \sim \int \mathrm{d}t e^{-t} t^n \sim n!$

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- No resolution to Borel ambiguity!
- No proof/conjecture that the divergence will survive or cancel out!

Solution:

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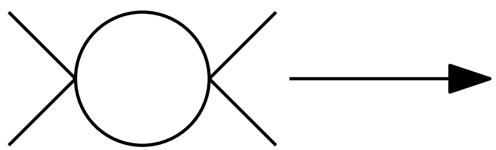
Outline:

- Review of $\delta^{(2)}(x)$ scattering.
- Construction of Renormalons in QM.
- Resolution of Renormalon ambiguity.

Renormalization in QM

Perturbative Scattering: $S = S^{(0)} - iT$ where $T = V \sum (GV)^n$

Consider 2-body scattering with $V = \delta^{(2)}(x, y)$

1 Loop:  $T^{(1)} = \frac{\lambda_0^2}{4\pi} \int_0^\infty \frac{du^2}{u_f^2 - u^2}$

Renormalization: $T^{(1)}_{\text{ren}} = \lambda^2 \, l(E_f)$, $l(E_f) = \frac{1}{4\pi} \log \frac{e^{i\pi} z}{\mu}$

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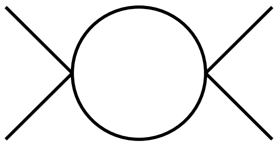
1 Loop:  $\longrightarrow$ $T^{(1)} = \frac{\lambda_0^2}{4\pi}\int_0^\infty \frac{du^2}{u_f^2 - u^2}$

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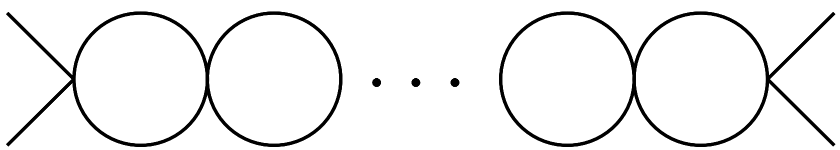
Generalization to higher loops is immediate!

N-1 Loop:  (Only diagram)

Renormalization: $T^{(n)}_{\mathrm{ren}} = \lambda^n\,(l(E_f))^{n-1}$ $\longrightarrow$ Easy to sum!

Renormalization in QM

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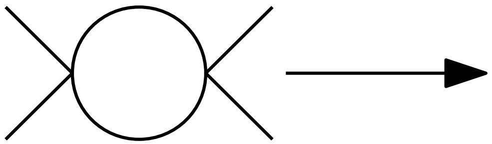
1 Loop:  $T^{(1)} = \frac{\lambda_0^2}{4\pi}\int_0^\infty \frac{du^2}{u_f^2 - u^2}$

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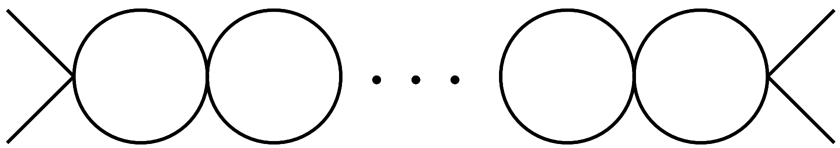
N-1 Loop:  (Only diagram)

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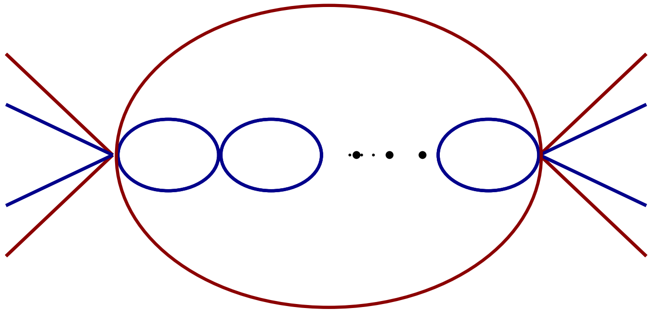
$$T = \frac{\lambda}{1 - \frac{\lambda}{4\pi}(\log\frac{E_f}{\mu} + i\pi)}$$

- NP Bound State: $E = -\mu e^{4\pi/\lambda}$
- This mathces with exact solution
- No Renormalon Integral

Renormalons In Quantum Mechanics

4 Point Interaction



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- Particle number conserves
- But still complicated

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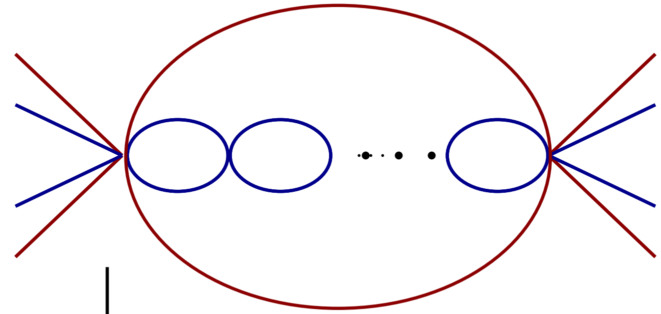

Write the problem with a combinations of background potentials

$*-\star-\star-\ldots-\star-*$

3D Model:

$$H = \frac{P^2}{2} + \lambda_0\, \delta^2(x,y) + V(x,y,z)$$

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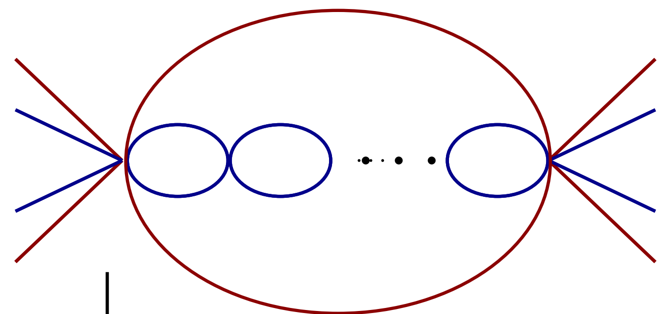

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$$T^{(n+3)} = \lambda_* \int du_2 \left(\frac{\lambda_*}{4\pi} \left[\log\left(\frac{E_f - u_2^2}{\mu_*^2} \right) - i\pi \right] \right)^n \int \frac{\mathrm{d}^2 Q_2}{(2\pi)^2} \frac{\mathrm{d}^2 Q_{n+3}}{(2\pi)^2} \mathcal{F}\left(P_2, P_{n+3}\right) \Big|_{u_2 = u_{n+3}}$$

renormalon integral $\sim n!$

- $(x,y,z) \to (Q,u)$

$$\mathcal{F}\left(P_2, P_{n+3}\right) = \frac{\tilde{V}(P_f - P_{n+3})\tilde{V}(P_2 - P_i)}{(E_f - E_{n+3})(E_f - E_2)}$$

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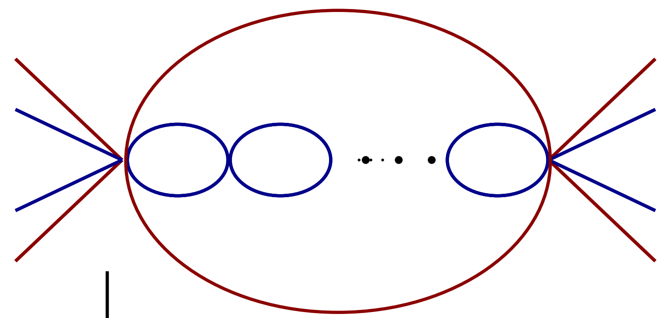

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- $(x,y,z) \to (Q,u)$
- This implies 3^{rd} direction leads to the renormalon integral

Renormalons In Quantum Mechanics

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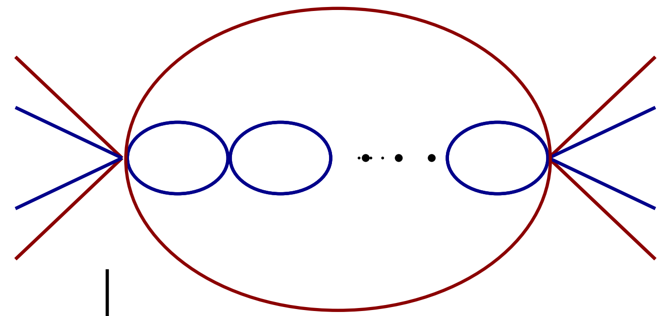

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Questions

- Does it survive in the full expansion?
- Physical implication?

Renormalons In Quantum Mechanics

$H = \frac{P^2}{2} + \lambda_0\, \delta^2(x,y) + V(x,y,z)$ $\qquad V(x,y,z) = \kappa\delta(z\cos\theta - y\sin\theta)$

$\theta \to 0 \Longrightarrow S_{\theta=0} = S_{2\mathrm{d}} S_{1\mathrm{d}} \longrightarrow$ No renormalon

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All Diagrams at Order κ^2

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$$T_{n,2} = \frac{9}{2}(\cos\theta\ \log\cos^2\theta)^2 \kappa^2 \mu^{-\frac{3}{2}} \left(\frac{\lambda}{6\pi}\right)^n (n-3)!$$

$\downarrow$ Standart Borel summation

Ambiguity: $\mathrm{Im}\, T_2 = \mp \frac{9\pi i}{2}(\cos\theta\ \log\cos^2\theta)^2 \kappa^2 \mu^{-\frac{3}{2}} e^{-\frac{6\pi}{\lambda}} \left(\frac{\lambda}{6\pi}\right)^2 \xrightarrow{\theta = 0} 0$

Resolution to Renormalon Ambiguity

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Take a step back $\longrightarrow T = \sum_n \int dq\, f(q)\, l(p_f^2 - q^2)^{n-1} \lambda^n$

Here there are two options:

1. First integrate, then sum (what we've done)
2. First sum, then integrate (Observe we have a geometric series!)

$T = \int dq\, f(q) \frac{\lambda}{1 - \lambda\, l(p_f^2 - q^2)} \longrightarrow$ Pole requires an analytical continuation

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$(E \to E + i\epsilon)$

this is choosing forward time evolution!

$\longrightarrow \text{Im}\, T > 0$

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Feynman prescription $\longrightarrow T = \int dq\, f(q) \frac{\lambda}{1-\lambda\, l(p_f^2 - q^2 - i\epsilon)}$ this is choosing backward time evolution!

$(E \to E - i\epsilon)$

$\longrightarrow \text{Im}\, T > 0 \longrightarrow$ Same sign!

Resolution to Renormalon Ambiguity

- What is the difference with standart Borel summation?

$$T = \int_{-\infty}^{\infty} dq\, f(q) \frac{\lambda}{1-\lambda\, l(p_f^2 - q^2 \pm i\epsilon)} = 2\int_0^{\infty} dq\, f_{\text{even}}(q) \frac{\lambda}{1-\lambda\, l(p_f^2 - q^2 \pm i\epsilon)}$$

No Ambiguity: $\text{Im}\, T_2 = +\frac{9\pi i}{2}(\cos\theta\, \log\cos^2\theta)^2 \kappa^2 \mu^{-\frac{3}{2}} e^{-\frac{6\pi}{\lambda}} \left(\frac{\lambda}{6\pi}\right)^2$

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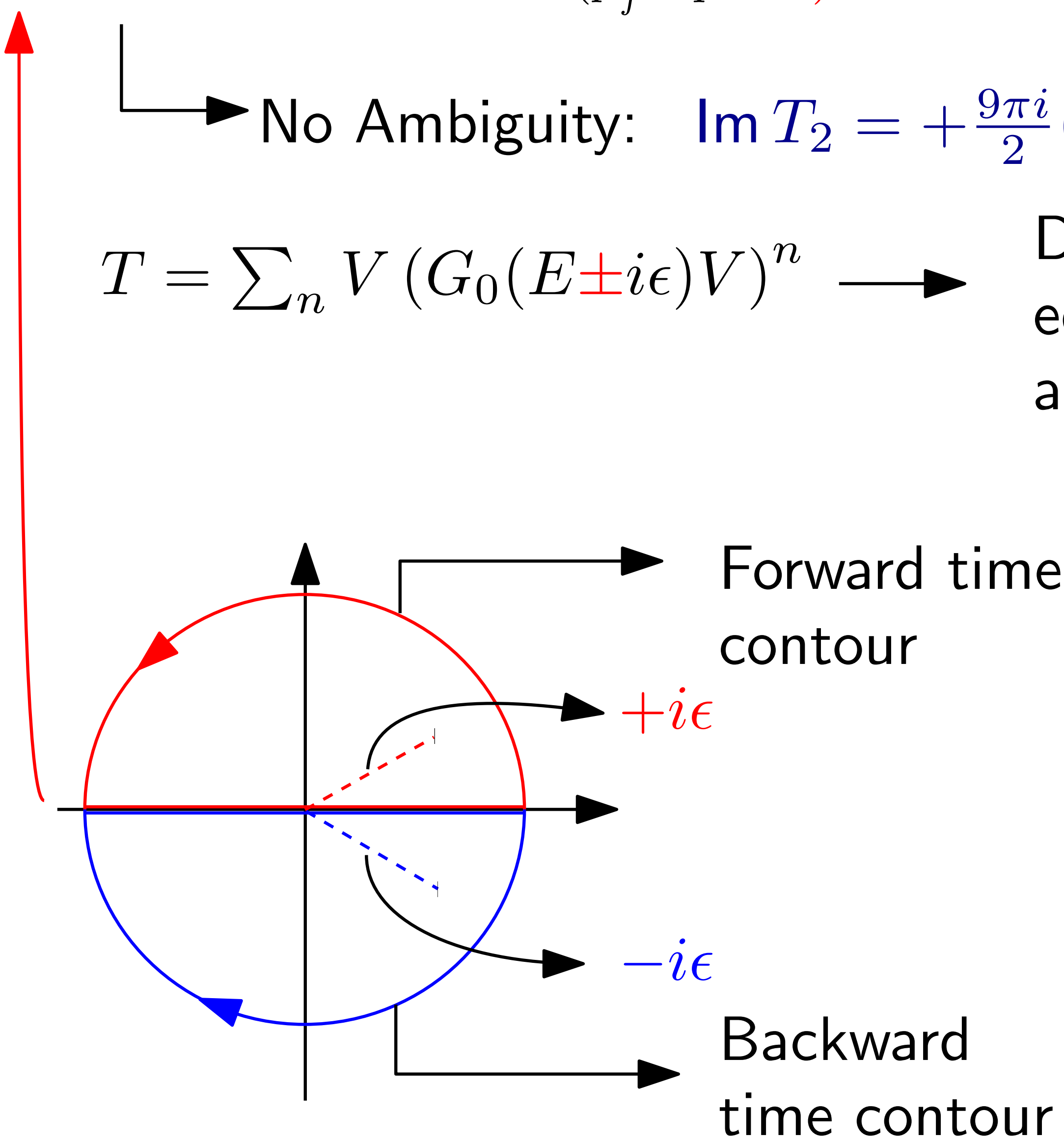
Defining time direction is equivalent to choosing the analytic half-plane of $G(E)$



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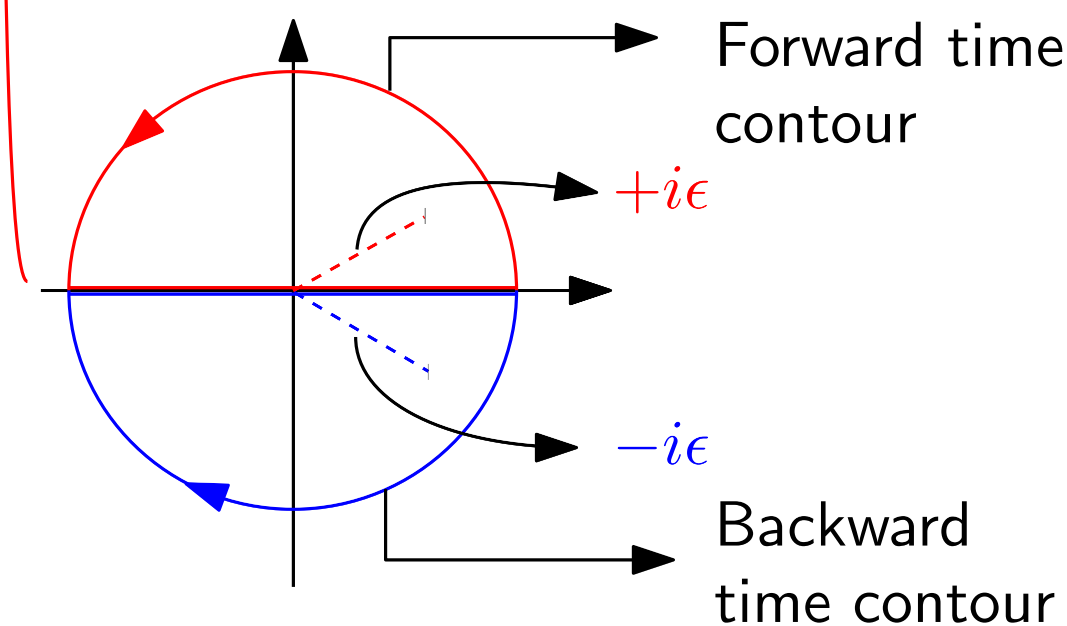
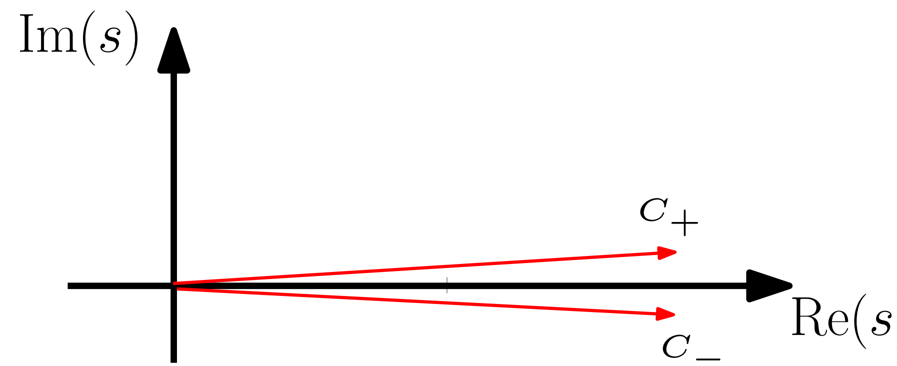


Borel Summation



- No distinction between C_+ and C_-

Conclusion

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Outlook

- Independent semi-classical interpretation? (Helps in QFT)
- New look at the Borel summation!