Virtual topological invariants of moduli spaces of sheaves on surfaces

(0) Introduction

We work over $\mathbb{C}$. Study topology of some moduli spaces.

[Conventions/Notations : A projective variety X is a subset $X \subset \mathbb{P}^N$, which is common zero set $Z(F_1, \dots F_n)$ of homog. polyn. $F_i \in \mathbb{C}[x_0, \dots x_n]$

A smooth projective surface is a nonsing. proj. variety of dim 2

A line bundle L on S is called ample if for some k the sections of $L^{\otimes k}$ define an embedding $S \hookrightarrow \mathbb{P}^N$.]

What is a moduli space?

Algebraic variety M that parametrizes naturally certain interesting objects in alg. geometry.

"parametrizes": $M(\mathbb{C}) = \{\text{points of } M\} \overset{\text{bijection}}{\longleftrightarrow} \{\text{Objects in question}\}$

"naturally": If $\begin{array}{c} \mathcal{E} \\ \downarrow \pi \\ T \end{array}$ family of objects parametrized by scheme T

$$\Rightarrow \text{map } \quad T \longrightarrow M \qquad \text{morphism of schemes.}$$
$$t \longmapsto \pi^{-1}(t)$$

<u>Example</u>: Hilbert scheme of points.

$S^{[n]} = \mathrm{Hilb}^n S$ on a smooth surface S.

$S^{[n]} = \{[Z] \subset S \mid \text{zero dim. subscheme of degree } n\}$

General point $Z = \{p_1, \dots, p_n\} \subset S$

But points can have multiple structure

$[Z] \in S^{[n]}$ is given by ideal sheaf.

$I_Z \subset \mathcal{O}_S$ (holom. functions vanishing on Z)

such that $\mathcal{O}_Z = \mathcal{O}_S / \mathcal{I}_Z$ has finite support

and $\deg(Z) = \dim_{\mathbb{C}} H^0(\mathcal{O}_Z) = \sum_{p \in \mathrm{supp}(Z)} \dim_{\mathbb{C}} (\mathcal{O}_{Z,p}) = n.$

Example: $S^{[0]} = \{[\emptyset]\}$

$$S^{[1]} = S = \{[p] \mid p \in S\}$$

$$S^{[2]} = \{p_1 \amalg p_2\} \cup \{(p + \text{tgt vector})\} = Bl_{\Delta}(S \times S) / \mathfrak{S}_2$$

Related to symmetric power

$$S^{(n)} = \mathrm{Sym}^n S = S^n / \mathfrak{S}_n = \left\{ \sum_i n_i p_i \;\middle|\; \begin{array}{l} p_i \in S \text{ distinct} \\ \sum_i n_i = n \end{array} \right\}.$$

Hilbert Chow morphism

$$\pi : S^{[n]} \to S^{(n)}, \quad Z \mapsto \mathrm{supp}(Z) = \sum_{p \in \mathrm{supp}(Z)} \dim(\mathcal{O}_{Z,p}) \cdot p.$$

Theorem (Fogarty):

(a) $S^{[n]}$ is nonsingular of dim $2n$

(b) $\pi : S^{[n]} \to S^{(n)}$ is a resolution of singularities.

Topological invariants

Set $e(X) = \sum_{i=0}^{\dim(X)} (-1)^i \dim H^i(X,\mathbb{R})$ topological Euler number.

Theorem: $\sum_{n\geq 0} e(S^{[n]})\, t^n = \dfrac{1}{\prod_{n>0}(1-t^n)^{e(S)}}$

$$= 1 + e(S)t + \tfrac{1}{2}\left(e(S)^2 + 3e(S)\right)t^2 + \dots$$

Aim: Get a similar formula for moduli spaces of sheaves on S.

(1) Moduli spaces of sheaves on surfaces

Let S be smooth proj. surface over $\mathbb{C}$.

Let H ample line bundle on S.

Fix $r \in \mathbb{Z}_{>0}$, $c_1 \in H^2(S, \mathbb{Z})$, $c_2 \in H^4(S, \mathbb{Z})$!

(Recall Chern classes of a vector bundle or sheaf E on X are classes

$$c(E) = 1 + c_1(E) + c_2(E) + \dots \quad \text{on } X, \ c_i(E) \in H^{2i}(X, \mathbb{Z})$$

measuring how far E is from being trivial

i.e. $c(\mathcal{O}^{\oplus r}) = 1.$)

Want to study moduli space of rank r torsion free coherent sheaves on S with $c_1(E) = c_1$, $c_2(E) = c_2$

<u>Note</u>: Sheaves can be viewed as vector bundles with singularities (i.e. dimension of fibre can jump)

There is no moduli space for all these sheaves, but for good (semistable) sheaves.

Definition: Let $\mathcal{E}$ be a coherent sheaf on a scheme X. Denote $H^i(X,\mathcal{E})$ i^{th} sheaf cohomology ($H^0(X,\mathcal{E})$ = global sections)

The holom. Euler characteristic of $\mathcal{E}$ is

$$\chi(X,\mathcal{E}) = \sum_{i=0}^{\dim X} (-1)^i \dim H^i(S,\mathcal{E})$$

Now let $\mathcal{E}$ coherent sheaf on S, H on S ample.

The Hilbert polynomial of $\mathcal{E}$ is

$$P_H(\mathcal{E},m) = \chi(X, \mathcal{E}\otimes H^{\otimes m}) \quad \text{(polynomial in } m\text{)}$$

Definition: A torsion free coherent sheaf $\mathcal{E}$ on S is called H-semistable if for all subsheaves

$$0 \neq \mathcal{F} \subsetneq \mathcal{E}$$

we have

$$\frac{P_H(\mathcal{F},m)}{rk(\mathcal{F})} \leq \frac{P_H(\mathcal{E},m)}{rk(\mathcal{E})} \quad \text{for } m \gg 0.$$

$\mathcal{E}$ is called stable if inequality is always strict.

($\mathcal{E}$ is called slope stable if $\frac{c_1(\mathcal{F})\cdot H}{rk(\mathcal{F})} < \frac{c_1(\mathcal{E})\cdot H}{rk(\mathcal{E})}$)

for all subsheaves $\mathcal{F} \subset \mathcal{E}$ by Riemann-Roch

slope stable $\Rightarrow$ stable ($\Rightarrow$ semistable)

(<u>Remark</u>: A torsion free sheaf is a "vector bundle with finitely many singularities"
It is stable if it does not have too big subsheaves.)

<u>Theorem</u>: (Maruyama-Gieseker)

There exists a (coarse) moduli space $M_S^H(r, c_1, c_2)$ of H semistable torsion coherent sheaves E on S of rank r, with Chern classes $c_1(E) = c_1$, $c_2(E) = c_2$.

$M_S^H(r, c_1, c_2)$ is projective.

The open subset $M_S^H(r, c_1, c_2)^s \subset M_S^H(r, c_1, c_2)$ parametrizes H-stable sheaves.

Expected dimension: Assume $b_1(S) = h^1(S,\mathbb{Q}) = 0$.

$M = M_S^H(r, c_1, c_2)$ is usually very singular but has an expected dimension

$$\mathrm{vd}(M) = 2rc_2 - (r-1)c_1^2 + (r^2-1)\chi(\mathcal{O}_S)$$

Expected dimension $\mathrm{vd}(M)$ = dimension M would have if it was good

Kuranishi: locally in the analytic topology $M_S^H(r, c_1, c_2)$ is the zero set of a holom. map

$$\mathbb{C}^m \xrightarrow{K} \mathbb{C}^k \quad \text{with} \quad \mathrm{vd}(M) = m - k$$

If K was smooth, M would be nonsingular of dimension $\mathrm{vd}(M)$.

(2) Vafa-Witten formula

S proj. alg surface with ample line bundle H.

Assume $b_1(S)=0$, $p_g(S) = \dim H^0(S, K_S) > 0$

($\exists$ holom. 2 form on S).

(e.g. S K3 surface, S elliptic surface, S surface of general type)

Choose H ample, c_1, c_2 s.th.

$$M_S^H(c_1,c_2) = M_S^H(c_1,c_2)^s$$

Write $K_S^2 = \int_S K_S^2 \in \mathbb{Z}$, $\chi(\mathcal{O}_S) = 1 + p_g(S)$

holom. Euler characteristic

Assume there is an irreducible curve in $|K_S|$

(i.e. $C = Z(s)$, s holom. 2 form on S)

Vafa-Witten formula: Put

$$\tilde{\eta}(x) = \prod_{n>0}(1-x^n).$$

$$\Theta(x) = \sum_{n\in\mathbb{Z}} x^{n^2} = (1+2x+2x^4+\dots) \qquad \text{(modular forms)}$$

Put $$\Psi_S = 8\left(\frac{1}{2\tilde{\eta}(x^2)^{12}}\right)^{\chi(\mathcal{O}_S)} \left(\frac{2\eta(x^4)^2}{\Theta(x)}\right)^{K_S^2}$$

Vafa-Witten formula:

$$e(M_S^H(c_1,c_2)) = \operatorname{Coeff}_{x^{vd}}(\Psi_S(x)).$$

Actual Vafa-Witten invariant is more general.

One considers invariants of Higgs moduli space.

$$N_S^H(c_1,c_2) = \left\{ (E,\phi) \;\middle|\; \begin{array}{l} E \text{ rk } 2 \text{ sheaf} \\ \phi : E \to E\otimes K_S \\ \operatorname{tr}\phi = 0 \end{array} \right\}$$

This has $\mathbb{C}^*$-action by

$$\lambda\cdot(E,\phi) = (E,\lambda\phi) \quad \text{(rescaling)}.$$

The full Vafa-Witten invariants are (Tanaka-Thomas)

$$VW_{c_1,c_2} = e^{vir}\left(N_S^H(c_1,c_2)^{\mathbb{C}^*}\right)$$

The fixpoint set $N_S^H(c_1,c_2)^{\mathbb{C}^*}$ has different components. The component $\Phi=0$ is $M_S^H(c_1,c_2)$.

This is often called "instanton branch."

The contribution of the other components Monopole branch.

<u>We will want to interpret these and generalize this formula</u>

(3) Virtual topological invariants

$M = M_S^H(c_1, c_2)$ is usually singular, not of expected dimension. $vd(M)$

But it is virtually smooth of dimension $vd(M)$

This means one can define virtual analogues of the invariants of a smooth proj. variety on M.

<u>Technically</u>: M has a 1-perfect obstruction theory

<u>Definition</u>: Let M be a scheme with an embedding $M \xhookrightarrow{i} X$ into a smooth scheme (e.g. M projective)

$I = I_{M/X}$ ideal sheaf.

A perfect obstruction theory on M is a complex

$E^\bullet = [E^{-1} \xrightarrow{d} E^0]$ of vector bundles on M

with a morphism of complexes

$$\begin{tikzcd} E^{-1} \arrow[r, "d"] \arrow[d, "\varphi"'] & E^{0} \arrow[d, "\varphi"] \\ I/I^2 \arrow[r, "d"] & \Omega_X|_M \end{tikzcd}$$

s.th (1) $\varphi : \mathrm{Coker}(d) \to \mathrm{Coker}(d)$ is an isomorphism.

(2) $\varphi : \mathrm{Ker}(d) \to \mathrm{Ker}(d)$ is surjective

Denote $vd(M) := rk(E^\bullet) = rk(E^0) - rk(E^{-1})$ the expected dimension of $(M, E^\bullet)$.

<u>Theorem</u>: (Behrend–Fantechi, Li–Tian..)

Let M be a scheme with a 1-perfect obstruction theory.

(1) M has a virtual fundamental class

$[M]^{vir} \in H_{2vd(M)}(M, \mathbb{Z})$.

(Thus for $\alpha \in H^*(M, \mathbb{Q})$, can define virtual intersection numbers $\int_{[M]^{vir}} \alpha \in \mathbb{Q}$,)

(2) M has virtual structure sheaf.

$$\mathcal{O}_M^{vir} \in K_0(M)$$

<u>Review of K-groups:</u> Let X variety

$$K^0(X) = \{ \text{formal linear combinations } a_1 E_1 + \dots + a_n E_n \mid n \geq 0 \quad a_i \in \mathbb{Z}, \; E_i \text{ vector bundle on } X \}$$

Equivalence relation

$$F \sim F + G \text{ whenever}$$

$$0 \to E \to F \to G \to 0 \quad \text{exact sequence.}$$

$K_0(X)$ is the same with vector bundles replaced by coherent sheaves.

Chern classes are defined for elements of $K^0(X)$

$$c\left(\sum_{i=1}^{n} a_i E_i\right) = \prod_{i=1}^{n} c(E_i)^{a_i}$$

<u>Definition:</u> The vertical tangent bundle of M is

$$T_M^{vir} = E_0 - E_1 \in K^0(M) \qquad E_i := (E^{-i})^\vee$$

The virtual Euler number of M is

$$e^{vir}(M) := \int_{[M]^{vir}} c_{vd}(T_M^{vir})$$

The case of moduli spaces of sheaves

Let $\mathcal{E}/_{S\times M_S^H(r,c_1,c_2)}$ univ. sheaf.

i.e $\mathcal{E}|_{S\times[E]} = E$ for $E\in M$.

$\pi: S\times M \to M$ projection

The dual of the obstruction theory for M is

$$R\pi_* R\mathcal{H}om(\mathcal{E},\mathcal{E})_0[1] \in D^b(M).$$

traceless

(i.e. Tangent space $Ext^1(E,E)_0$, obstruction space $Ext^2(E,E)_0$)

One can represent $R\pi_* R\mathcal{H}om(\mathcal{E},\mathcal{E})_0$ by a complex $E^0 \to E^1$ of vector bundles on M

This gives the following

(1) M has a virtual fundamental class
$$[M]^{vir} \in H_{2vd}(M)$$

(2) M has a virtual tangent bundle $T_M^{vir} = E^0 - E^1 \in K^0(M)$

Can define virtual Euler number.
$$e^{vir}(M) = \int_{[M]^{vir}} c_{vd}(M) \in \mathbb{Z}.$$

(3) M has a virtual structure sheaf.
$$\mathcal{O}_M^{vir} \in K_0(M).$$
For any vector bundle V on M define
$$\chi^{vir}(M, V) := \chi(M, V \otimes \mathcal{O}_M^{vir})$$

Conjecture: Vafa-Witten conjecture holds for virtual Euler number ($h_1(S)=0$, $p_g(S)>0$, H-stable = H-semistable, $|K_S|$ contains smooth curve).

Then
$$e^{vir}(M_S^H(c_1,c_2)) = \mathrm{Coeff}_{x^{vd}}\left[8 \left(\frac{1}{\bar{\eta}(x^2)^{12}}\right)^{\chi(\mathcal{O}_S)} \left(\frac{2\bar{\eta}(x^4)^2}{\theta(x)}\right)^{K_S^2}\right.$$

Check this conjecture and refine it.

Refinements of conjecture

Replace $e(M)$ by finer topological invariants

χ_{-y}-genus: X smooth proj. variety, has Hodge numbers $h^{p,q}(X) = h^q(X, \Omega^p_X)$

The χ_{-y}-genus of X is

$$\chi_{-y}(X) = \sum_{p,q} (-1)^{p+q} h^{p,q}(X) y^q = \sum_q (-y)^q \chi(X, \Omega^q_X).$$

Easy to see $\chi_{-y}(X)|_{y=1} = e(X)$.

e.g. K3 surface

$$\chi_{-y}(S) = 2 + 20y + 2y^2, \quad e(S) = 24$$

Virtual χ_{-y}-genus

Now let M be virtually smooth.

Let $\Omega^{p,vir}_M = \Lambda^p (T^{vir}_M)^\vee$ virtual p-forms.

(note $\Lambda^p(E_0 - E_1) = \bigoplus_{i+j=p} (-1)^j \Lambda^i E_0 \otimes \mathrm{Sym}^j(E_1)$)

The vertical χ_{-y}-genus of M is

$$\chi_{-y}^{vir}(M) := y^{-vd(M)/2} \sum_p (-y)^p \chi^{vir}(M, \Omega_M^{p,vir})$$

<u>Can show</u>: $\chi_{-y}^{vir}(M)\big|_{y=1} = e^{vir}(M)$.

Use virtual Riemann-Roch theorem

$$\chi^{vir}(M,V) = \int_{[M]^{vir}} ch(V) \cdot td(T_M^{vir}).$$

Conjecture for virtual χ_{-y}-genus.

Put $\Theta(x,y) := \sum_{n\in\mathbb{Z}} x^{n^2} y^n$, $\tilde{\eta}(x) = \prod_{n>0} (1-x^n)$.

$$\Psi_S(x,y) := 2^{3-\chi(\mathcal{O}_S)+K_S^2} \left(\frac{1}{\prod_{n>0} (1-x^{2n})^{10} (1-x^{2n}y)(1-x^{2n}/y)} \right)^{\chi(\mathcal{O}_S)} \cdot \left(\frac{\tilde{\eta}(x^4)^2}{\Theta(x, y^{1/2})} \right)^{K_S^2}$$

<u>Conjecture</u>: S surface with $b_1(S), p_g(S) > 0$.

$|K_S|$ contains smooth irred curve

$$M_S^H(c_1, c_2) = M_S^H(c_1, c_2)^s$$

Then $$\chi_{-y}^{vir}(M_S^H(c_1, c_2)) = \operatorname{Coeff}_{x^{vd(M)}}(\Psi_S(x, y))$$

<u>Example</u>: S K3 surface, then $M_S^H(c_1, c_2)$ is smooth

and $$\chi_{-y}(M_S^H(c_1, c_2)) = \chi_{-y}(S^{[vd(M)/2]})$$

(Yoshioka).

<u>Further refinements</u>: Formula for elliptic genus of M.

Partial formula for cobordism class of M.

(4) Mochizuki's formula

The main tool is Mochizuki's formula computing intersection numbers on moduli spaces of sheaves $M_S^H(c_1, c_2)$ in terms of intersection numbers on Hilbert schemes of points on S and Seiberg-Witten invariants

Seiberg-Witten invariants

Seiberg-Witten invariants are C^∞ invariants of 4 manifolds.
For projective algebraic surfaces they are quite simple.
Let S smooth proj. surface, with $p_g(S)>0$, $b_1(S)=0$

Seiberg-Witten invariants: $SW : H^2(S, \mathbb{Z}) \to \mathbb{Z}$
$$a \mapsto SW(a)$$

If $SW(a) \neq 0$, a is called a Seiberg-Witten class.

<u>Facts</u> (1) If S is a K3 surface, the only SW class is 0, with $SW(0)=1$.

(2) If $|K_S|$ contains a smooth irreducible curve SW classes are $0, K_S$ with $SW(0)=1$
$$SW(K_S)=(-1)^{\chi(\mathcal{O}_S)}$$

(3) If $\hat{S} \xrightarrow{\pi} S$ is the blowup of S in a point with exceptional divisor E the SW classes of $\hat{S}$ are
$$\{\pi^*(a), \pi^*(a)+E \mid a \text{ SW class of } S\}$$
and $SW(\pi^*(a)) = SW(\pi^*(a)+E) = SW(a)$.

(4) $E(n) \to \mathbb{P}^1$ elliptic surface with $12n$ nodal fibres F class of a fibre, then $K_S=(n-2)F$, $\chi(\mathcal{O}_S)=n$

Seiberg-Witten classes are kF $\quad 0 \le k \le n-2$

with $SW(kF)=(-1)^k \binom{n-2}{k}$

(b) Setup for Mochizuki's formula

S proj. surface with $b_1(S)=0$, $p_g(S)>0$.

$$\begin{tikzcd} & S\times S^{[n_1]}\times S^{[n_2]} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ S & & S^{[n_1]}\times S^{[n_2]} \end{tikzcd}$$

Universal sheaves: Let $Z_n(S)=\{(x,z)\in S\times S^{[n]} \mid x\subset z\}$

univ. subscheme.

$I_{Z_n(S)}/S\times S^{[n]}$ universal ideal sheaf.

For $L\in \mathrm{Pic}(S)$ put

$$I_i(L) := p^*_{S\times S^{[n_i]}}\left(I_{Z_{n_i}(S)}\otimes p_S^* L\right) \qquad i=1,2.$$

sheaf on $S\times S^{[n_1]}\times S^{[n_2]}$ with $I_i(L)\big|_{S\times[Z_1]\times[Z_2]} = I_{Z_i}\otimes L$

Tautological sheaves

$$\begin{tikzcd} & Z_n(S) \arrow[dl, "q"'] \arrow[dr, "p"] & \\ S & & S^{[n]} \end{tikzcd}$$

$G^{[n]}(L) = p_* q^*(L)$ is vector bundle of rk n on $S^{[n]}$

with fibre $G^{[n]}(L)([Z]) = H^0(L|_Z)$.

Define $G_1(L)$, $G_2(L)$ on $S^{[n_1]} \times S^{[n_2]}$ by $G_i(L) = p^*_{S^{[n_i]}} G^{[n_i]}(L)$

Classes on $M = M_S^H(c_1, c_2)$

Assume on $S \times M$ we have a universal sheaf $\mathcal{E}$.

(i.e. $\mathcal{E}|_{S \times [E]} = E$)

$$\begin{tikzcd} & S \times M \arrow[dl, "\pi_S"'] \arrow[dr, "\pi_M"] & \\ S & & M \end{tikzcd}$$

For $\alpha \in H^k(S)$ put $z_i(\alpha) := \pi_{M*}(c_i(\mathcal{E}) \cap \pi_S^*(\alpha))$

$\in H^{2i-4+k}(M)$.

Let $P(\mathcal{E})$ be a polynomial in the

$(z_i(\alpha))_{i \geq 0, \alpha \in H^*(S)}$.

Mochizuki's formula: Compute

$$\int_{[M]^{vir}} P(\mathcal{E}) \text{ in terms of intersection numbers}$$

on $S^{[n_1]} \times S^{[n_2]}$.

For $a_1, a_2 \in \mathrm{Pic}(S)$ put $\Psi_P(a_1, a_2, n_1, n_2, s) =$

Expression in s, Chern classes of $I_1(a_1), I_2(a_2)$
$\mathcal{O}_1(a_1)$, $\mathcal{O}_2(a_2)$ depending on P

Explicitly: $\pi: S \times S^{[n_1]} \times S^{[n_2]} \longrightarrow S^{[n_1]} \times S^{[n_2]}$

For sheaves $\mathcal{E}_1, \mathcal{E}_2$ on $S \times S^{[n_1]} \times S^{[n_2]}$ put

$$Q(\mathcal{E}_1, \mathcal{E}_2) = \mathcal{E}u\left(-R\pi_* R\mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2)\right).$$

Let s be a variable. If E is a vector bundle of $\mathrm{rk}\, E$ put

$$c_k(E(s)) = \sum_{i=0}^{r} \binom{r-i}{k-i} c_i(E)\, s^{k-i}$$ (i.e. treat s like the first Chern class of a line bundle)

[In fact s is the equivariant first Chern class of an equivariant line bundle]

For $a_1, a_2 \in H^2(S)$ we put

$$\Psi(a_1,a_2,n_1,n_2,s) = \frac{P(I_1(a_1-s)\oplus I_2(a_2+s))\, Eu(\mathcal{O}_1(a_1))\, Eu(\mathcal{O}_2(a_2+2s))}{Q(I_1(a_1-s), I_2(a_2-s))\quad (2s)^{n_1+n_2-\chi(\mathcal{O}_S)}}$$

$$\in H^*(S^{[n_1]}\times S^{[n_2]}, \mathbb{Q})[s^{-1}][[s]].$$

Put $A(a_1,a_2,n,s) := \sum_{n_1+n_2=n} \int_{S^{[n_1]}\times S^{[n_2]}} \Psi(a_1,a_2,n_1,n_2,s)$

<u>Theorem</u> (Mochizuki)

Assume (1) all $E \in M_S^H(c_1,c_2)$ are H-stable

(2) $\chi(S,E) > 0 \quad \forall E \in M_S^H(c_1,c_2)$

Then for every P as above

$$\int_{[M_S^H(c_1,c_2)]^{vir}} P(E) = \sum_{\substack{c_1=a_1+a_2\\ a_1H<a_2H}} SW(a_1)\, \mathrm{Coeff}_{s^0}\left(A(a_1,a_2,c_2-a_1a_2,s)\right)$$

How does this help us?

We know little about $M_S^H(c_1, c_2)$, so cannot compute any intersection numbers on it.

Mochizuki's formula replaces the simple $P(\mathcal{E})$ by a much more complicated expression on $S^{[n_1]} \times S^{[n_2]}$

But while before it was impossible to compute now it is just complicated.

(5) Application to virtual Euler number

Take $P(\mathcal{E}) = c_{vd}(M) = c_{vd}(T_M^{vir})$

with $T_M^{vir} = -R\pi_* R\mathcal{H}om(\mathcal{E}, \mathcal{E})_0$.

Applying Grothendieck-Riemann Roch to projection

$\pi : S \times M \to M$ gives $c_{vd}(T_M^{vir})$ is a polynomial in the $z_i(\alpha)$.

So Mochizuki's formula applies

Similarly apply the virtual Riemann–Roch formula

$$\chi^{vir}(M, V) = \int_{[M]^{vir}} ch(V)\, td(T_M^{vir})$$

expresses $\chi_{-y}^{vir}(M)$ as a polynomial in the $z_i(\alpha)$.

For the rest of this section we will assume

$$P(E) = c_{vd}(T_M^{vir})$$

The refinement to χ_{-y}, elliptic genus and cobordism class are similar.

Thus to compute

$$e^{vir}(M) := \int_{[M_S^H(c_1,c_2)]^{vir}} c_{vd}(T_M^{vir}) =$$

$$= \sum_{\substack{c_1 = a_1 + a_2 \\ a_1 H < a_2 H}} SW(a_1)\, \mathrm{Coeff}_{s^0}\left[A(a_1, a_2, c_2 - a_1 a_2, s)\right]$$

This seems impossible to compute directly, but we will show some nice properties

Cobordism invariance and multiplicativity

Write down generating function

$$Z_S'(a_1,a_2,S,q) := \sum_{n\geq 0} A(a_1,a_2,n,S)\,q^n.$$

(partition function)

<u>Proposition</u>: There is a polynomial $\tilde{P}$ in $a^2, ab, b^2, aK_S, bK_S, K_S^2, \chi(\mathcal{O}_S)$ s.th. for all surfaces S, all $a,b\in \mathrm{Pic}(S)$ we have

$$A(a,b,n,S) = \tilde{P}(a^2, ab, b^2, aK_S, bK_S, K_S^2, \chi(\mathcal{O}_S))$$

This is a modification of an argument of Ellingsrud-G-Strømme, using an inductive scheme to understand intersection numbers on $S^{[n]}$!

Based on

Theorem: Let $Z_n(S) = \{(x,Z) \in S \times S^{[n]} \mid x \in Z\}$ univ. subscheme.

$$S^{[n,n+1]} = \{(Z,W) \in S^{[n]} \times S^{[n+1]} / Z \subset W\}$$

Then $S^{[n,n+1]}$ is the (smooth) blowup of $S \times S^{[n]}$ along $Z_n(S)$.

Idea of proof: Have natural map

$$\begin{aligned} \rho_{n+1} : S \times S^{[n]} &\dashrightarrow S^{[n+1]} \\ (x, Z) &\longmapsto x \amalg Z \end{aligned} \quad \text{defined outside } Z_n(S).$$

To expand it to $Z_n(S)$ need to extend $Z \in S^{[n]}$ to $W \in S^{[n+1]}$ in a point $p \in Z$.

Thus we have exact sequence

$$0 \to I_W \to I_Z \xrightarrow{\lambda} \underset{p}{k(x)} \to 0$$

skyscraper at x.

Then λ is a 1-dim quotient of fibre of $I_{Z_n(S)}$ over (x,Z). This gives $S^{[n,n+1]} = P(I_{Z_n(S)})$

$$= Bl_{Z_n(S)}(S \times S^{[n]}).$$

Inductive scheme

$$\begin{tikzcd}[column sep=small]
 & S^{[n-1,n]} \arrow[dl, "g_n"'] \arrow[dr, "p_{n-1}"] & & S\times S^{[n-2,n-1]} \arrow[dl, "g_{n-1}"'] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr, "p_1"] & \\
S^{[n]} & & S\times S^{[n-1]} & & \cdots & & S^n.
\end{tikzcd}$$

For $\alpha\in H^*(S^{[n]})$ have

$$\int_{S^{[n]}}\alpha = \frac{1}{n}\int_{S^{[n-1,n]}} g_n^*(\alpha) = \frac{1}{n}\int_{S\times S^{[n-1]}} p_{n-1*}\, g_n^*\alpha = \dots$$

$$= \dots = \int_{S^n} p_{1*}\, g_1^* \cdots\cdots p_{n-1*}\, g^*\alpha$$

We do have an understanding of $S^{[n-1,n]} \xrightarrow[p_{n-1}]{} S\times S^{[n-1]}$

Thus we can compute with this.

This gives

Theorem: Let P a polynomial in the Chern classes of $G^{[n]}(L)$ and $T_{S^{[n]}}$
$\Rightarrow \exists$ polyn. $\tilde{P}$ in $L^2, LK_S, K_S^2, \chi(\mathcal{O}_S)$ s.th.

$$\int_{S^{[n]}} P = \tilde{P}.$$

The method of proof can be adapted to show above proposition.

<u>Multiplicativity</u>:

Write $Z'(a,b,s,q) := (2s)^{\chi(\mathcal{O}_S)} \left(\frac{2s}{1+2s}\right)^{\chi(S,b-a)}$

$$\left(\frac{-2s}{1-2s}\right)^{\chi(S,a-b)} \cdot Z(a,b,s,q)$$

$\Longrightarrow$ $Z_S(a,b,s,q)$ is a power series in q starting with 1 (instanton part of partition function).

<u>Theorem</u>: There are power series $A_1, \dots A_7 \in \mathbb{Q}((s))[[q]]$ with

$$Z_S(a,b,s,q) = A_1^{a^2} A_2^{ab} A_3^{b^2} A_4^{aK_S} A_5^{bK_S} A_6^{K_S^2} A_7^{\chi(\mathcal{O}_S)}$$

<u>Idea of proof</u>:

Put $v(S,a,b) = (a^2, ab, b^2, aK_S, bK_S, K_S^2, \chi(\mathcal{O}_S)) \in \mathbb{Z}^7$.

Write $(S_1, a_1, b_1) \oplus (S_2, a_2, b_2)$ for $(S_1 \amalg S_2, a, b)$ with $a|_{S_i} = a_i$, $b|_{S_i} = b_i$

Then $\nu\big((S_1,a_1,b_1)\oplus(S_2,a_2,b_2)\big) = \nu(S_1,a_1,a_2) + \nu(S_2,a_2,b_2)$.

Write $Z_{(S,a,b)}$ for $Z_S(a,b,s,q)$.

Then we need to show

$$Z_{(S_1,a_1,b_1)\oplus(S_2,a_2,b_2)} = Z_{(S_1,a_1,b_1)} \cdot Z_{(S_2,a_2,b_2)}$$

Then the multiplicativity follows formally.

This will follow from

$$(S_1 \# S_2)^{[n]} = \coprod_{n_1+n_2=n} S_1^{[n_1]} \times S_2^{[n_2]}.$$

Because the $\Psi(a_1,a_2,n_1,n_2,S)$ are compatible.

Reduction to the case of $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$

$Z_S(a,b,s,q)$ depends only on the 7 tuple

$v(S,a,b) = (a^2, ab, b^2, aK_S, bK_S, K_S^2, \chi(\mathcal{O}_S))$

By multiplicativity in order to compute

$Z_S(a,b,s,q)$ for all S, $a,b \in \mathrm{Pic}(S)$ it is enough to compute it for 7 triples (S_i, a_i, b_i) such that the

$v(S_i, a_i, b_i)$ are linearly independent in $\mathbb{Q}^7$.

<u>Choose</u>: $(\mathbb{P}^2, \mathcal{O}, \mathcal{O}), (\mathcal{O}(1), \mathcal{O}), (\mathcal{O}, \mathcal{O}(1)), (\mathcal{O}(1), \mathcal{O}(1))$

$(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}, \mathcal{O}), (\mathcal{O}(1,0), \mathcal{O}), (\mathcal{O}, \mathcal{O}(1,0))$.

<u>Note</u> $\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$ are toric surfaces, i.e. have an action of $T = \mathbb{C}^* \times \mathbb{C}^*$ with finitely many

fixpoints.

$\Rightarrow$ Can compute $Z_S(a,b,s,q)$ by equivariant localization.

Equivariant localization

Let X smooth proj. variety with action of $T = (\mathbb{C}^*)^k$ with finitely many fixpoints.

$p_1, \dots p_e$

Let p be one of the fixpoints

Let E be a T equivariant vector bundle on X

Then the fibre $E(p)$ is a vector space

with a linear T-action

$\Rightarrow$ $E(p)$ has a Basis of eigenvectors.

$$E(p) = \bigoplus_{i=1}^{r} \mathbb{C} \cdot v_i \quad \text{with} \quad (t_1, \dots t_k) \cdot v_i = t_1^{n_{1i}} \cdots t_k^{n_{ki}} v_i$$

The weight of v_i is

$$w(v_i) = \sum_{j=1}^{k} n_{ji} \varepsilon_j \qquad \text{for variables } \varepsilon_1, \dots, \varepsilon_k$$

Localized equivariant Chern class of E at p

$$c^T(E)(p) = (1 + c_1^T(E)(p) + \dots + c_r^T(E(p)))$$

$$= \prod_{i=1}^{r} (1 + w(v_i)) \in \mathbb{Z}[\varepsilon_1, \dots, \varepsilon_s] = H_T^*(pt)$$

equiv. cohom.

c_i^T = part of degree i in the ε_i

<u>Theorem</u>: (Bott residue formula)

Let $E_1, \dots E_s$ T-equiv. vector bundles on X

$P(c_i(E_1), \dots c_i(E_s))$ pol in the Chern classes of the E_i.

$$\Longrightarrow \int_{[X]} P(c_i(E_1) \dots c_i(E_s)) = \sum_{j=1}^{\ell} \left. \frac{P(c_i^T(E_1)(p_j), \dots c_i^T(E_s))}{c_d^T(T_{p_j} X)} \right|_{\varepsilon_1 = \dots = \varepsilon_s = 0}$$

Localization on Hilbert schemes of points

Let S smooth projective toric surface, i.e.

i.e. $T = (\mathbb{C}^*)^2$ acts on S with finitely many fixed points

$p_1, \dots p_e$, and one dense orbit on which T acts freely, thus
$T \subset S$.

For simplicity $S = \mathbb{P}^2$.

homog. coord. X_0, X_1, X_2

action of $t = (t_1, t_2) \in T$ by

$$t \cdot [X_0, X_1, X_2] = [X_0, t_1 X_1, t_2 X_2].$$

Fixpoints are

$$p_0 = [1,0,0], \quad p_2 = [0,1,0], \quad p_2 = [0,0,1]$$

We have local coordinates

$$x_0 = \frac{X_1}{X_0}, \quad y_0 = \frac{X_2}{X_0}, \quad \text{at } p_0,$$

$$x_1 = \frac{X_0}{X_1}, \quad y_1 = \frac{X_2}{X_1} \quad \text{at } p_1$$

$$x_2 = \frac{X_0}{X_2}, \quad y_2 = \frac{X_1}{X_2} \quad \text{at } p_2$$

These are eigenvectors for the T action, e.g.

$t\cdot(x_0,y_0)=(t_1x_0,\ t_2y_0)$ $\quad w(x_0)=\varepsilon_1,\ w(y_0)=\varepsilon_2$

$t\cdot(x_1,y_1)=(t_1^{-1}x_1,\ t_1^{-1}t_2y_1)$ $\quad w(x_1)=-\varepsilon_1,\ w(y_1)=\varepsilon_2-\varepsilon_1$

$w(x_2)=-\varepsilon_2,\ w(y_2)=\varepsilon_1-\varepsilon_2$

The action lifts to $\mathbb{P}^{2[n]}$ by

$t\cdot[Z]=[tZ]$ $\qquad t:\mathbb{P}^2\to\mathbb{P}^2$ isom.

If $Z\in(\mathbb{P}^2)^{[n]}$ is a fixpoint for the T action,

then $\mathrm{supp}(Z)$ is fixed by T action, thus

$Z=Z_0\sqcup Z_1\sqcup Z_2$, $\ \mathrm{supp}(Z_i)=p_i$,

$\mathrm{len}(Z_i)=n_i$, $\quad n_0+n_1+n_2=n$

And in local coordinates x_i,y_i at p_i

I_{Z_i} is generated by monomials.

If $m=n_i$

$I_{Z_i}=(y^{m_0},xy^{m_1},\dots,x^ry^{m_r},x^{r+1})$ $\quad$ with $x=x_i$, $y=y_i$

Where $(m_0,m_1,\dots m_r)$ is a partition of m.

(In fact $\left\{x^iy^j \,\middle|\, \begin{matrix}0\le i\le r\\ 0\le j<m_i\end{matrix}\right\}$ is basis of $\mathcal{O}_{Z_i}=\mathcal{O}_{Z,p_i}$
$=\mathbb{C}[x,y]/I_Z$!

Putting this together we have a bijection

$$\text{Fixpoints } z \in ((\mathbb{P}^2)^{[n]})^T \longleftrightarrow \text{triples } \nu_0, \nu_1, \nu_2 \text{ of partitions of } n$$

We apply the Bott residue formula the Mochizuki formula using this description of the fixpoints.

(1) Fibres at fixpoints of all universal sheaves of $T_{S^{[n_1]} \times S^{[n_2]}}$, $Q(I_1(a_1), I_2(a_2))$ are described in terms of partitions

(2) Mochizuki formula is evaluated as a combinatorial formula in terms of partitions

(3) Relation to Nekrasov Partition function

Nekrasov partition function

Consider instanton moduli space

$$M(n) = \left\{ (E, \phi) \;\middle|\; \begin{array}{l} E \text{ rk } 2 \text{ sheaf on } \mathbb{P}^2,\ c_2(E) = n \\ \phi : E|_{\ell_\infty} \simeq \mathcal{O}^{\oplus 2} \end{array} \right\}$$

Torus action: $\mathbb{C}^* \times \mathbb{C}^*$ acts on $(\mathbb{P}^2, \ell_\infty)$

$$(t_1, t_2)(z_0 : z_1 : z_2) = (z_0 : t_1 z_1 : t_2 z_2)$$

Extra $\mathbb{C}^*$ acts by $s \cdot (E, \phi) = \left(E, \begin{pmatrix} s^{-1} & 0 \\ 0 & s \end{pmatrix} \circ \phi\right)$.

$T = (\mathbb{C}^*)^2 \times \mathbb{C}^*$

Fixpoints: $M(n)^T = \left\{ (I_{Z_1} \oplus I_{Z_2}, id) \;\middle|\; Z_i \in \mathrm{Hilb}^{n_i}(\mathbb{A}^2, 0) \right\}$

monomial ideal

Nekrasov partition function, pure theory

$$Z^{inst}(\varepsilon_1, \varepsilon_2, a, q) = \sum_{n \geq 0} \left(\int_{M(n)} 1 \right) q^n$$

$\varepsilon_1 = \log(t_1)$

$\varepsilon_2 = \log(t_2)$

$\log(s)$

(formal application of Bott residue formula: $M(n)$ is noncompact.)

Nekrasov conjecture = explicit formula for
$\log(Z^{\text{inst}})$.

$\Longrightarrow$ Wallcrossing formula for Donaldson inv. in case $b_+=1$.

$\mathcal{E}$ universal sheaf on $\mathbb{P}^2 \times M(n)$

$$\begin{tikzcd} & \mathbb{P}^2 \times M(n) \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathbb{P}^2 & & M(n) \end{tikzcd}$$

tautological bundle

$$V = R^1 p_*\left(\mathcal{E} \otimes q^* \mathcal{O}(-\ell_\infty)\right)$$

Nekrasov partition function with fundamental –

$$Z^{\text{inst}}(\varepsilon_1, \varepsilon_2, a, m, q) = \sum_{n=0}^{\infty} \int_{M(n)} \mathrm{Eu}\,(V(m))$$

m equiv. weight

Nekrasov conj. for this + Mochizuki formula

$\Longrightarrow$ Witten conj. SW = Donaldson

Adjoint matter: $\mathrm{Eu}\,(T_{M(n)}(M))$ M equiv. weight

Nekrasov part. function with adjoint and fundamental matter

$$Z^{\text{inst}}(\varepsilon_1,\varepsilon_2,a,m,M,q) = \sum_{n=0}^{\infty} \int_{M(n)} \mathrm{Eu}\left(T_{M(n)}(M)\right)\cdot \mathrm{Eu}\left(V(m)\right)$$

Now S projective toric surface $p_1,\dots p_e$ fixpoints.

Then $$Z_S(a_1,a_2,s,q) = \prod_{i=1}^{e} Z(w(x_i),w(y_i),m_i,M_i,q)$$

suitable expressions in s, a_1, a_2

(Note. On both sides the fixpoints for coeff of q^n are $2e$ tuples of partitions. Check contributions at each fixpoint is the same).

Thus: Nekrasov conjecture for adjoint + fundamental matter $\Rightarrow$ Vafa–Witten formula.

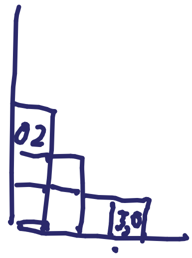
Problem: Nekrasov conj is not known, conformal dimension is negative.

So for now, we just evaluate combinatorial expression with computer, compute up to g^{30}.

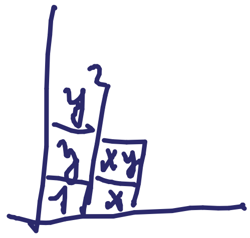
$\Rightarrow$ Vafa–Witten formula checked in many cases for $vd(M)$ up to 50.

<u>Young diagrams.</u>

A Young diagram of a partition $\nu = (n_0, \ldots n_r)$ is a diagram of boxes in the positive quadrant of the plane with columns of lengths $(n_0, \ldots n_r)$

e.g. $Y_{(3211)}$ 

We can think of the square (i,j) to correspond to the monomial $x^i y^j$.



Thus if $\nu = (n_0, .. n_r)$, $I_{Z_\nu} = (y^{n_0}, \dots x^r y^{n_r}, x^{r+1})$

then I_{Z_ν} has as Eigenbasis all monomials $x^j y^k$ with $(i,j) \notin Y_\nu$

$\mathcal{O}_{Z_\nu}$ has as Eigenbasis all monomials $x^i y^k$ with $i,j \in Y_\nu$.

Chern classes of tautological bundles.

Let $Z = Z_0 \amalg Z_1 \amalg Z_2 \in ((\mathbb{P}^2)^{[n]})^T$.

Let $I_{Z_i} = I_{Z_{\nu_i}}(x_i, y_i)$ for partitions ν_0, ν_1, ν_2.

Recall $\mathcal{O}^{[n]}(L)$ for a line bundle L on S is

The vector bundle of rk n with fibre $H^0(Z, \mathcal{O}_Z \otimes L)$ at Z

Take $S = \mathbb{P}^2$, $L = \mathcal{O}(m)$

<u>Proposition</u>: $$c^T(\mathcal{O}^{[n]}(m)(Z)) = \prod_{r=0}^{2} \prod_{(i,j)\in Y_{\nu_r}} \left(1 + i\,w(x_r) + j\,w(y_r) + m\,\varepsilon_r\right)$$

(where we put $\varepsilon_0 = 0$)

(weights on $\mathcal{O}(m)$ are 0 at p_0, $m\,\varepsilon_1$ at p_1, $m\,\varepsilon_2$ at p_2)

Tangent at Ext-bundles

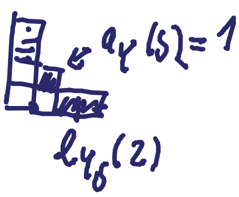
For a partition $\nu = (n_0, \dots n_r)$ with young diagram Y_ν.
The dual partition $\nu' = (n_0', \dots n_s')$ has $n_i' =$ length of i^{th} row of Y_ν.

Then $Y_{\nu'} = Y_\nu$ reflected along diagonal

e.g. $(3,2,1,1)' = (4,2,1)$

For $s = (i,j) \in Y_\nu$ the arm length is

$$a_{Y_\nu}(s) = n_i - j$$
$$\ell_{Y_\nu}(s) = n_j' - i$$



Theorem: As T-modules we have

$$T_{Z_{Y_\nu}}(\mathbb{A}^{2[n]}) = \sum_{s\in Y_\nu} x^{-l_{Y_\nu}(s)} y^{a_{Y_\nu}(s)+1} + x^{l_{Y_\nu}(s)+1} y^{-a_{Y_\nu}(s)}$$

Thus for $Z = Z_0 \sqcup Z_1 \sqcup Z_2$ with $I_{Z_i}(x_i, y_i)$.

$$c^T(T_Z \mathbb{P}^{2[n]}) = \prod_{i=1}^{3} \prod_{s\in Y_\nu} \left(1 - l_{Y_{\nu_i}}(s) w(x_i) + (a_{Y_{\nu_i}}(s)+1) w(y_i)\right) \cdot \left(1 + (l_{Y_{\nu_i}}(s)+1) w(x_i) - a_{Y_{\nu_i}}(s) w(y_i)\right)$$

There is a very similar formula for

$Ext^1(I_{Z_1}, I_W \otimes \mathcal{O}(n))$ on $\mathbb{P}^{2[n_1]} \times \mathbb{P}^{2[n_2]}$.

These are all the ingredient of the Mochizuki formula

In the same way we can compute on $\mathbb{P}^1 \times \mathbb{P}^1$.

So in the 7 cases above all terms of

$\Psi(a_1, a_2, n_1, n_2, s)$ have a formula in terms of

the combinatorics of partitions.

Apply Bott residue formula to obtain $Z_S(a_1, a_2, s, q)$

mod q^{31} (writing a program in PARI/GP)

$\Rightarrow$ Conjecture proven for many surfaces.

(elliptic surfaces, double covers of $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, complete intersections) up to high ord (30-50).

All arguments also work for χ_y-genus.

<u>Generalizations</u>: (1) Elliptic genus

(2) Cobordism class

(3) Case of moduli spaces of rank 3

(4) Verlinde formulas for algebraic surfaces.

Case of moduli spaces of rk 3 sheaves

There is a version of the Mochizuki formula for $M_S^H(r, c_1, c_2)$ for any r.

We apply this in the case of $r=3$.

In this case we have

$$vd = vd(M) = 6c_2 - 2c_1^2 - 8\chi(\mathcal{O}_S)$$

Again assume S alg surface with $b_1(S)=0$ and $p_g(S)>0$. For simplicity assume S contains an irreducible curve in $|K_S|$.

Theta functions for A_2-lattice

$$\Theta_{A,0} = \sum_{(n,m)\in\mathbb{Z}^2} x^{(n,m)\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}\begin{pmatrix}n\\ m\end{pmatrix}} = \sum_{n,m\in\mathbb{Z}^2} x^{2(n^2-nm+m^2)}$$

$$\Theta_{A,1} = \sum_{n,m\in\mathbb{Z}^2} \varepsilon^{n+m} x^{2(n^2-nm+m^2)} \qquad \varepsilon = e^{2\pi i/3}$$

(Modular forms of weight 1)

Define modular function

$$z(x) := \frac{\Theta_{A,0}(x)}{\Theta_{A,1}(x)} = 1 + 9x^2 + 27x^4 + \dots$$

Define $z_1(x)$, $z_2(x) = z_1(-x)$ as solutions of the equation

$$w^2 - 4z(x)^2 w + 4z(x) = 0.$$

Set

$$\Psi_{S,c_1}(x) := 9\left(\frac{1}{3\bar{\eta}(x^2)^{12}}\right)^{\chi(O_S)} \left(\frac{3\bar{\eta}(x^6)^3}{\Theta_{A,1}(x)}\right)^{K_S^2} \cdot \left(z_1(x)^{K_S^2} + z_2(x)^{K_S^2} + (-1)^{\chi(O_S)}\left(\varepsilon^{c_1 K_S} + \varepsilon^{-c_1 K_S}\right)\right)$$

<u>Conjecture</u>:

$$e^{vir}(M_S^H(3, c_1, c_2)) = \mathrm{Coeff}_{x^{vd}}(\Psi_{S,c_1}(x))$$

Similar formula for $\chi_{-y}^{vir}(M_S^H(3, c_1, c_2))$

This formula was again checked by the same methods for many examples