Context (Level 1)

Hard Lefschetz:

X - smooth projective of dim d

$w \in H^2(X)$ class of the hyperplane section ($w = c_1(L)$)

$\wedge w^k: H^{d-k}(X) \to H^{d+k}(X)$ is an isomorphism

Observation: something similar happens in the world of holomorphic symplectic affine varieties

Example 1 $T = (\mathbb{C}^*)^{2d}$ $w = \sum a_{ij} \frac{dt_i}{t_i} \wedge \frac{dt_j}{t_j}$

$a_{ij} = -a_{ji}$ (a_{ij}) non-degenerate called log-canonical

$\wedge w^k: H^{d-k}(T) \to H^{d+k}(T)$ is isomorphism.

Example 2 $\mathbb{C}^* \times \mathbb{C}^*$ x, y $w = \frac{dx}{x} \wedge \frac{dy}{y}$

affine blow up $(1,0)$

new variable z

$x - 1 = yz$ $x \in \mathbb{C}^*$

obtain a complement of $yz + 1 = 0$ in $\mathbb{C}^2$.

i) the form extends

ii) as a set $\mathbb{C}^* \times \mathbb{C}^* \sqcup \mathbb{C}$

Cohomology:

$H^0(T)$

$\mathbb{Z}^2 = H^1(T) \xrightarrow{res} H^0(\mathbb{C}) = \mathbb{Z}$

$H^2(T)$

Obtain $\left.\begin{array}{l} H^0 = \mathbb{Z} \\ H^1 = \mathbb{Z} \\ H^2 = \mathbb{Z} \end{array}\right\} \wedge w$

so far so good

Example 3 $\mathbb{C}^* \times \mathbb{C}^*$, blow up $(0,1)$ $(1,0)$:

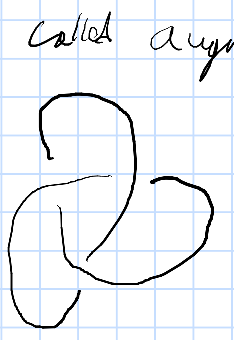
$x - 1 = y z_1$

$y - 1 = x z_2$

$y - 1 = (1 + y z_1) z_2$

$y z_1 z_2 + z_2 - y + 1 = 0$

(so called augmentation variety of



) (~ triply graded Khovanov homology at $a=0$)

$\left.\begin{array}{l} H^0 = \mathbb{Z} \\ H^2 = \mathbb{Z} \end{array}\right\} \wedge w$ $a + q + t$

So far so good

Example 4 blow up 6 pts

(blow up 6 pts on $\mathbb{P}^2$ get a cubic)

~ character variety of $\mathbb{P}^1 \setminus 4$ pts

Fricke

$\mathbb{Z}$

$2\mathbb{Z} \to 6\mathbb{Z}$

$\mathbb{Z}$

$\mathbb{Z}$ H^0

H^2 $4\mathbb{Z}$ $\mathbb{Z}$

not nice from the point of view of the cohomology index, but weight filtration is preserved by the operations, so we expect a Hard Lefschetz for weight index

Theorem Fix d, suppose X is filtered by closed subspaces $X_i \subset X_{i+1} \subset \ldots$

$X_i \setminus X_{i-1} \simeq (\mathbb{C}^*)^{2d-2a_i} \times \mathbb{C}^{a_i}$.

Suppose X has a closed holomorphic 2-form w such that restriction to each $(\mathbb{C}^*)^{2d-2a_i} \times \mathbb{C}^{a_i}$ is a pullback via projection of a non-degenerate log-canonical form from $(\mathbb{C}^*)^{2d-2a_i}$. Then curious hard Lefschetz holds:

$Gr^W_{2d-2i} H^j \xrightarrow{(\wedge w)^i} Gr^W_{2d+2i} H^{j+2i}$ is an isomorphism.

I won't explain this. In simple terms: same thing happens as in Examples 1-4.

Context (Level 2)

Character variety of $\Sigma_{g,k}$ (parabolic) (affine variety) — non-abelian Hodge — Moduli space of Higgs bundles

Hitchin system $X^{2d} \to \mathbb{C}^d$

Observation 1

Hausel - Letellier - Rodriguez-Villegas made a conjecture on explicit formula for the refined Poincaré polynomial of the char. variety (taking the weight filtration into account) $\in \mathbb{Z}[q,t]$

Mysterious $q \leftrightarrow t$ symmetry has been observed.

Observation 2 (Follow-up) + de Cataldo, Migliorini

P=W conjecture says that $W_{2i} = P_i$ where P is the perverse Leray filtration on $H(X)$ induced by the map $X \to \mathbb{C}^d$.

relative Hard Lefschetz when translated from P to W imply curious Hard Lefschetz.

Remark

Curious Hard Lefschetz + $W_{2i} \subset P_i$ implies $W_{2i} = P_i$, which is probably the method to prove P=W.

Results

1) $\tilde{X} \xrightarrow{\mathbb{C}^{\binom{n}{2}}\text{-bundle}}$ GL_n-Parabolic character variety (with at least 1 parabolic point)

$\tilde{X}$ admits a stratification so that each piece is a vector bundle over certain variety Y_p associated to a braid $p \in Br_n$.

2) Each Y_p admits a stratification so that each piece is of the form $(\mathbb{C}^*)^{2d-2i} \times \mathbb{C}^i$

3) Restriction of Goldman's holomorphic symplectic form on the character variety to Y_p produces a certain natural form on Y_p

4) Further restriction to $(\mathbb{C}^*)^{2d-2i} \times \mathbb{C}^i$ is a pullback via projection of a non-degen log-canonical form from $(\mathbb{C}^*)^{2d-2i}$

Hence curious H.L. for the parabolic character variety.

5) Moving around the eigenvalues around a "good point" cohomology does not jump (i.e. form local system on the torus $T \subset SL_n$)

"good point" = "generic" in the notations of Hausel-....

This + Springer theory implies 6), 6) ⇒ 7)

6) $H(X_{non-parabolic}) = H(X_{parabolic})^{S_n}$

7) curious hard Lefschetz for X, not necessarily with parabolic point.

≥ 2 parabolic pts ⇒ ∃ dual algebraic torus