Introduction to Beam Optics

Joint ICTP-IAEA Workshop on Accelerator Technologies, Basic Instruments and Analytical Techniques

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The aim of beam transport

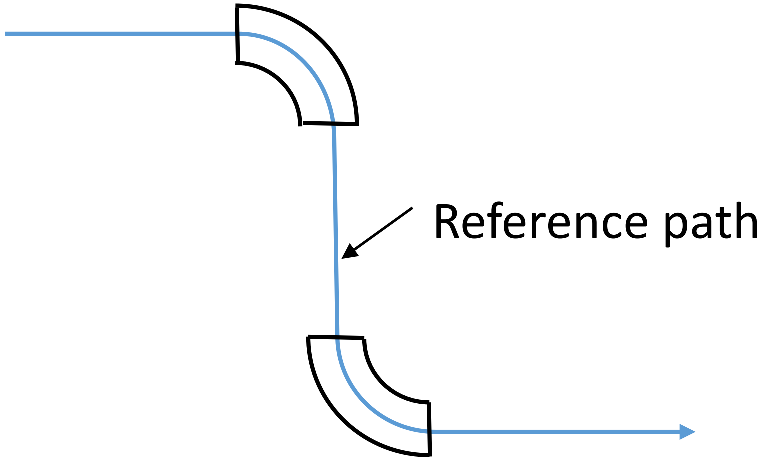
- There is a beam of charged particles. Each particle has a mass, charge, and a kinetic energy.

- We want to transport the beam from one place (e.g. an accelerator) to another (e.g. an experiment)

- This requires:
    1. The beam must follow the right path
    2. The beam must remain focussed around the path

- This is achieved using magnetic or electric fields

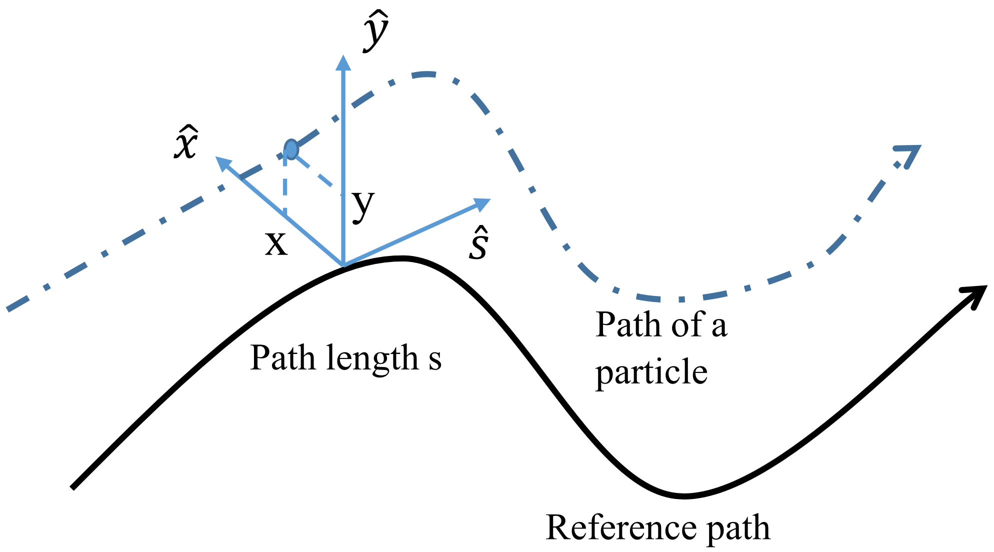
The reference path



- Reference path (or central trajectory): The ideal designed path, followed by the middle of the beam. Usually in the horizontal plane
- Consists of straight lines connected with bends:
- Straight line: called a drift space. It is just a length of beam pipe without any optical components
- Bends: A magnetic or electric field is used to change the direction of the entire beam

Coordinates for off-reference particles

- Real particles do not follow the exact reference path

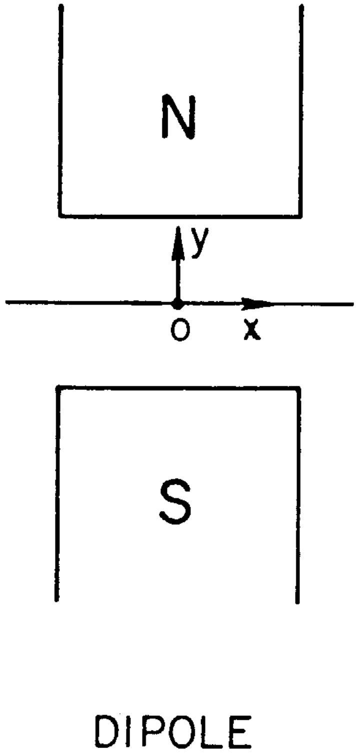


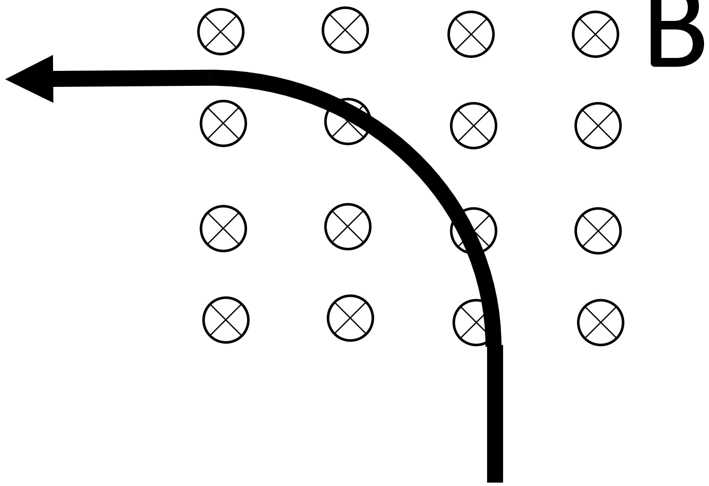
At each path length s, the deviation from the reference path is described by:

x(s)	The horizontal displacement
y(s)	The vertical displacement
ℓ(s)	The displacement in the forward direction, relative to a reference particle

$x' = \frac{dx}{ds} = \frac{P_x}{P_s}$	The horizontal angle ($\tan x' \approx x'$)
$y' = \frac{dy}{ds} = \frac{P_y}{P_s}$	The vertical angle
$\delta = \frac{(P_s - P_0)}{P_0}$	The longitudinal momentum deviation

Reference path: Bending Magnets





$$Br = \frac{P}{q}$$

B – magnetic field

r – radius of curvature

q – particle charge

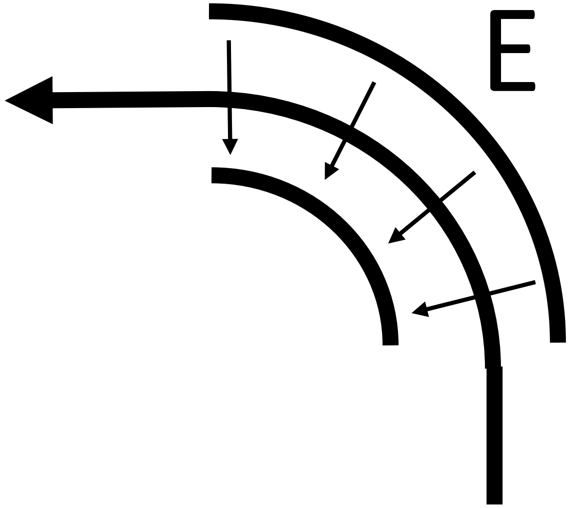
P – relativistic momentum:

$$P = \gamma m_0 \mathrm{v}$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Reference path: Electrostatic Deflectors

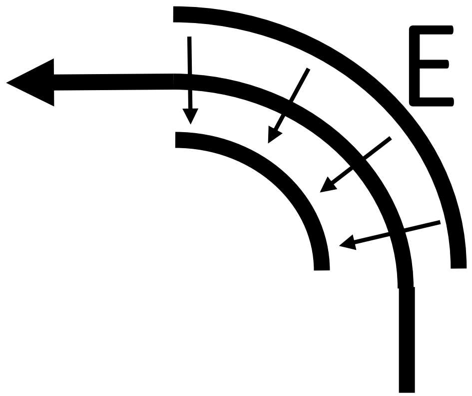
$$\mathrm{E}r = \frac{Pv}{q}$$

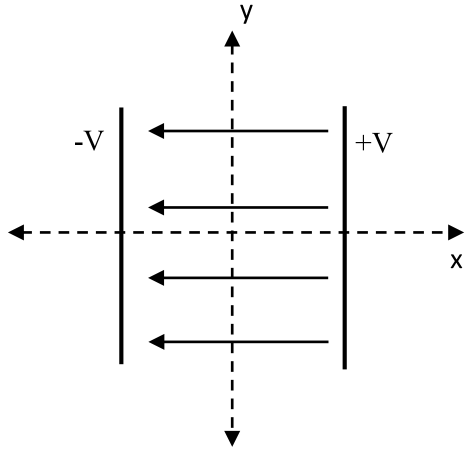


E – Electric field
r – radius of curvature
q – particle charge
v – speed
P – momentum

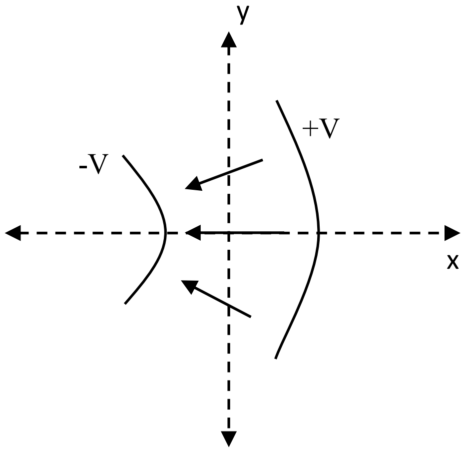
Electrostatic deflectors are only used at low speeds (E scales with Pv while B only scales with P)

Electrostatic Bend





- Electrodes are parts of cylinders
- No vertical forces



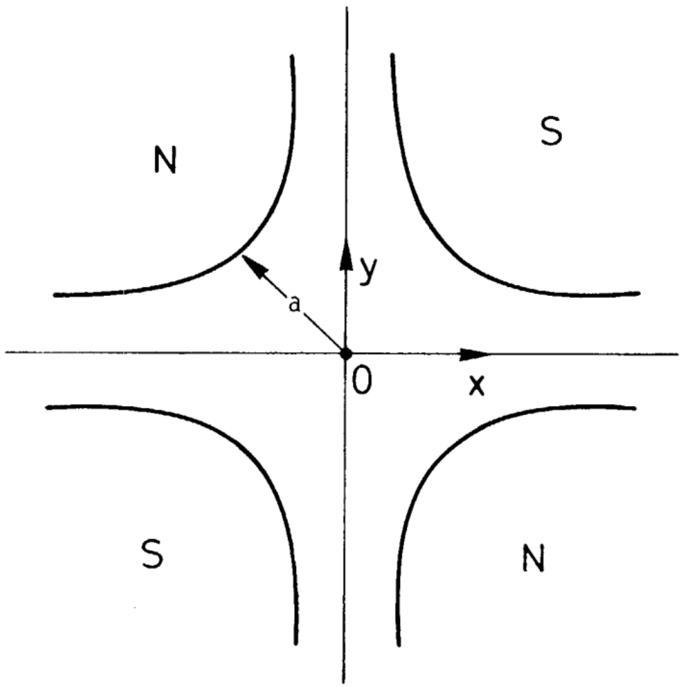
- Electrodes are parts of spheres
- Vertical focussing, horizontal defocussing
- Can make a double focussing device

Focussing described by the field index:

$$n_E = -\frac{r_0}{E_0}\frac{\partial E_x}{\partial x}$$

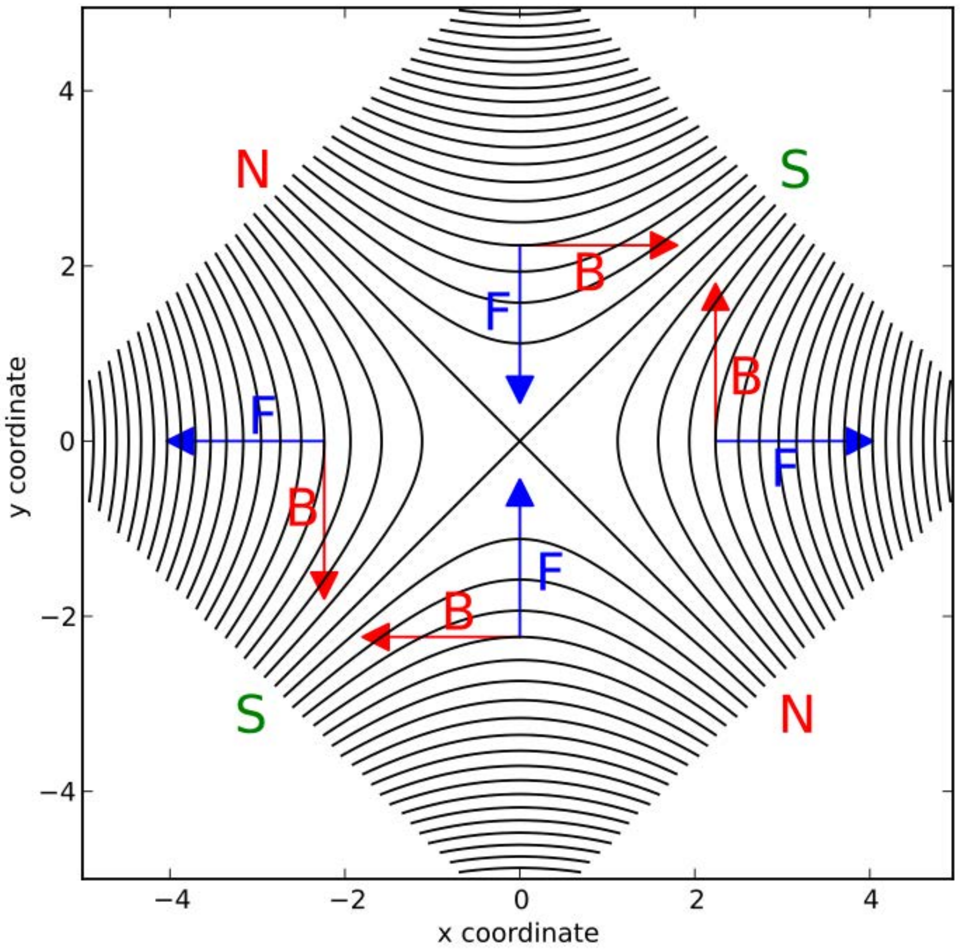
r_0 – bending radius of central path
E_0 – field on central path

Magnetic Quadrupole (1)



QUADRUPOLE

Aperture of quadrupole = 2a



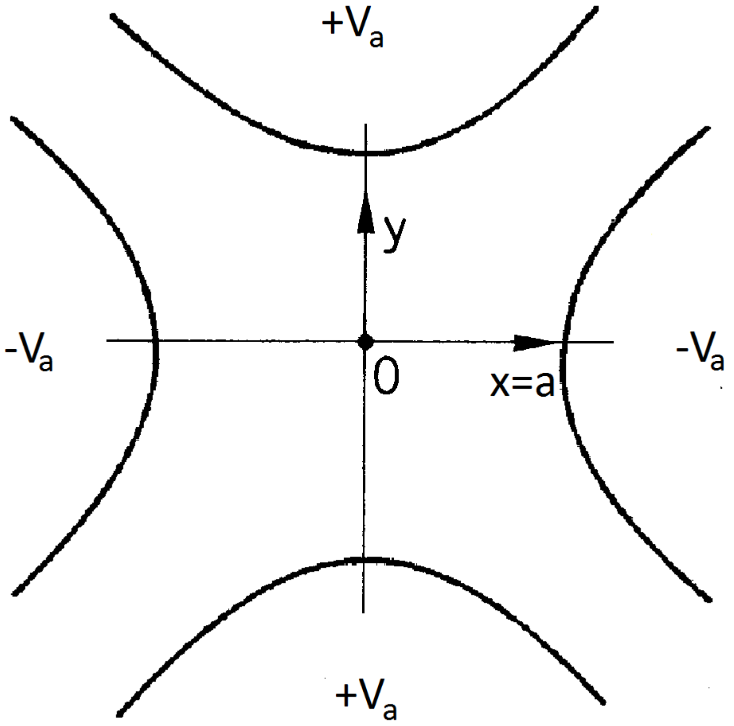
- A quadrupole focusses in one direction (y in picture) and defocusses in the other direction (x in picture)

- The magnetic field is given by:

$$B_x = \frac{B_a}{a} y \qquad B_y = \frac{B_a}{a} x$$

- Where a is the pole radius, and B_a the field strength at the pole tip

Electrostatic Quadrupole



$$E_x = \frac{2V_a}{a^2} x$$

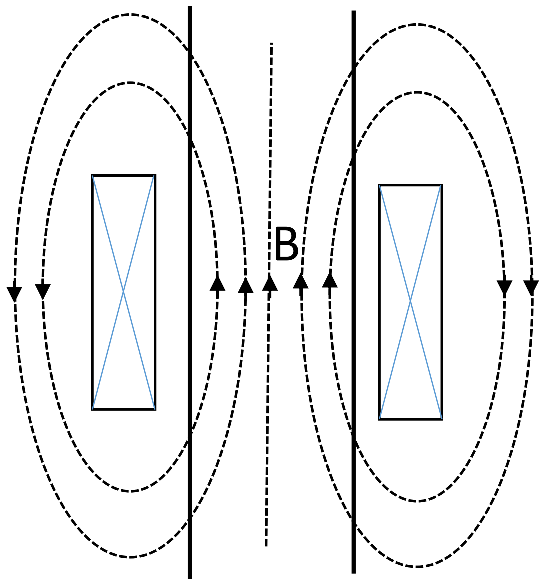
$$E_y = -\frac{2V_a}{a^2} y$$

- Poles faces are rotated 45 degrees compared to magnetic quadrupole
- As drawn in picture: focusses vertically, defocusses horizontally
- Transfer matrix: same as magnetic quadrupole, but:

$$k = \sqrt{\frac{q}{Pv} \frac{2V_a}{a^2}}$$

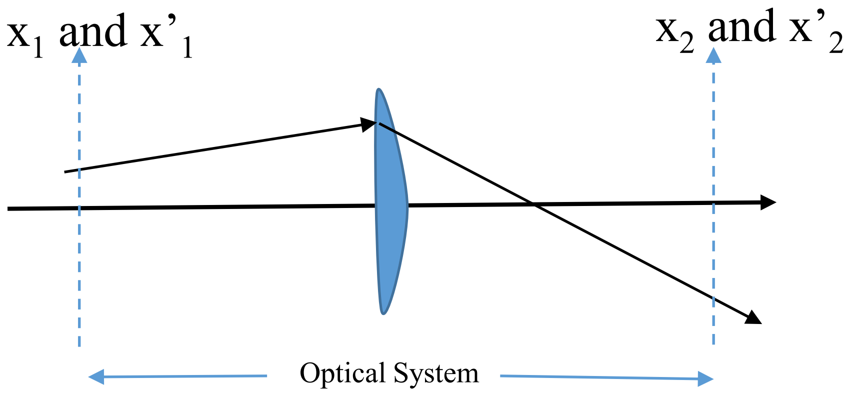
P – momentum, v – speed, q – charge

Solenoid Lens



- Magnetic field parallel to beam pipe
- Focusses in both x and y
- Rotates the beam in (x,y) plane
- Action depends on B and effective length L

Transfer function



$$\boldsymbol{x} = \begin{bmatrix} x \\ x' \\ y \\ y' \\ \ell \\ \delta \end{bmatrix}$$

- For any optical system, the deviation of a particle at the exit ($\mathbf{x_2}$) is a function of the deviation at the entrance ($\mathbf{x_1}$):

$$\boldsymbol{x}_2 = T(\boldsymbol{x}_1)$$

- Where T is the transfer function
- If the deviation is small, then we can approximate this linearly:

$$\boldsymbol{x}_2 = R\boldsymbol{x}_1$$

- Where the transfer matrix R is a 6x6 matrix

Transfer Matrix R

$$\begin{bmatrix} x \\ x' \\ y \\ y' \\ \ell \\ \delta \end{bmatrix}_2 = \mathrm{R} \begin{bmatrix} x \\ x' \\ y \\ y' \\ \ell \\ \delta \end{bmatrix}_1$$

General Transfer Matrix for a magnetic system with mid-plane symmetry

$$\mathrm{R} = \begin{bmatrix} R_{11} & R_{12} & 0 & 0 & 0 & R_{16} \\ R_{21} & R_{22} & 0 & 0 & 0 & R_{26} \\ 0 & 0 & R_{33} & R_{34} & 0 & 0 \\ 0 & 0 & R_{43} & R_{44} & 0 & 0 \\ R_{51} & R_{52} & 0 & 0 & 1 & R_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$x_2 = R_{11}x_1 + R_{12}x_1' + R_{16}\delta_1$$

$$x'_2 = R_{21}x_1 + R_{22}x_1' + R_{26}\delta_1$$

$$y_2 = R_{33}y_1 + R_{34}y_1'$$

$$y'_2 = R_{43}y_1 + R_{44}y_1'$$

Horizontal and Vertical Transfer Matrix

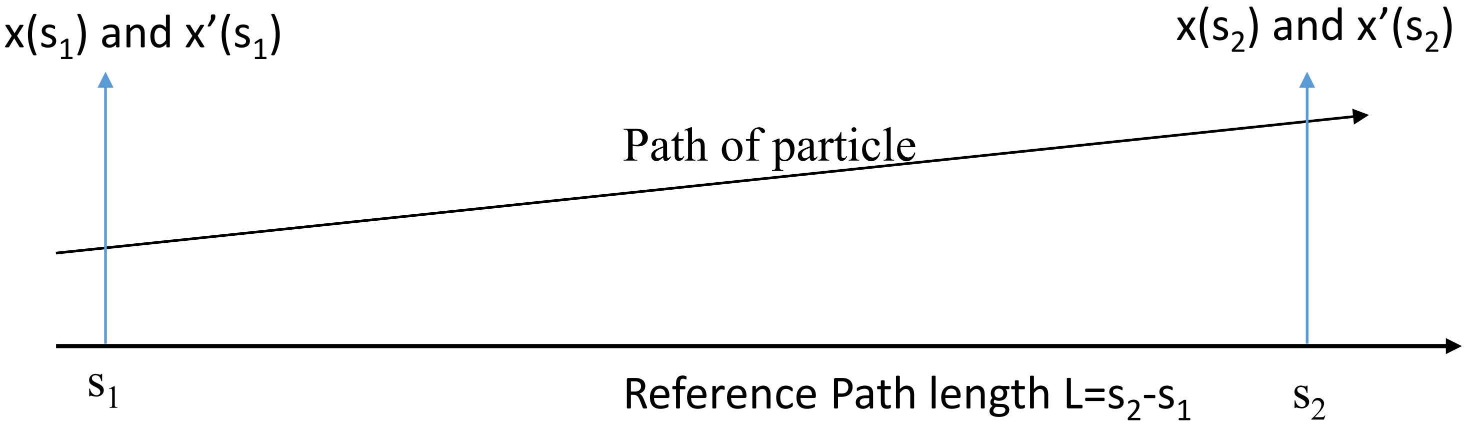
$$R = \begin{bmatrix} R_{11} & R_{12} & 0 & 0 & 0 & R_{16} \\ R_{21} & R_{22} & 0 & 0 & 0 & R_{26} \\ 0 & 0 & R_{33} & R_{34} & 0 & 0 \\ 0 & 0 & R_{43} & R_{44} & 0 & 0 \\ R_{51} & R_{52} & 0 & 0 & 1 & R_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

In systems with mid-plane symmetry, the horizontal and vertical motions are decoupled. So they can be expressed separately:

$$\begin{bmatrix} x \\ x' \\ \delta \end{bmatrix}_2 = R_H \begin{bmatrix} x \\ x' \\ \delta \end{bmatrix}_1 \qquad \begin{bmatrix} y \\ y' \end{bmatrix}_2 = R_V \begin{bmatrix} y \\ y' \end{bmatrix}_1$$

$$R_H = \begin{bmatrix} R_{11} R_{12} R_{16} \\ R_{21} R_{22} R_{26} \\ 0 \quad 0 \quad 1 \end{bmatrix} \qquad R_V = \begin{bmatrix} R_{33} R_{34} \\ R_{43} R_{44} \end{bmatrix}$$

Example: Drift Space



Straight line motion:

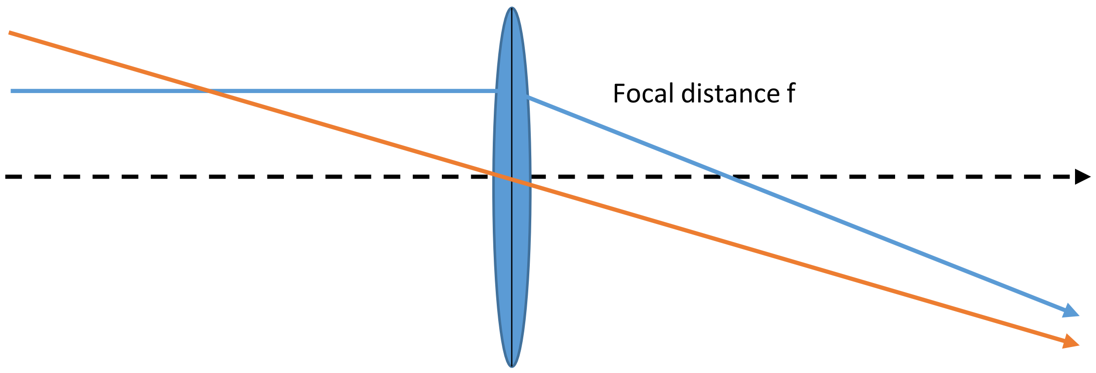
$$x(s_2) = x(s_1) + x'(s_1)L$$

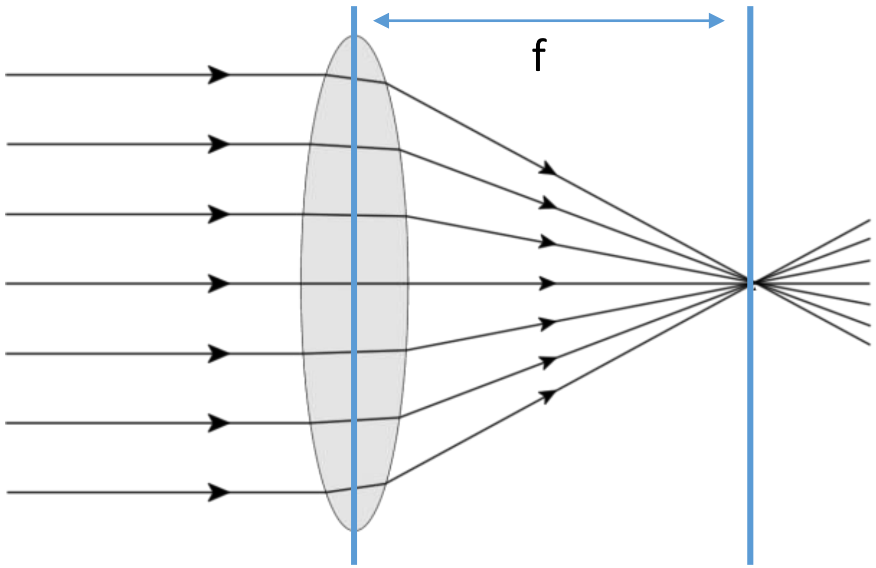
$$x'(s_2) = x'(s_1)$$

This can be written in a Matrix form:

$$\begin{bmatrix} x_2 \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_1' \end{bmatrix}$$

Example of a transfer matrix: A thin lens





- Thin lens: $x_2 = x_1$
- x=0 path (red): $x_2' = x_1{}'$
- x'=0 path (blue): $x_2' = -\frac{x_1}{f}$

This means that the transfer matrix for the lens is:

$$\begin{bmatrix} x_2 \\ x_2{}' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_1' \end{bmatrix}$$

Magnetic Quadrupole (2)

- When focussing vertically:

$$\begin{bmatrix} x_2 \\ x_2{}' \end{bmatrix} = \begin{bmatrix} \cosh kL & k^{-1} \sinh kL \\ k \sin kL & \cosh kL \end{bmatrix} \begin{bmatrix} x_1 \\ x_1' \end{bmatrix}$$

$$\begin{bmatrix} y_2 \\ y_2' \end{bmatrix} = \begin{bmatrix} \cos kL & k^{-1} \sin kL \\ -k \sin kL & \cos kL \end{bmatrix} \begin{bmatrix} y_1 \\ y_1' \end{bmatrix}$$

- When focussing horizontally:

$$\begin{bmatrix} x_2 \\ x_2{}' \end{bmatrix} = \begin{bmatrix} \cos kL & k^{-1} \sin kL \\ -k \sin kL & \cos kL \end{bmatrix} \begin{bmatrix} x_1 \\ x_1' \end{bmatrix}$$

$$\begin{bmatrix} y_2 \\ y_2' \end{bmatrix} = \begin{bmatrix} \cosh kL & k^{-1} \sinh kL \\ k \sin kL & \cosh kL \end{bmatrix} \begin{bmatrix} y_1 \\ y_1' \end{bmatrix}$$

Where:

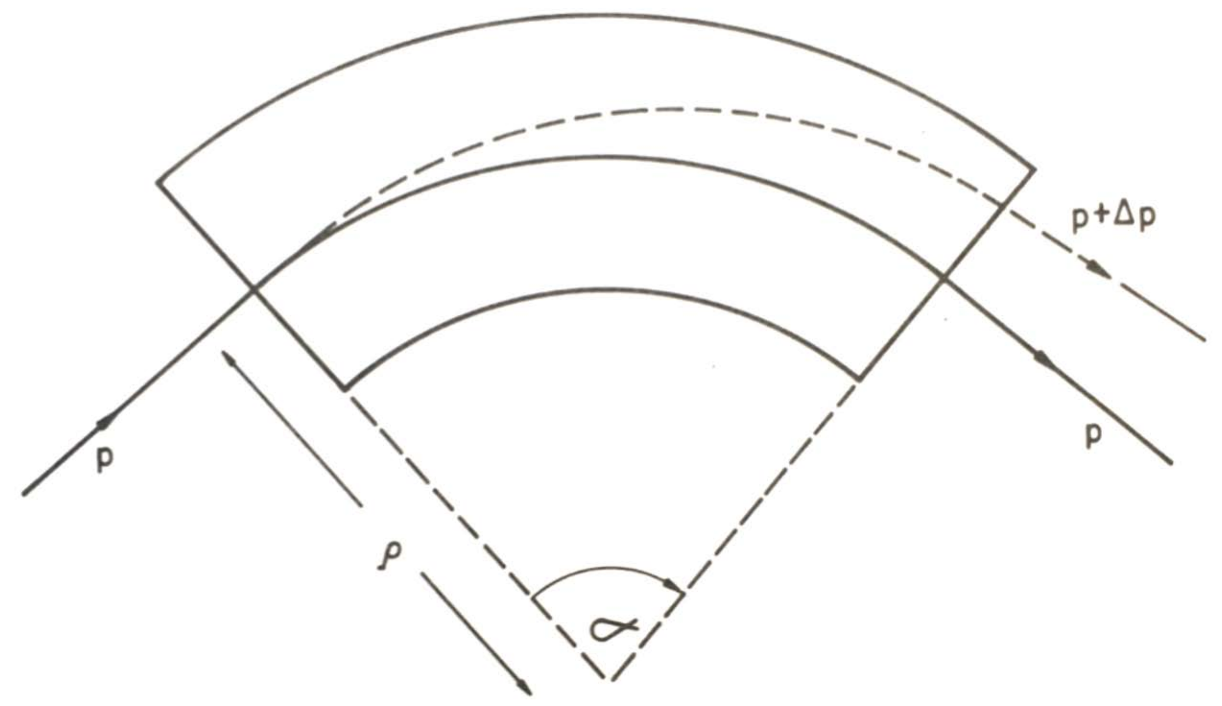
L is the equivalent path length through the quadrupole

$$k = \sqrt{\frac{B_a}{a} \frac{q}{P}}$$

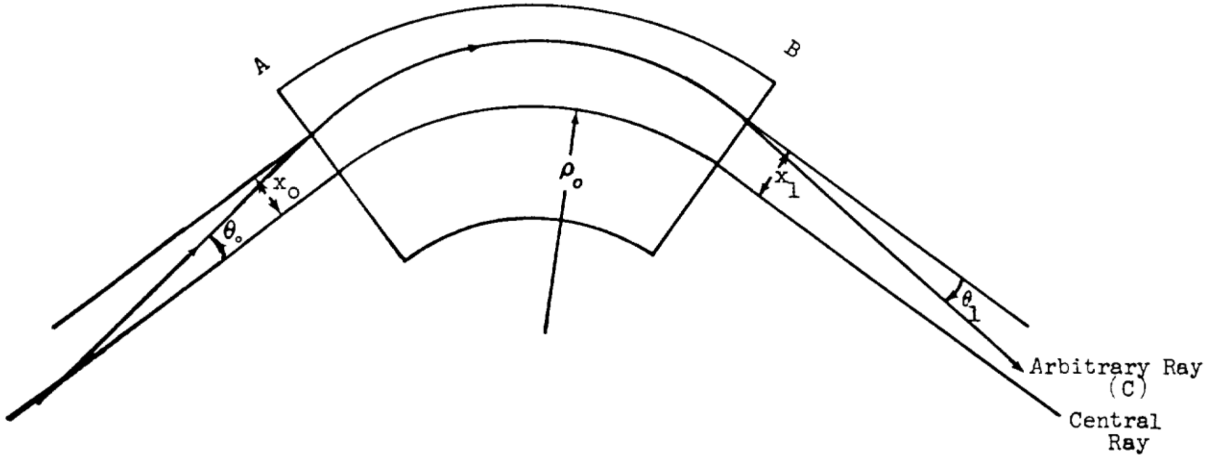
P – momentum, q – charge

Bending magnet: Dispersion

Dipole with entry and exit pole face rotation at right angle to the beam with momentum p. The path of the dispersed ray with momentum p + Δp is also shown



Bending Magnet (1)



- Dipole field
- Only produces horizontal forces
- Shape described by:

  r – radius of curvature

  L - path length along central trajectory

  $k_x = 1/r$

$$\begin{bmatrix} y \\ y' \end{bmatrix}_2 = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix}_1$$

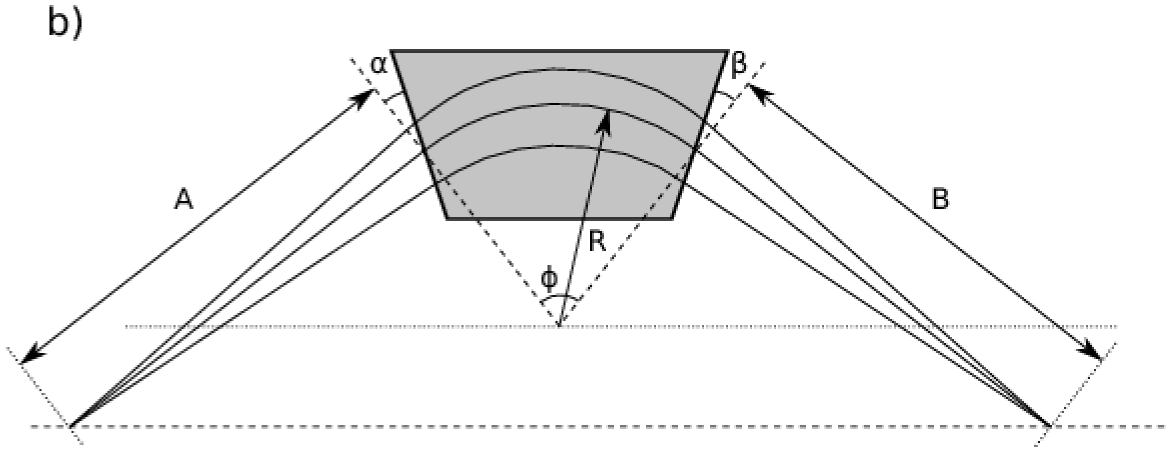
$$\begin{bmatrix} x \\ x' \\ \delta \end{bmatrix}_2 = \begin{bmatrix} \cos k_x L & k_x^{-1} \sin k_x L & k_x^{-1}(1 - \cos k_x L) \\ -k_x \sin k_x L & \cos k_x L & \sin k_x L \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ x' \\ \delta \end{bmatrix}_1$$

Vertically: just a drift space

Horizontal: depends on the momentum δ

This is called chromatic dispersion

Bending magnet: Edge focussing



From: *Beam Extraction and Transport*, T.Kalvas

$$\begin{bmatrix} x_2 \\ x_2{}' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ r^{-1}\tan\beta & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_1' \end{bmatrix}$$

$$\begin{bmatrix} y_2 \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -r^{-1}\tan\beta & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_1' \end{bmatrix}$$

- The entrance and exit edge are not always perpendicular to the beam path (angles α and β in the picture)
- This produces quadrupoles in the fringe field regions
- Usually used to focus vertically and defocus horizontally, to produce double-focussing bending magnets
- For double focussing:
  see next slide

Bending magnet: Double focussing

Edge angles may be chosen to bring both the horizontal and vertical point-rays to a focus at the same point. The dipole is then said to be “double-focussing. A pair of identical edge-angles is often used, to form a symmetrical dipole. The double focussing edge-angle Θ_1 and Θ_2 may then be calculated from the equation

$$\tan \Theta_1 = \tan \Theta_2 = ½ \tan \alpha/2$$

And the image and object distance, *D*, from the magnet

$$D = \frac{2\rho}{tan\, \alpha/2}$$

With α the bending angle and ρ the bending radius of the magnet

Combining Optical Components

- Start with a particle $\begin{bmatrix} x_1 \\ x_1' \end{bmatrix}$

- After the first optical component the particle is at: $\begin{bmatrix} x_2 \\ x_2' \end{bmatrix} = M_1 \begin{bmatrix} x_1 \\ x_1' \end{bmatrix}$

- After the next optical component the particle is at: $\begin{bmatrix} x_3 \\ x_3' \end{bmatrix} = M_2 \begin{bmatrix} x_2 \\ x_2' \end{bmatrix} = M_2 M_1 \begin{bmatrix} x_1 \\ x_1' \end{bmatrix}$

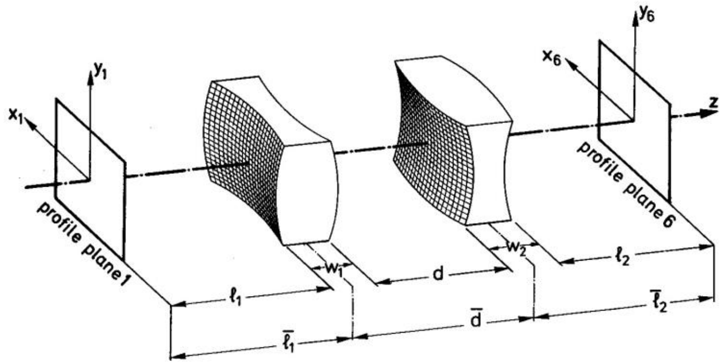
So: $M_{1\ then\ 2} = M_2 M_1$ (order is important)

- So, the transfer matrices of the individual components can be multiplied together to get the total transfer matrix of the entire system. For N components:

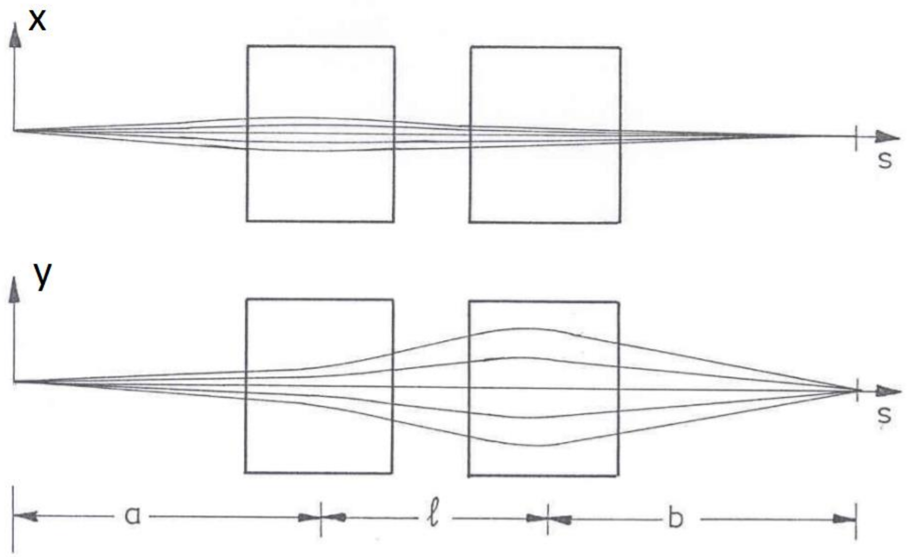
  - $M_{total} = M_n M_{n-1} \ldots M_2 M_1$

Quadrupole Doublet

- A single quadrupole cannot focus in both x and y
- But combining two quadrupoles with opposite polarity can
- Known as strong focussing, or alternating gradient focussing



Taken from: An introduction to ion optics, Georg P Berg



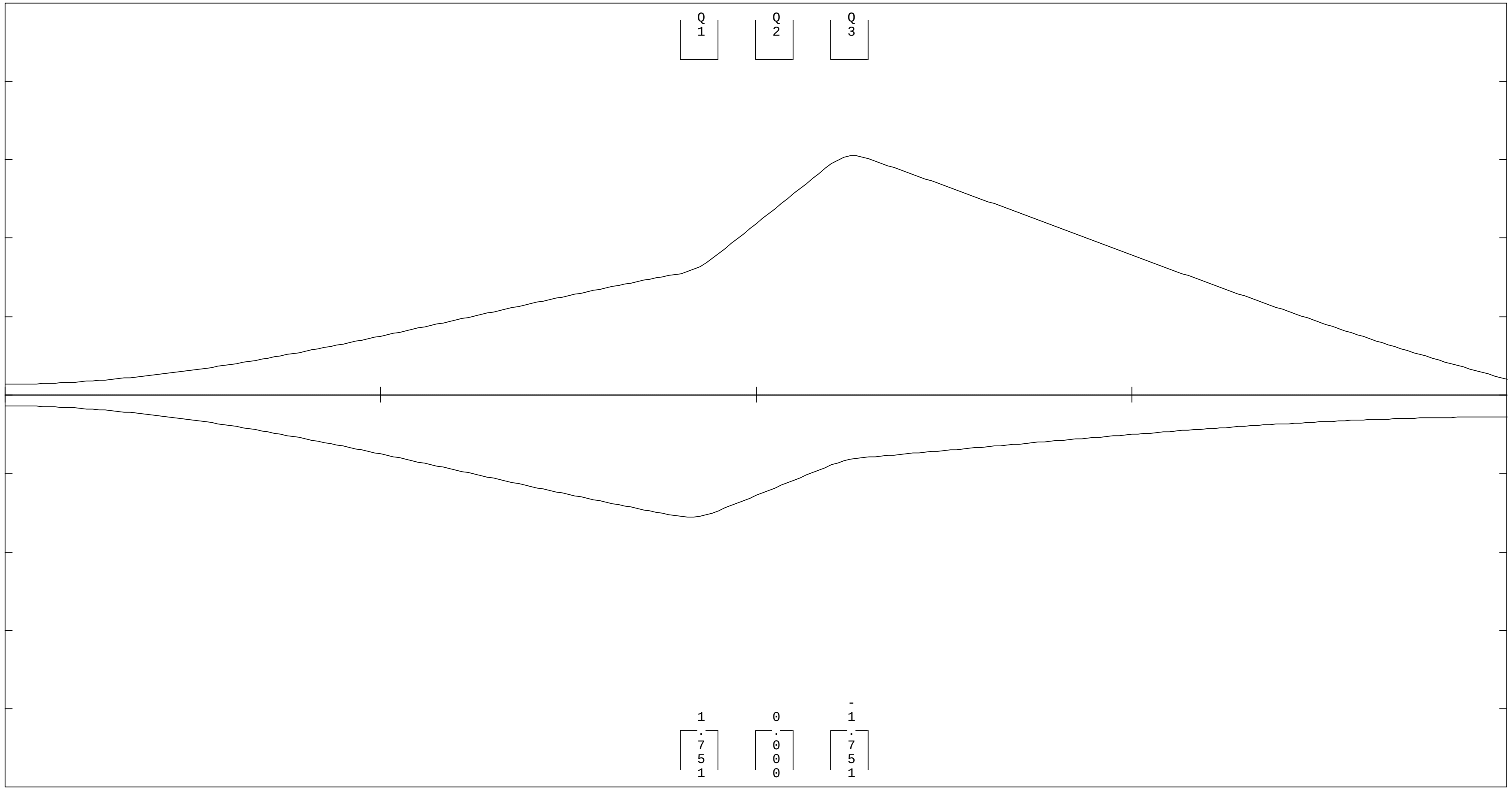
In x direction:

First quadrupole focusses

Second quadrupole defocusses

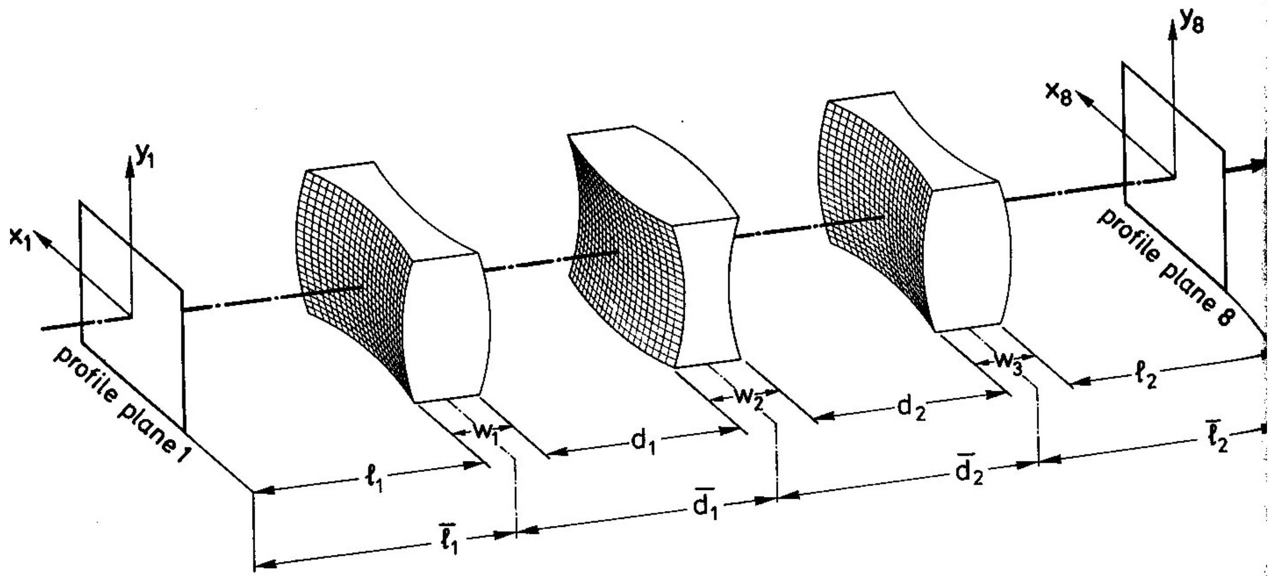
In y direction: The opposite

Beam profile for a quadrupole doublet



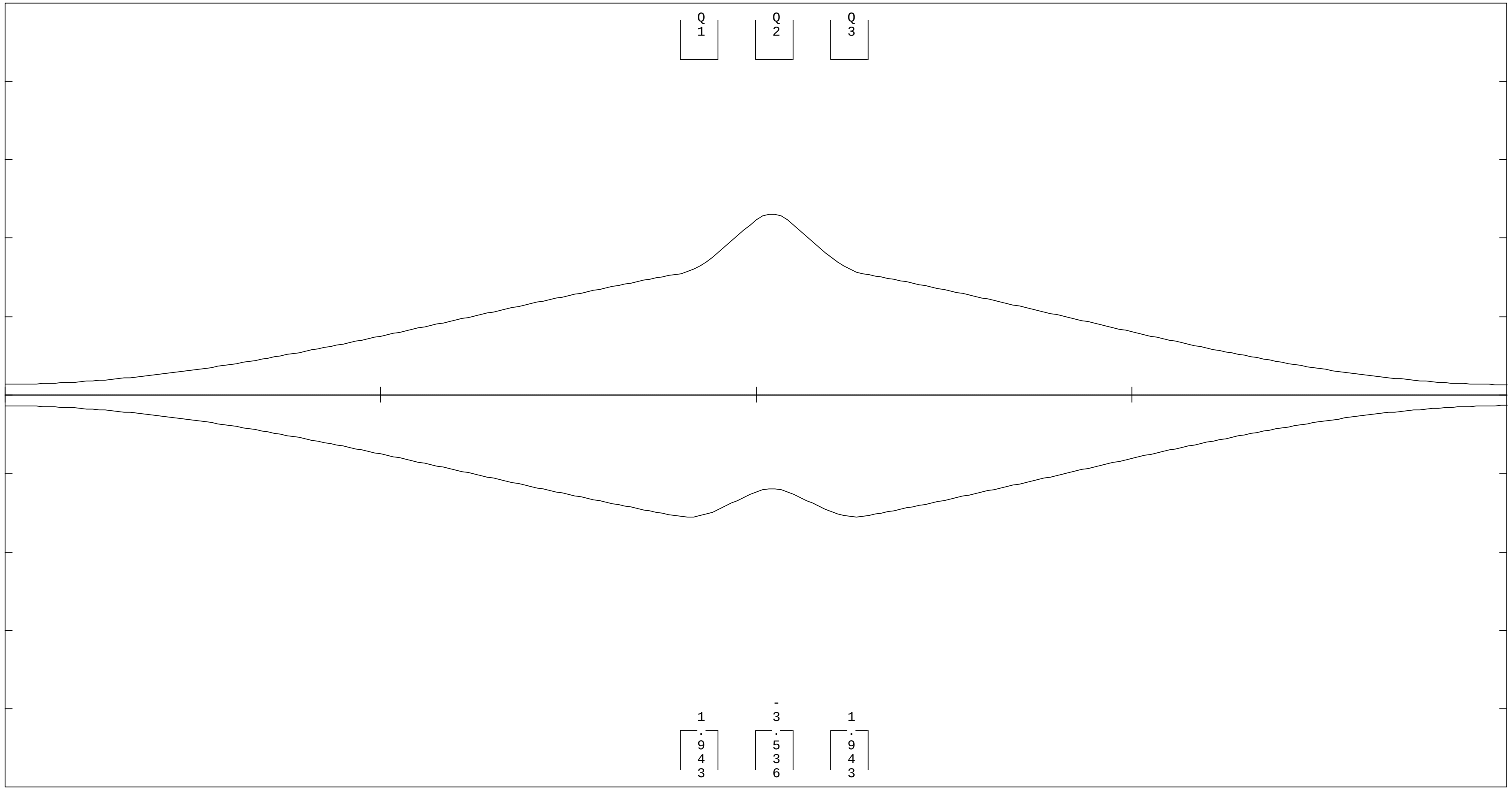
Quadrupole Triplet

- Similar to a quadrupole doublet, but it is symmetric
- The middle lens has the opposite polarity to the others lenses
- The beam is not as wide as in a doublet

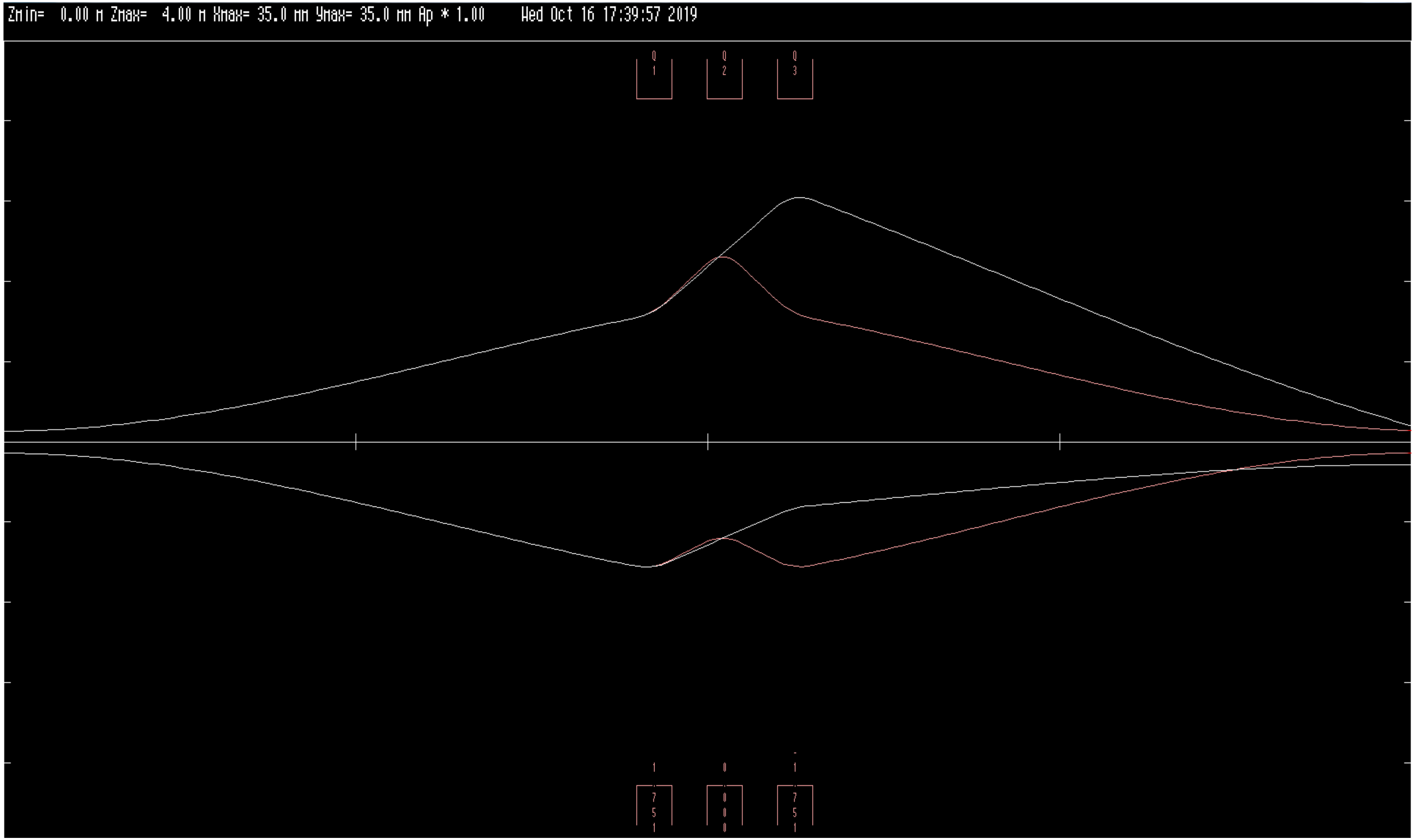


Taken from: An introduction to ion optics, Georg P Berg

Beam profile for symmetric quadrupole triplet

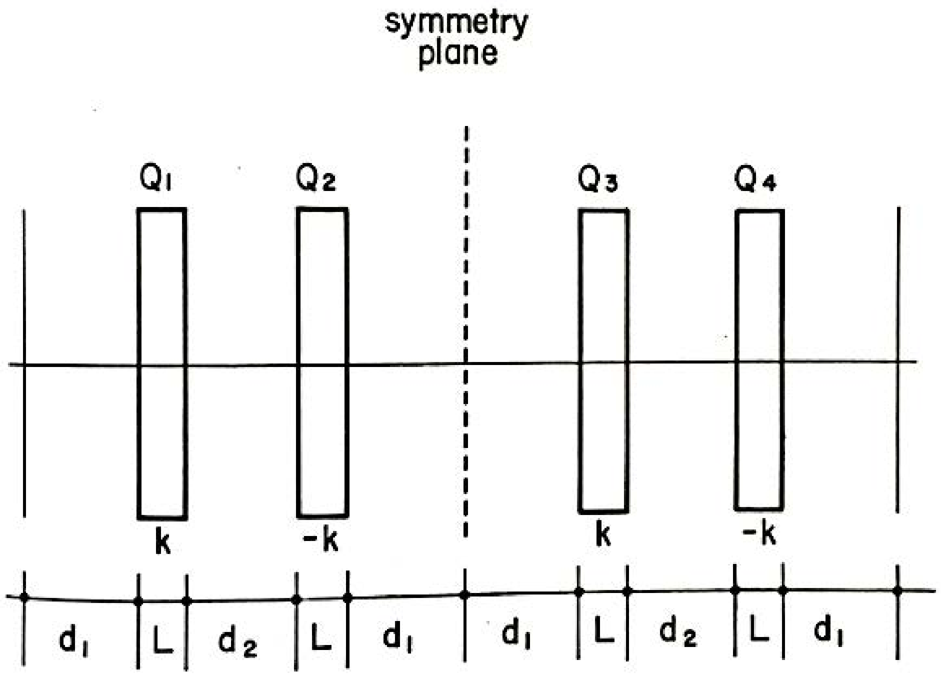


Difference of beam profile for a quadrupole doublet and quadrupole triplet

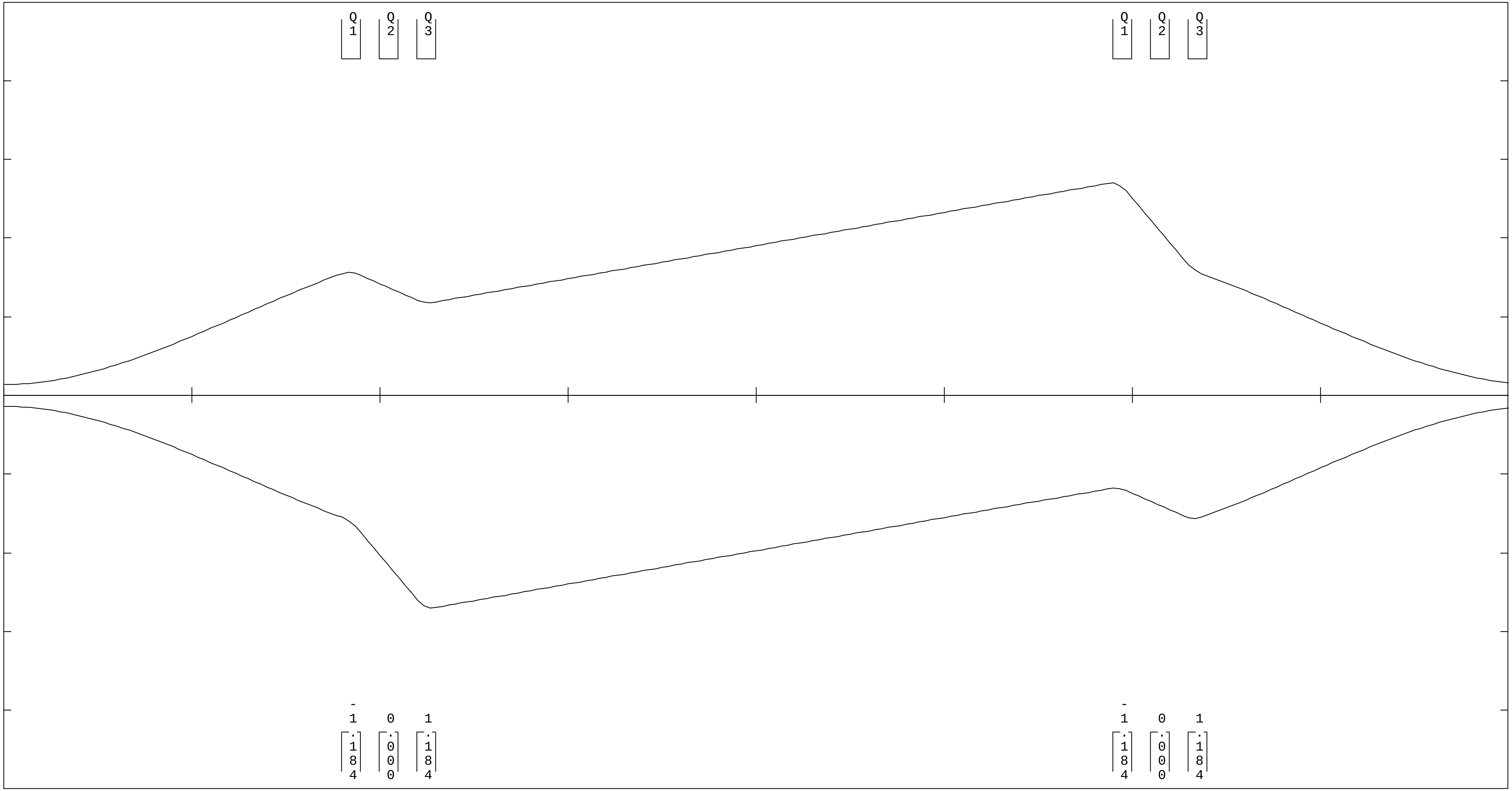


Combining Quadrupoles: Telescopic Doublet

A symmetric two doublet system

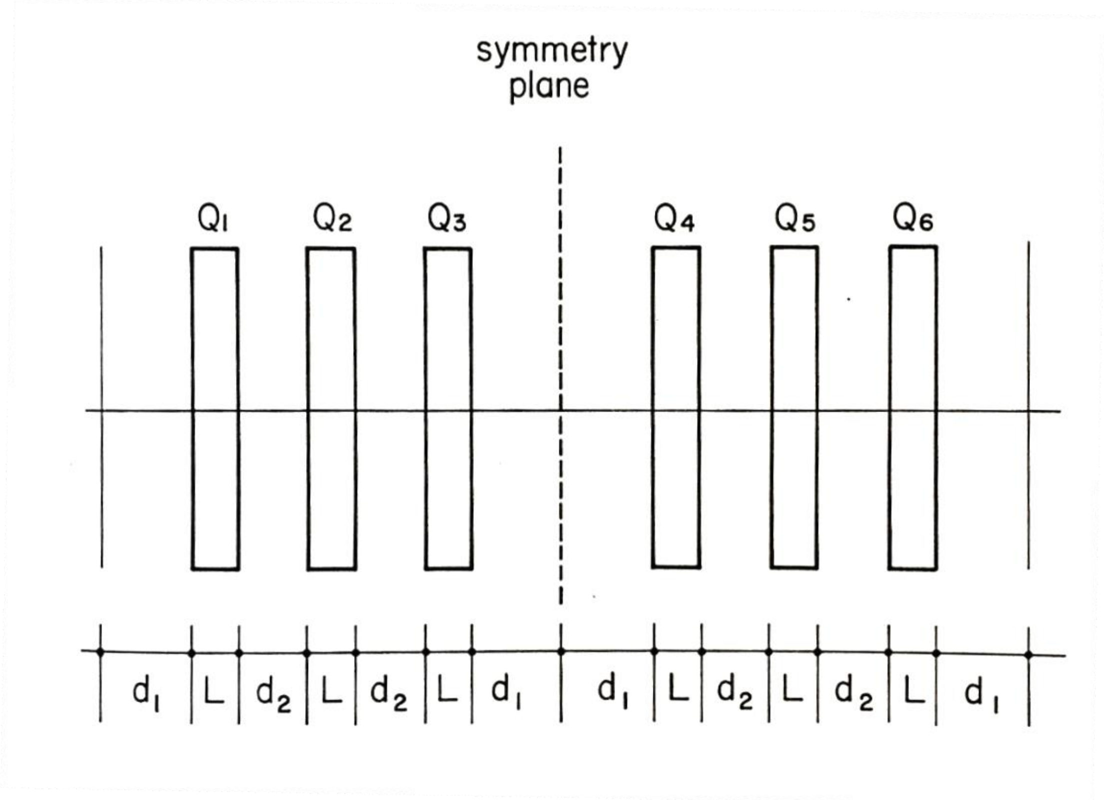


A two doublet quadrupole telescope

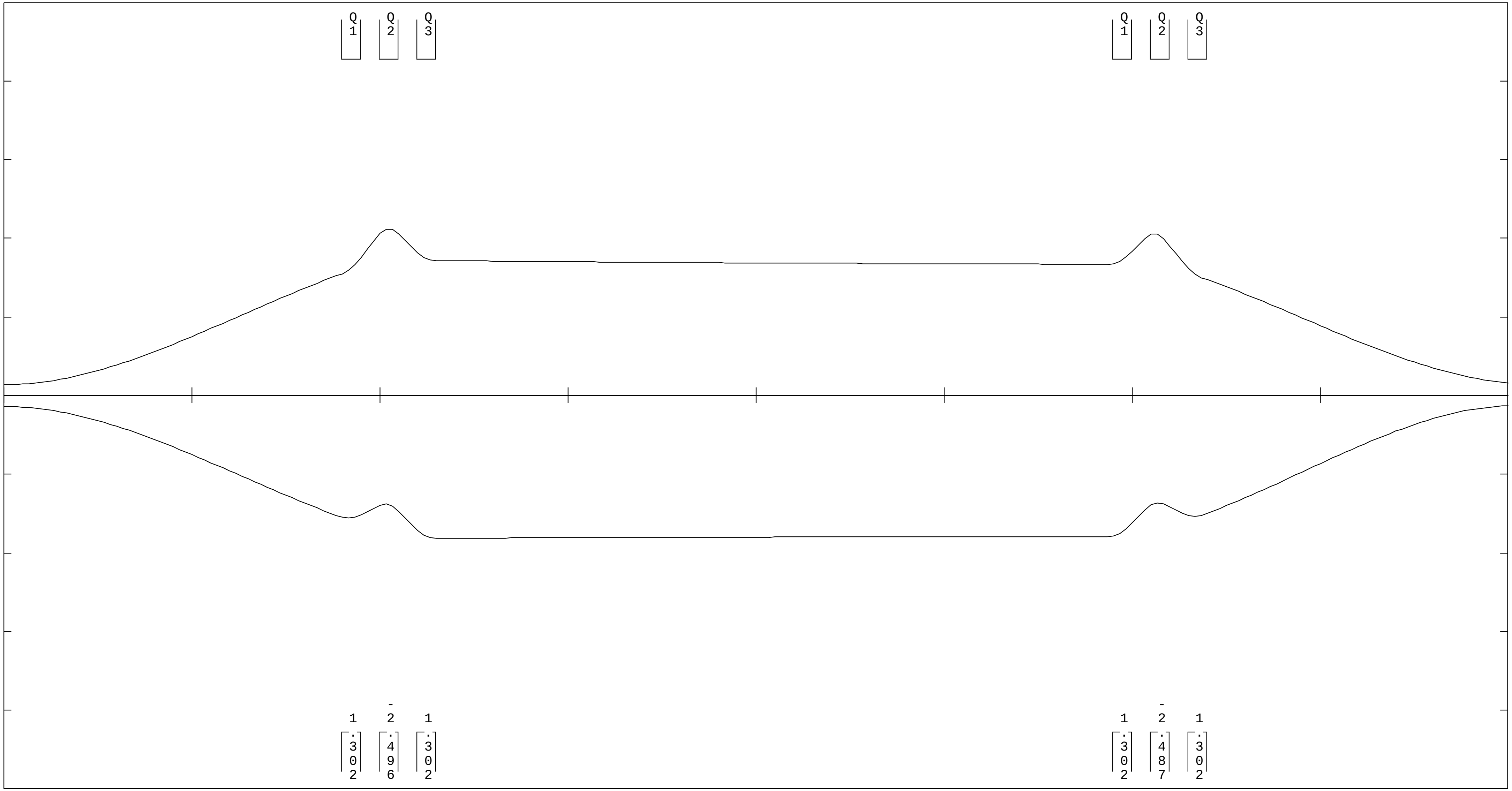


Combining Quadrupoles: Telescopic Triplet

A symmetric two triplet system

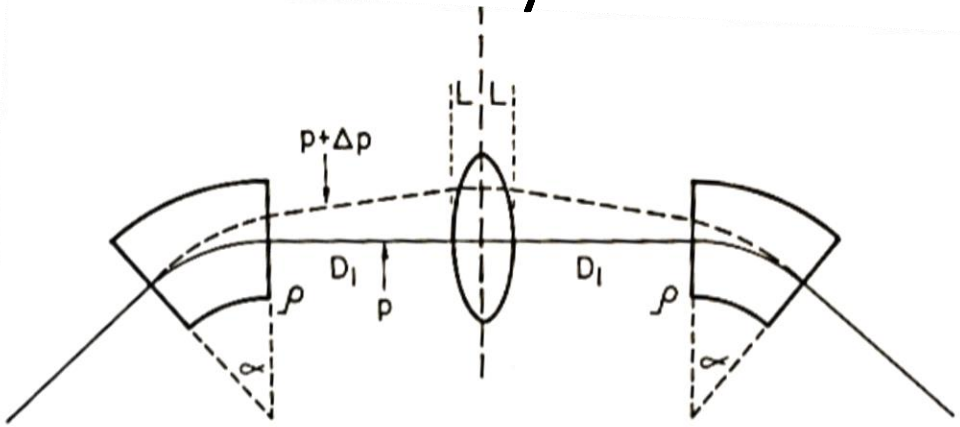


A two triplet quadrupole telescope

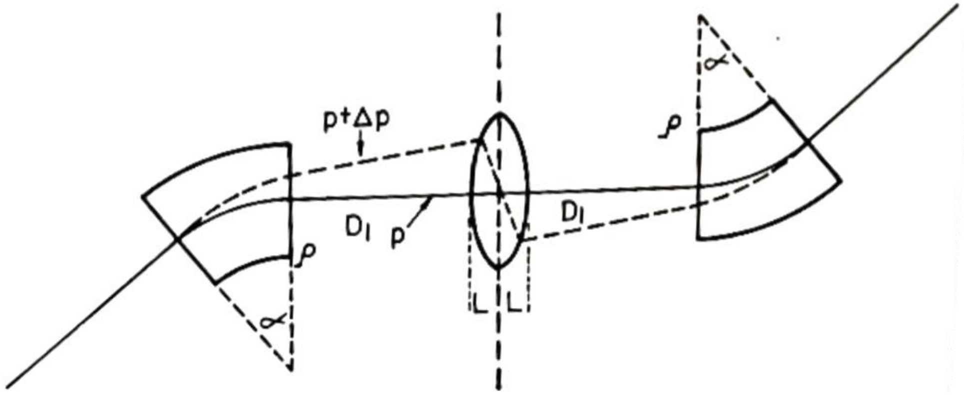


Achromatic system with one quadrupole between the dipoles with a mirror and anti-mirror symmetrical systems

Mirror symmetry

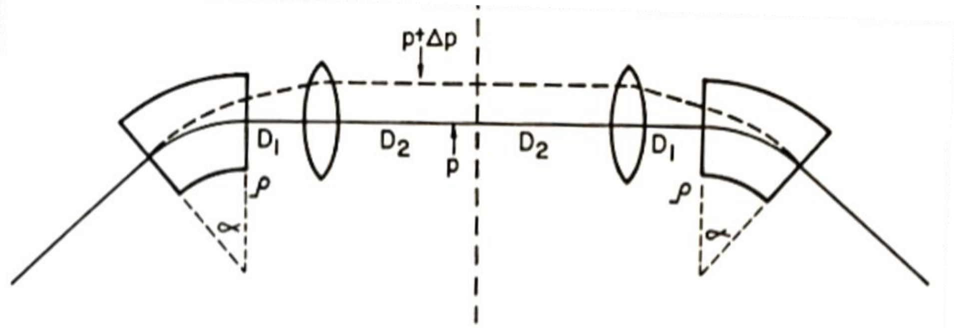


Anti-mirror symmetry

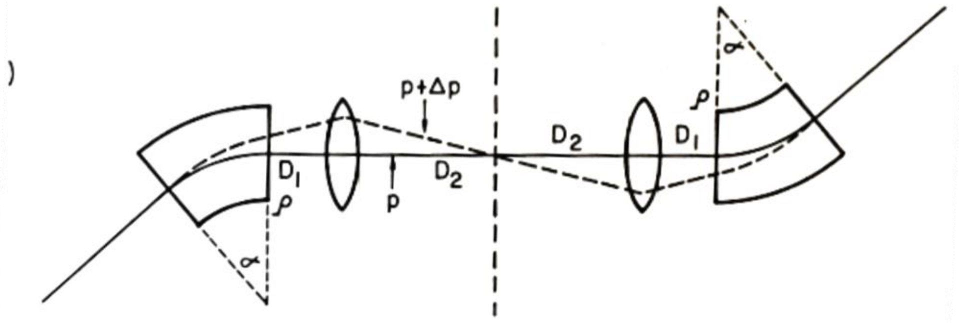


Achromatic system with two quadrupole between the dipoles with a mirror and anti-mirror symmetrical systems

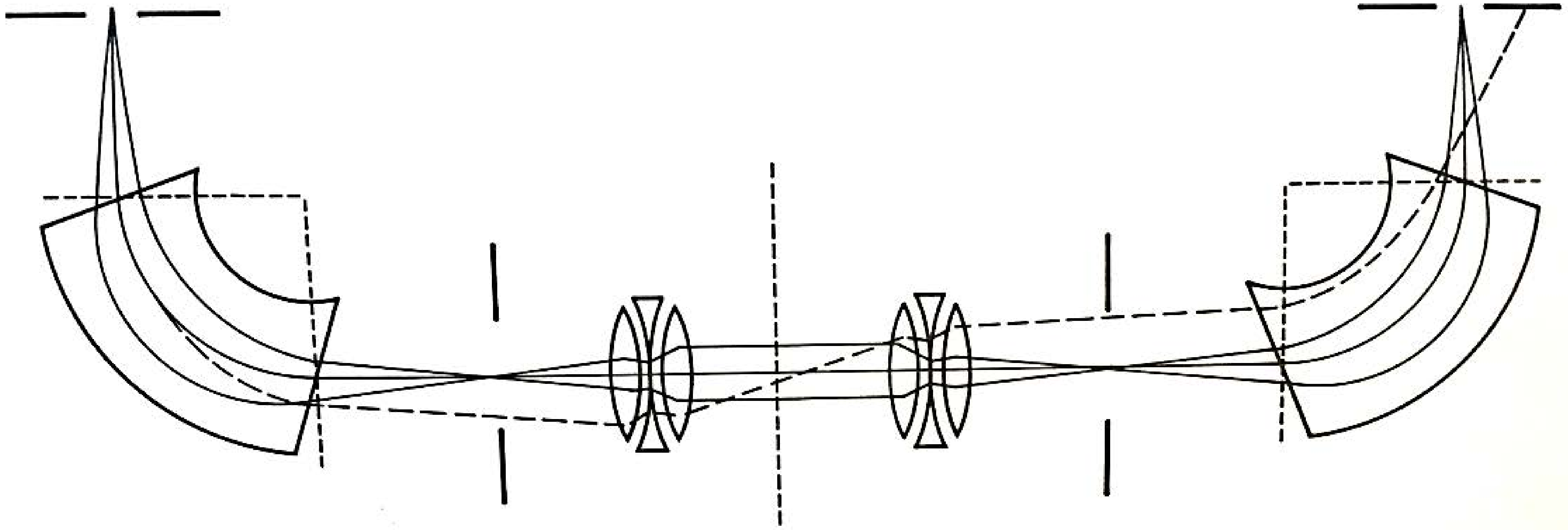
Mirror symmetry



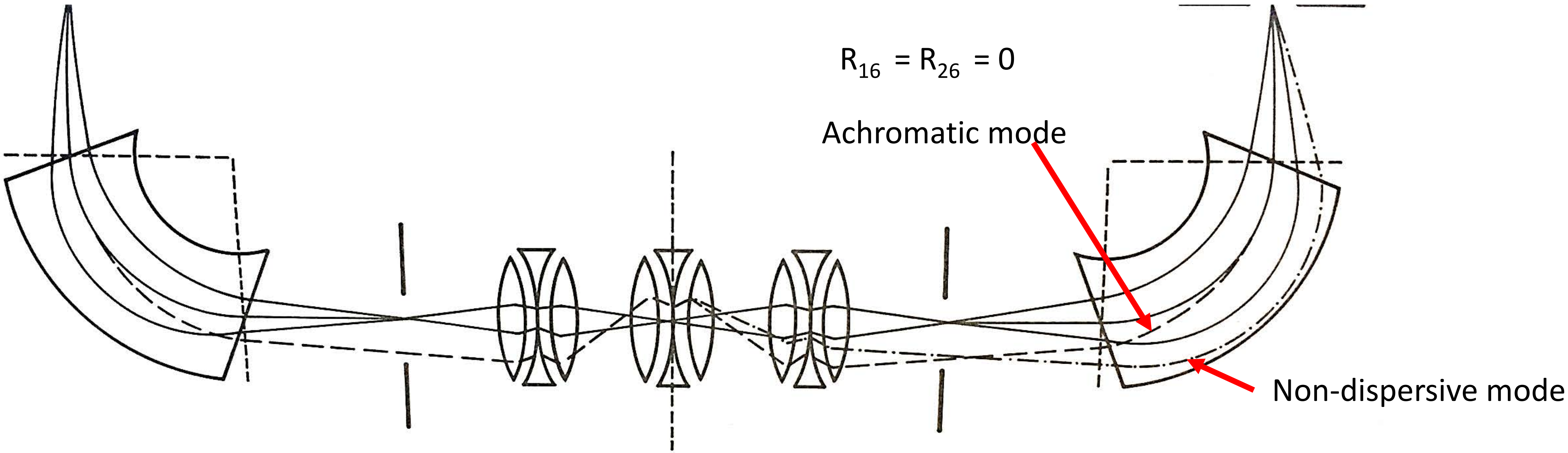
Anti-mirror symmetry



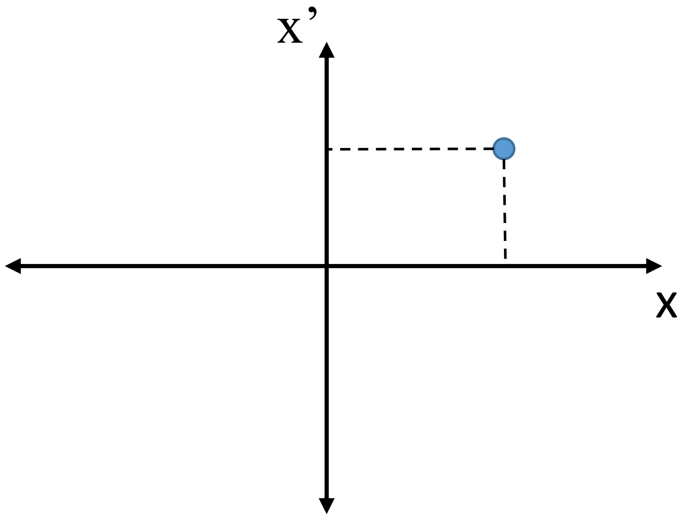
Double dispersive system, showing the horizontal point rays and off-momentum rays (broken line) for double dispersion mode



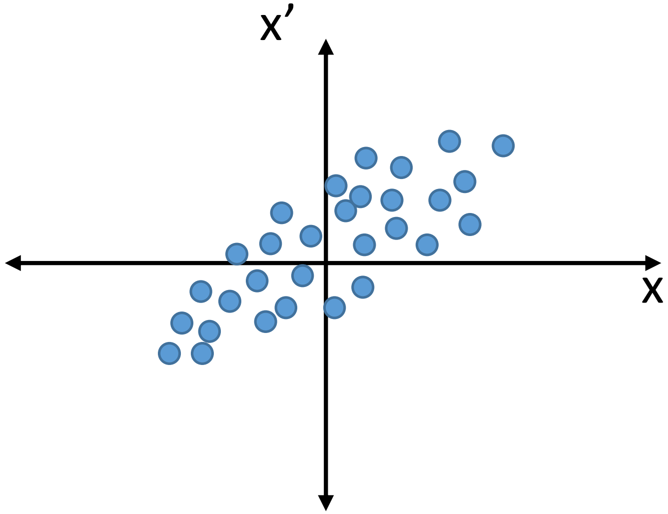
Beam line layout that can produce a achromatic beam and a non-dispersive beam at the end of the beam line. This system can also produce a double dispersive mode. The point rays and off momentum rays are shown.



Describing an entire beam

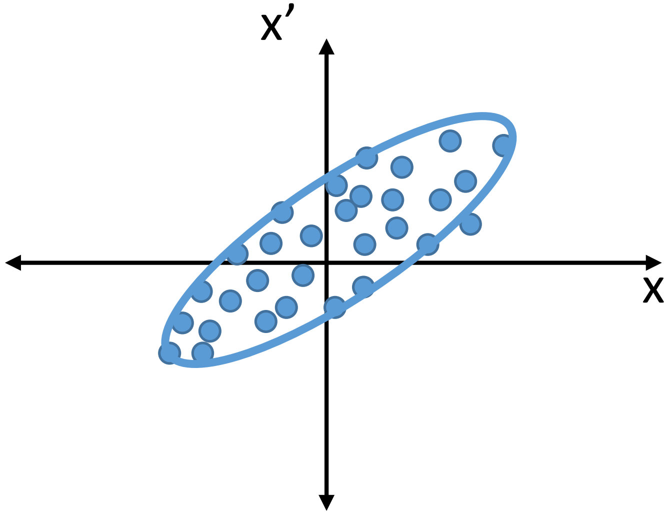


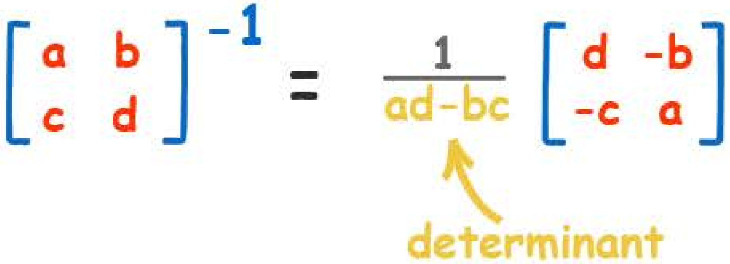
At any point along the path, the state of a particle can be shown on a (x,x’) phase space plot



But the beam consists of many particles

A simple way to describe the beam, is with an ellipse:





Beam ellipse – Sigma Beam Matrix

We describe the beam ellipse with a **sigma beam matrix**

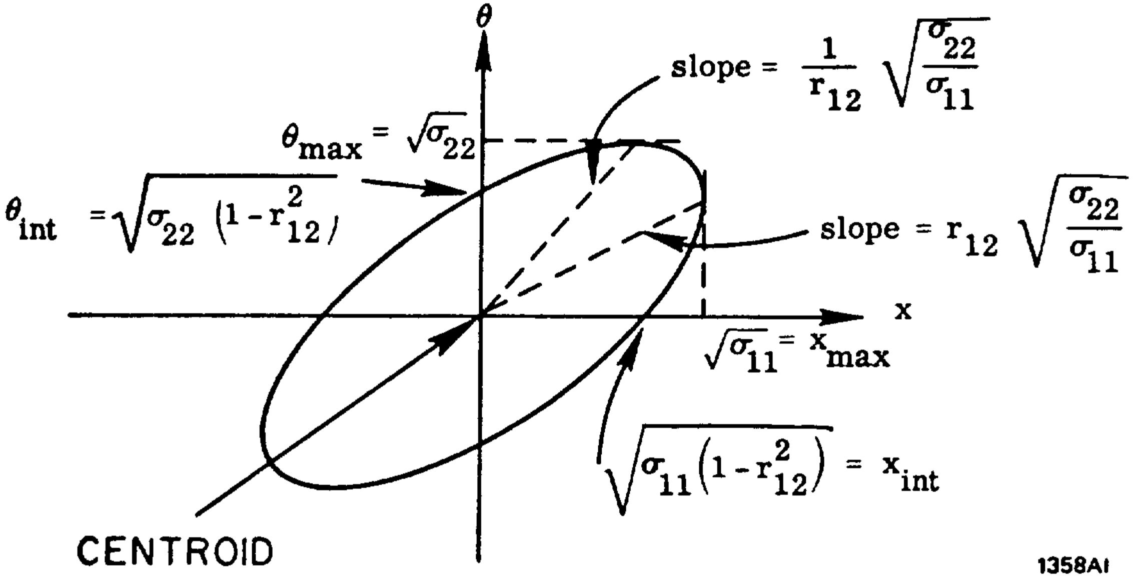
To get the transpose of the X matrix, write the rows of R matrix as the columns of X^T

If we write $\boldsymbol{x} = \begin{bmatrix} x \\ x' \end{bmatrix}$ then the equation for an ellipse in phase space is:

$$\boldsymbol{x}^T \sigma^{-1} \boldsymbol{x} = 1$$

To see that this is true, write $\sigma = \begin{bmatrix} \sigma_{11} \sigma_{12} \\ \sigma_{21} \sigma_{22} \end{bmatrix}$ then $\sigma^{-1} = \frac{1}{\det \sigma} \begin{bmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{21} & \sigma_{11} \end{bmatrix}$ so the equation becomes:

$$\sigma_{22} x^2 - (\sigma_{12} + \sigma_{21}) x x' + \sigma_{11} x'^2 = \det \sigma$$



Which is an ellipse in the (x,x') plane. The area of the ellipse is $A = \pi\sqrt{\det \sigma}$

Skewness expressed by the correlation r_{12}:

$$r_{12} = \frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}$$

How beam sigma matrix and beam transfer matrix change

Ray X_0 from location 0 is transported by a 6 x 6 Matrix R to location 1 by: $X_1 = RX_0$ (1)

Ellipsoid in Matrix notation, generalized to e.g. 6-dim. using σ Matrix: $X_0^{\,T}\sigma_0^{-1}X_0 = 1$ (2)

Inserting Unity Matrix $I = RR^{-1}$ in equation (2) it follows $X_0^{\,T}(R^T R^{T-1})\,\sigma_0^{-1}(R^{-1}R)\,X_0 = 1$

from which we derive $(RX_0)^T\,(R\sigma_0 R^T)^{-1}\,(RX_0) = 1$

The equation of the new ellipsoid after transformation becomes $X_1^{\,T}\sigma_1^{-1}X_1 = 1$

where $\sigma_1 = R\sigma_0 R^T$ (3)

Knowing the TRANSPORT matrix R that transports one ray through an ion-optical system using (1) we can now also transport the phase space ellipse describing the initial beam using (3)

The beam sigma matrix is symmetric so that only a triangle of elements is needed

	x	θ	y	ϕ	ℓ	δ
x	$\sigma(11)$					
θ	$\sigma(21)$	$\sigma(22)$				
y	$\sigma(31)$	$\sigma(32)$	$\sigma(33)$			
θ	$\sigma(41)$	$\sigma(42)$	$\sigma(43)$	$\sigma(44)$		
ℓ	$\sigma(51)$	$\sigma(52)$	$\sigma(53)$	$\sigma(54)$	$\sigma(55)$	
δ	$\sigma(61)$	$\sigma(62)$	$\sigma(63)$	$\sigma(64)$	$\sigma(65)$	$\sigma(66)$

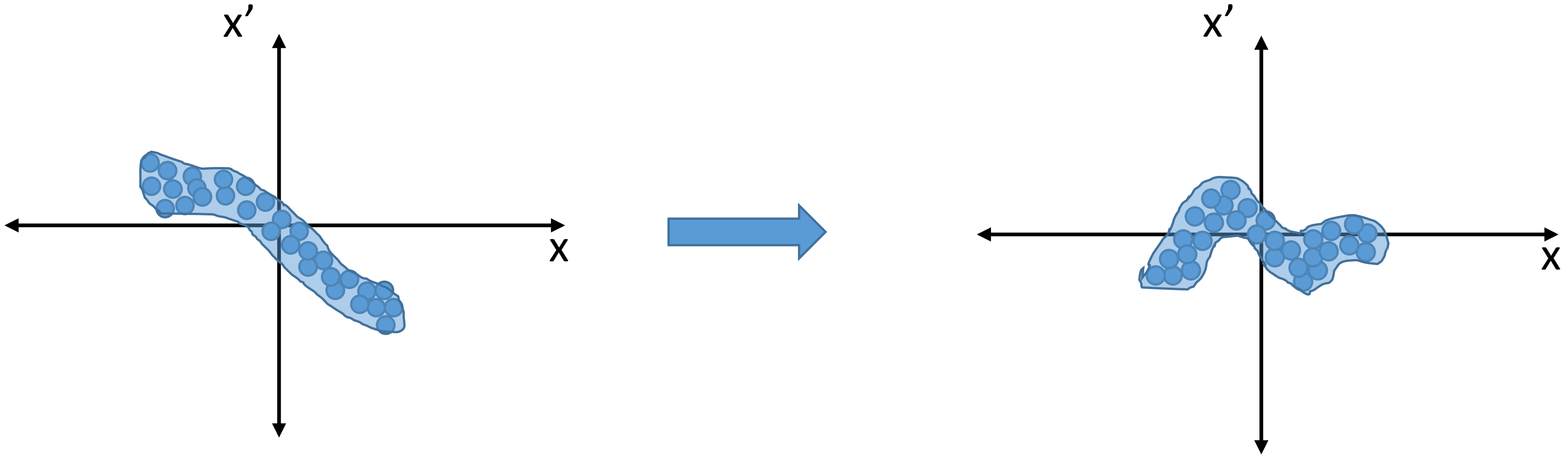
In the program TRANSPORT the printed output of the beam sigma matrix has a different format for ease of interpretation

			x	θ	y	ϕ	ℓ
x	$\sqrt{\sigma}(11)$	CM					
θ	$\sqrt{\sigma}(22)$	MR	$r(21)$				
y	$\sqrt{\sigma}(33)$	CM	$r(31)$	$r(32)$			
ϕ	$\sqrt{\sigma}(44)$	MR	$r(41)$	$r(42)$	$r(43)$		
ℓ	$\sqrt{\sigma}(55)$	CM	$r(51)$	$r(52)$	$r(53)$	$r(54)$	
δ	$\sqrt{\sigma}(66)$	PC	$r(61)$	$r(62)$	$r(63)$	$r(64)$	$r(65)$

where:

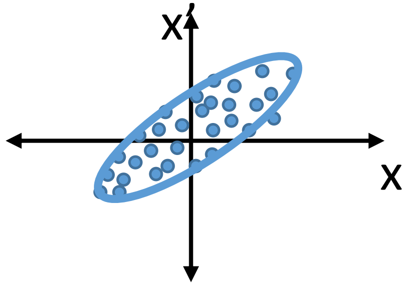
$$r(ij) = \frac{\sigma(ij)}{\left[\sigma(ii)\sigma(jj)\right]^{\frac{1}{2}}} .$$

Emittance



From Hamiltonian mechanics (Liouville's theorem): the area of phase space is conserved.

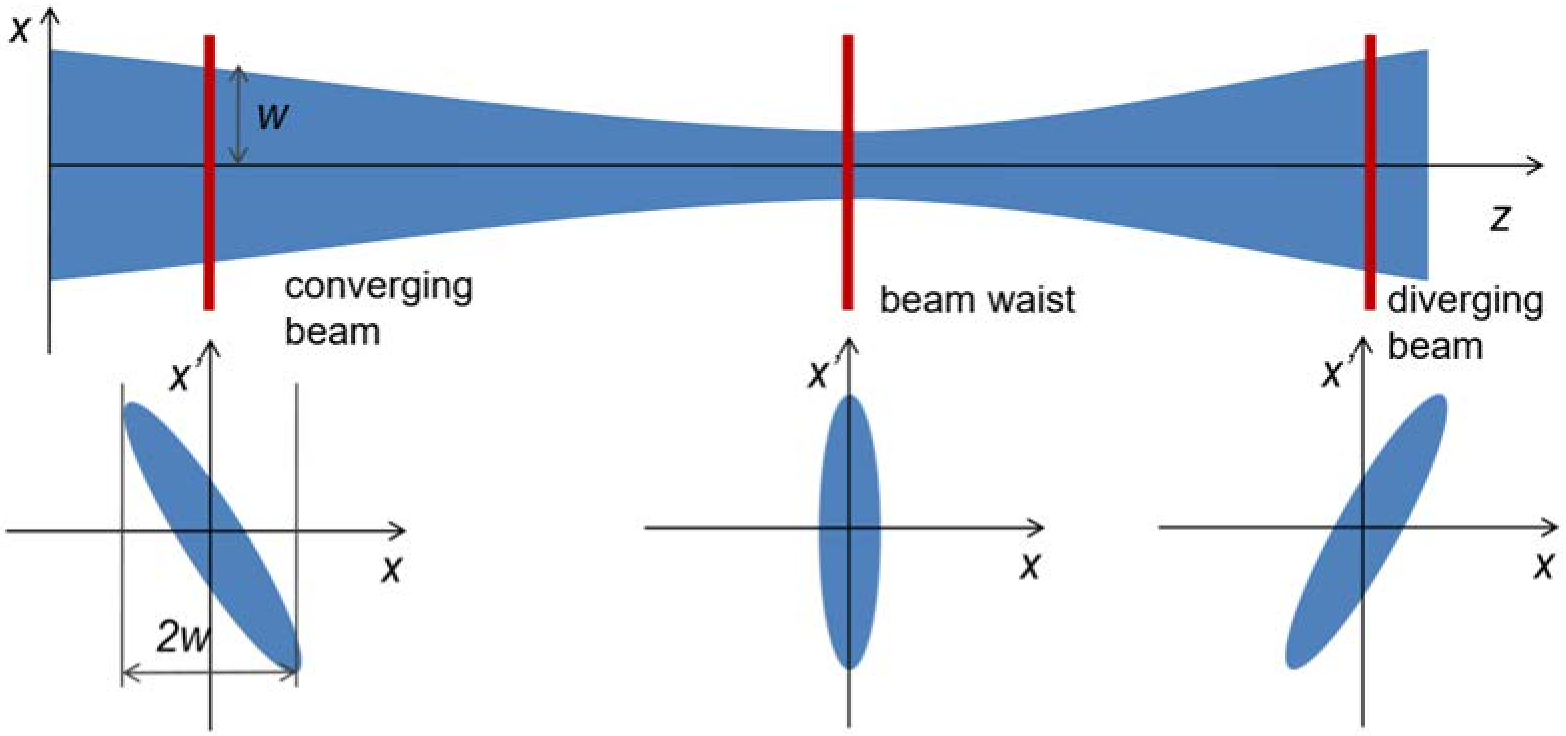
This area is described by the **emittance** : $\varepsilon = \sqrt{< x^2 >< x'^2 > - < xx' >^2}$



For an elliptical beam: Area = πε

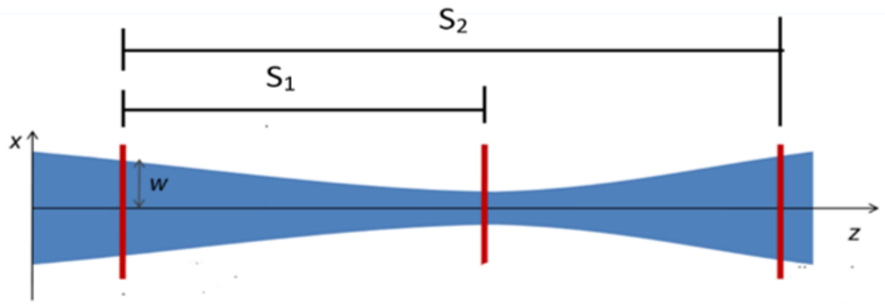
A big emittance means the beam is “wide” and “spread out” – it is difficult to work with. A small emittance means the beam is “narrow” and easy to work with.

Elliptical beam in a drift space



The shape of the beam changes, but the emittance stays the same

Beam emittance measurement with three profile monitors in a drift space



The beam transfer Matrix R(s) for drift space:

$$R(s) = \begin{bmatrix} 1 & S \\ 0 & 1 \end{bmatrix}$$

The beam matrix at any location s with respect to the location of the first profiler (s_0=0) is given by the formula:

$$\sigma(s) = R(s)\sigma(0)R(s)^T$$

From above follow at position s_1:

$$\sigma 11(s_1) = \sigma 22(0){s_1}^2 + 2s_1\sigma 12(0) + \sigma 11(0)$$

At position s_2

$$\sigma 11(s_2) = \sigma 22(0){s_2}^2 + 2s_2\sigma 12(0) + \sigma 11(0)$$

With $\sigma 11(0)$, $\sigma 11(s_1)$ *and* $\sigma 11(s_2)$ know from the width of the beam at the three profile monitors, $\sigma 22(0)$ and $\sigma 12(0)$ can be calculated from the equation above. The emittance is given by:

$$\varepsilon = \sqrt{\sigma 11(0)\sigma 22(0) - \sigma 12(0)^2}$$

Thank you