Minimal surfaces by way of complex analysis

Franc Forstnerič

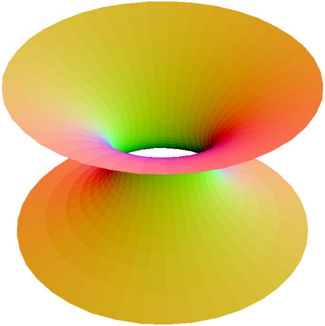
Univerza *v Ljubljani*

Institute of Mathematics, Physics and Mechanics

ICTP, Trieste
7 December 2018

From Euler's surfaces of rotation...

1744 **Euler** A minimal surface is one that locally minimizes area among all nearby surfaces with the same boundary. The only area minimizing surfaces of rotation are planes and catenoids.



...via Lagrange's equation of minimal graphs...

1760 **Lagrange** Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain. Then a smooth graph $(x, y, f(x, y)) \subset \overline{\Omega} \times \mathbb{R}$ is a critical point of the area functional with prescribed boundary values iff

$$\mathrm{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) = 0.$$

This is known as the **equation of minimal graphs**.

...to the modern concept of a minimal surface

1776 **Meusnier** A smooth surface $M \subset \mathbb{R}^3$ satisfies locally the above equation iff its mean curvature function vanishes identically.

Definition

A smoothly immersed surface $M \to \mathbb{R}^3$ is a **minimal surface** if its **mean curvature function** $\mathrm{H} : M \to \mathbb{R}$ is identically zero: $\mathrm{H} = 0$.

We have

$$\mathrm{H} = \frac{\kappa_1 + \kappa_2}{2}$$

where κ_1, κ_2 are the **principal curvatures**. Their product

$$\mathrm{K} = \kappa_1 \kappa_2 : M \to \mathbb{R}$$

is the **Gauss curvature** function of M. Note that $\mathrm{H} = 0 \Rightarrow \mathrm{K} \leq 0$.

The helicoid (Archimedes' screw)

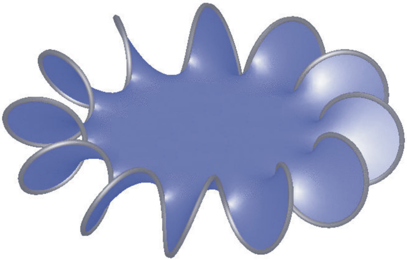
1776 **Meusnier** The helicoid is a minimal surface.

$$x = \rho\cos(\alpha\theta), \quad y = \rho\sin(\alpha\theta), \quad z = \theta$$

1842 **Catalan** The helicoid and the plane are the only ruled minimal surfaces (unions of straight lines) in $\mathbb{R}^3$.

The Plateau Problem

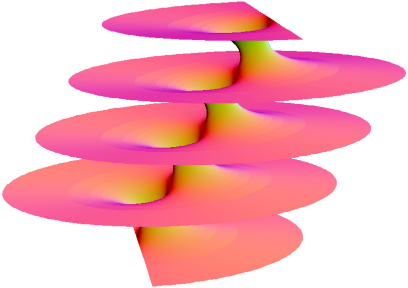
1873 **Plateau** Minimal surfaces can be obtained as soap films.



1932 **Douglas, Radó** Every continuous simple closed curve — a Jordan curve — in $\mathbb{R}^3$ spans a minimal surface.

Riemann's minimal examples

1865 **Riemann** and others discovered new examples using the **Weierstrass representation of minimal surfaces**.



Riemann's minimal examples are properly embedded minimal surfaces in $\mathbb{R}^3$ with countably many parallel planar ends and such that every horizontal plane intersects each of them in either a circle or a straight line. Topologically, they are planar domains.

Conformal minimal surfaces in $\mathbb{R}^3$

Assume that M is an **open Riemann surface**, i.e., a smooth noncompact orientable surface with a choice of a conformal (=complex) structure.

A smooth immersion $X = (X_1, X_2 \ldots, X_n) : M \to \mathbb{R}^n$ is **conformal** if it preserves angles. Denote by $H : M \to \mathbb{R}^n$ its mean curvature vector. Then,

$$\Delta X = 2\,H \quad \cdots \quad \textbf{the basic formula}$$

Here, Δ is the metric Laplacian. In any isothermal coordinate $z = x + \imath y$,

$$g = X^*(ds^2) = \lambda(dx^2 + dy^2), \quad \Delta_g = \frac{1}{\lambda}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

Hence, a conformal immersion $M \to \mathbb{R}^n$ is minimal if and only if it is harmonic.

Such immersions are stationary points of the area functional. Small pieces of it minimize area among all surfaces with the same boundary.

Connection with complex analysis

Let $X(\zeta)$ be a smooth function of a complex variable $\zeta = u + \mathrm{i}v$ (which we think of as a local coordinate on a Riemann surface M). Set

$$\partial X = \frac{1}{2}\left(\frac{\partial X}{\partial u} - \mathrm{i}\frac{\partial X}{\partial v}\right) d\zeta, \qquad \bar{\partial} X = \frac{1}{2}\left(\frac{\partial X}{\partial u} + \mathrm{i}\frac{\partial X}{\partial v}\right) d\bar{\zeta}.$$

- X is **holomorphic** iff $\bar{\partial} X = 0$; equivalently, if $dX = \partial X$.
- X is **harmonic** iff

$$\Delta X = 2\mathrm{i}\,\partial\bar{\partial} X = -2\mathrm{i}\,\bar{\partial}\partial X = 0 \quad \Longleftrightarrow \quad \partial X \text{ is holomorphic.}$$

- An immersion $X = (X_1, X_2, \cdots, X_n) : M \to \mathbb{R}^n$ is **conformal** iff

$$X_u \cdot X_v = 0,\ |X_u|^2 = |X_v|^2 \quad \Longleftrightarrow \quad \sum_{k=1}^{n} (\partial X_k)^2 = 0$$

The Weierstrass formula

Hence, a smooth immersion $X = (X_1, X_2, \cdots, X_n) : M \to \mathbb{R}^n$ is a conformal minimal immersion (a minimal surface) if and only if

$$\partial X = (\partial X_1, \ldots \partial X_n) \text{ is holomorphic and } \sum_{k=1}^{n} (\partial X_k)^2 = 0.$$

Fix a nowhere vanishing holomorphic 1-form θ on M. Let

$$\mathcal{A} = \left\{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_{j=1}^{n} z_j^2 = 0 \right\} \cdots \textbf{ the null quadric}.$$

Hence, every conformal minimal immersion $X : M \to \mathbb{R}^n$ is of the form

$$X(p) = X(p_0) + \int_{p_0}^{p} \Re(f\theta); \qquad p, p_0 \in M,$$

where $f\colon M \to \mathcal{A}_* = \mathcal{A} \setminus \{0\}$ is a holomorphic map such that

$$\int_C \Re(f\theta) = 0 \quad \text{for all closed curves } C \text{ in } M.$$

Holomorphic null curves

A holomorphic immersion $Z = (Z_1, \ldots, z_n) : M \to \mathbb{C}^n$ $(n \geq 3)$ is said to be a **holomorphic null curve** if

$$\sum_{k=1}^{n} (\partial Z_k)^2 = 0.$$

Every such curve is of the form

$$Z(p) = Z(p_0) + \int_{p_0}^{p} f\theta; \qquad p, p_0 \in M,$$

where $f\colon M \to \mathcal{A}_* = \mathcal{A} \setminus \{0\}$ is a holomorphic map such that

$$\int_C f\theta = 0 \quad \text{for all closed curves } C \text{ in } M.$$

Hence, the real and the imaginary part of a null curve are conformal minimal surfaces. Conversely, every conformal minimal surfaces is locally (on simply connected domains) the real part of a holomorphic null curve.

Catenoid and helicoid

Example

The *catenoid* and the *helicoid* are conjugate minimal surfaces — the real and the imaginary part of the same null curve $Z\colon \mathbb{C} \to \mathbb{C}^3$ given by

$$Z(z) = (\cos z, \sin z, -\mathrm{i}z), \qquad z = x + \mathrm{i}y \in \mathbb{C}.$$

Consider the family of minimal surfaces ($t \in \mathbb{R}$):

$$\begin{aligned} X_t(z) &= \Re\left(e^{\mathrm{i}t} Z(z)\right) \\ &= \cos t \begin{pmatrix} \cos x \cdot \cosh y \\ \sin x \cdot \cosh y \\ y \end{pmatrix} + \sin t \begin{pmatrix} \sin x \cdot \sinh y \\ -\cos x \cdot \sinh y \\ x \end{pmatrix}. \end{aligned}$$

At $t = 0$ we have a parametrization of a catenoid, and at $t = \pm\pi/2$ we have a (left or right handed) helicoid.

Robert Osserman, 1926–2011

This connection between complex analysis and minimal surface theory goes back to **Bernhard Riemann** and **Karl Weierstrass**.

Robert Osserman was a modern pioneer of this field. His book *A survey of minimal surfaces* (Dover, New York,1986) remains a classic.

However, this connection was fully explored only in the last few years.

A summary of topics

I will present new results on the following topics:

- Runge-Mergelyan approximation theorems for conformal minimal immersions (CMI's)
- Proper CMI's in $\mathbb{R}^n$, and in minimally convex domains in $\mathbb{R}^n$
- The Calabi-Yau problem: existence of bounded complete CMI's
- New results on the Gauss map

They have been obtained during the period 2013–2017 in collaboration with **Antonio Alarcón** and **Francisco J. López** (University of Granada); some also with **Barbara Drinovec Drnovšek** (University of Ljubljana).

Runge's theorem for minimal surfaces

A compact set K in an open Riemann surface M is **holomorphically convex** if $M \setminus K$ has no relatively compact connected components. If $M = \mathbb{C}$ then $\mathbb{C} \setminus K$ is connected and K is **polynomially convex**.

1885 **Runge** Every holomorphic function on a neighbourhood of a compact polynomially convex set $K \subset \mathbb{C}$ can be approximated uniformly on K by entire functions on $\mathbb{C}$.

1949 **Behnke-Stein** If K is a compact holomorphically convex set in an open Riemann surface M then every holomorphic function on a neighbourhood of K can be approximated uniformly on K by holomorphic functions on M.

Theorem (Alarón, López, F., 2012–2016)

Let K be a compact holomorphically convex set in an open Riemann surface M. Then, every conformal minimal immersion $U \to \mathbb{R}^n$ $(n \geq 3)$ on a neighborhood of K can be approximated uniformly on K by conformal minimal immersions $M \to \mathbb{R}^n$.

Sketch of proof

Assume that $X : U \to \mathbb{R}^n$ is a conformal minimal immersion on a connected open set $U \subset M$ containing K. By the Weierstrass formula,

$$X(p) = X(p_0) + \int_{p_0}^{p} \Re\,(f\theta)\,, \qquad p_0 \in K,\ p \in U,$$

where $f\colon U \to \mathcal{A}_*$ is holomorphic and the real periods of $f\theta$ vanish.

Pick a smoothly bounded compact domain D with $K \subset \mathring{D} \subset D \subset U$. Given a basis $\{C_j\}_{j=1}^{l}$ of $H_1(D;\mathbb{Z}) \cong \mathbb{Z}^l$, let

$$\mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_l) : \mathscr{O}(D, \mathbb{C}^n) \to (\mathbb{C}^n)^l = \mathbb{C}^{ln}$$

denote the **period map** whose j-th component equals

$$\mathcal{P}_j(f) = \oint_{C_j} f\theta \in \mathbb{C}^n, \qquad f \in \mathscr{O}(D, \mathbb{C}^n).$$

The 1-form $f\theta$ is exact iff $\mathcal{P}(f) = 0$, and $\Re(f\theta)$ is exact iff $\Re\mathcal{P}(f) = 0$.

Holomorphic period dominating sprays

Lemma

Given a nonflat holomorphic map $f \in \mathcal{O}(D, \mathcal{A}_*)$ (i.e., one whose image is not contained in a ray of the null quadric $\mathcal{A}_*$), there exist an open neighborhood V of the origin in $(\mathbb{C}^n)^l$ and a holomorphic map $\Phi_f : D \times V \to \mathcal{A}_*$ such that $\Phi_f(\cdot\,, 0) = f$ and

$$\frac{\partial}{\partial t}\bigg|_{t=0} \mathcal{P}(\Phi_f(\cdot\,, t)) : (\mathbb{C}^n)^l \to (\mathbb{C}^n)^l \quad \textit{is an isomorphism.}$$

Furthermore, there is a neighborhood Ω_f of f in $\mathscr{O}(M, \mathcal{A}_*)$ such that the map $\Omega_f \ni g \mapsto \Phi_g$ depends holomorphically on g.

This lemma does not apply to flat CMI's. However, it is easily seen that a flat CMI can be approximated by nonflat ones.

Sketch of proof of the lemma

Since the convex hull of the null quadric $\mathcal{A}$ equals $\mathbb{C}^n$, it is easy to show that for every loop $C \subset M$,

$$\text{integrals } \int_C g\theta \text{ over loops } g : C \to \mathcal{A}_* \text{ assume all values in } \mathbb{C}^n.$$

This uses the basic idea of Gromov's convex integration theory.

By considering such deformations for loops $C_1, \ldots, C_l$ in a period basis of $H_1(D; \mathbb{Z})$, we create a smooth period dominating spray ϕ_f over the set $C = \bigcup_{j=1}^{l} C_j$ with the core $\phi_f(\cdot\,, 0)|_C = f|_C$.

It is standard that loops in a period basis can be chosen such that C is holomorphically convex in D.

Hence, Mergelyan's theorem allows us to approximate ϕ_f by a holomorphic period dominating spray Φ_f as in the lemma.

Application of Oka-Grauert theory

Since the null quadric $\mathcal{A}_*$ is a homogeneous space of the complex orthogonal group $O_n(\mathbb{C}) = \{A \in GL_n(\mathbb{C}) : AA^t = I\}$, holomorphic maps from Stein manifolds to $\mathcal{A}_*$ satisfy the Runge approximation theorem in the absence of topological obstructions (**Grauert, 1958**).

Thus, we can approximate a period dominating spray $\Phi_f : D \times V \to \mathcal{A}_*$, furnished by the lemma, by holomorphic maps $F : M \times V \to \mathcal{A}_*$.

Since $dX = \Re(f\theta)$, we have $\Re\mathcal{P}(f) = 0$. If F is close enough to Φ_f, there is $t \in V$ near 0 such that

$$\text{the map } \tilde{f} = F(\cdot\,, t) : M \to \mathcal{A}_* \text{ satisfies } \Re\mathcal{P}(\tilde{f}) = 0.$$

Hence, given any domain $D' \supset D$ in M such that D is a deformation retract of D', the map

$$\widetilde{X}(p) = X(p_0) + \int_{p_0}^{p} \Re\left(\tilde{f}\theta\right), \quad p \in D',$$

is a conformal minimal immersion $D' \to \mathbb{R}^n$ approximating X on D.

Completion of proof of the approximation theorem

Choose a strongly subharmonic Morse exhaustion function $\rho : M \to \mathbb{R}_+$. Let $D_j = \{\rho \le c_j\}$ be regular sublevel sets, where $0 < c_1 < c_2 < \cdots$, $\lim_i c_i = +\infty$, and ρ has at most one critical point $p_i \in D_i \setminus D_{i-1}$.

Suppose that $X_i : D_i \to \mathbb{R}^n$ is a CMI. The lemma allows us to extend it by approximation to $\{\rho \le t\}$ for any $c_i < t < \rho(p_{i+1})$.

If p_{i+1} is a minimum of ρ, a new connected component of $\{\rho \le t\}$ appears when passing $\rho(p_{i+1})$, so we can extend X_i to this component and then apply the lemma to extend it further to D_{i+1}.

Otherwise, p_{i+1} has Morse index 1. Then, D_{i+1} retracts onto $D_i \cup E$ for a suitable chosen arc $E \subset M \setminus D_i$, attached with its endpoints p, q to D_{i-1}. Let $dX_i = f\theta$ on D_i, where $f : D_i \to \mathcal{A}_*$. We extend f to a smooth map $f : D_i \cup E \to \mathcal{A}_*$ such that $\int_E f\theta = X_i(q) - X_i(p)$. By integration, this extends X_i to $D_i \cup E$. Then, forming a period dominating spray with the core f and applying Mergelyan's theorem allows us to approximate X_i in $\mathscr{C}^1(D_i \cup E)$ by a CMI $X_{i+1} : D_{i+1} \to \mathbb{R}^n$.

The limit $X = \lim_i X_i : M \to \mathbb{R}^n$ satisfies the theorem.

Proper minimal surfaces in Euclidean spaces

By using this approximation theorem, a new general position result, and a well known strategy for constructing proper maps, we proved:

Theorem (Alarcón, López, F., Math. Z. 2016)

Let M be an open Riemann surface.

(a) *There is a proper conformal minimal immersion $M \to \mathbb{R}^3$. Moreover, proper immersions are dense in the space of all conformal minimal immersions $M \to \mathbb{R}^3$ in the compact-open topology.*

(b) *There exists a proper conformal minimal immersion $M \to \mathbb{R}^4$ with simple double points, and such immersions are dense in the space of all conformal minimal immersions $M \to \mathbb{R}^4$.*

(c) *There is a proper conformal minimal embedding $M \hookrightarrow \mathbb{R}^5$. Moreover, proper conformal minimal embeddings are dense in the space of all conformal minimal immersions $M \to \mathbb{R}^n$ for any $n \geq 5$.*

Part (a) was first proved by Alarcón and López (2012, 2014).

What is the minimal embedding dimension?

2015 **Meeks, Pérez, Ros** (Ann. of Math. 2015) **Planes, catenoids, helicoids, and Riemann's minimal examples are the only properly embedded minimal planar domains in $\mathbb{R}^3$.**

Problem

Does every open Riemann surface admit a (proper) embedding into $\mathbb{R}^4$ as a conformal minimal surface?

Note that every complex curve in $\mathbb{C}^n$ is also a minimal surface.

Problem (Bell-Forster-Narasimhan Conjecture)

Does every open Riemann surface admit a (proper) holomorphic embedding into $\mathbb{C}^2$?

2013 **Wold, F.** Every circled domain (possibly infinitely connected) in $\mathbb{C}$, or in a complex torus, embeds properly holomorphically into $\mathbb{C}^2$.

A properly embedded minimal Möbius strip in $\mathbb{R}^4$

2017 **Alarcón, López, F.** (Memoirs AMS, in press):

The harmonic map $X\colon \mathbb{C}^* = \mathbb{C}\setminus\{0\} \to \mathbb{R}^4$ given by

$$X(z) = \Re\left(\mathrm{i}(z+\frac{1}{z}), z-\frac{1}{z}, \frac{\mathrm{i}}{2}(z^2-\frac{1}{z^2}), \frac{1}{2}(z^2+\frac{1}{z^2})\right)$$

is a proper conformal minimal immersion such that

$$X(z_1) = X(z_2) \iff z_1 = z_2 \text{ or } z_1 = -1/\bar{z}_2.$$

Note that

$$\mathfrak{I}(z) = -1/\bar{z}$$

is a fixed-point-free antiholomorphic involution on $\mathbb{C}^*$, and $\mathbb{C}^*/\mathfrak{I}$ is a Möbius strip.

Hence, $X(\mathbb{C}^*)$ **is a properly embedded minimal Möbius strip in $\mathbb{R}^4$.**

Proper CMI's in minimally convex domains

Let $D \subset \mathbb{R}^n$ be a domain. A smooth function $\rho : D \to \mathbb{R}$ is **minimal plurisubharmonic** (MPSH) if, at every point $p \in \mathbb{D}$, we have

$$\lambda_1 + \lambda_2 \geq 0$$

where $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are the eigenvalues of the Hessian of ρ at p.

It is easy to see that ρ is MPSH iff $\rho \circ F$ is subharmonic on M for every conformal minimal surface $F : M \to D$.

Theorem (Alarcón, Drinovec, F., López, Trans. AMS 2018)

Let M be a compact bordered Riemann surface. If a domain $D \subset \mathbb{R}^3$ admits a MPSH exhaustion function (such D is said to be **minimally convex***), then every conformal minimal immersion $M \to D$ can be approximated on compacts in $\mathring{M}$ by proper CMI's $\mathring{M} \to D$.*

A domain $D \subset \mathbb{R}^3$ with smooth boundary bD is minimally convex iff bD is mean-convex, i.e., its mean curvature is ≥ 0 at every point. This is the biggest class of domains in $\mathbb{R}^3$ for which the result holds.

The Riemann-Hilbert problem for minimal surfaces

The proof depends on another tool from complex analysis.
Assume that M is a compact bordered Riemann surface and $X\colon M \to \mathbb{R}^3$ is a conformal minimal immersion. Let $I \subset bM$ be an arc.

Let $Y\colon bM \times \overline{\mathbb{D}} \to \mathbb{R}^3$ be a continuous map of the form

$$Y(p,\xi) = X(p) + F(p,\xi), \quad p \in I,\ \xi \in \overline{\mathbb{D}},$$

where $F(p,\cdot\,) : \overline{\mathbb{D}} \to \mathbb{R}^3$ is a conformal minimal immersion for each $p \in I$, and $F(p,\cdot\,) = 0$ for $p \in bM \setminus I$.

Then, there are conformal minimal immersions $\widetilde{X}\colon M \to \mathbb{R}^3$ such that

- $\widetilde{X}$ approximates X outside a small neighbourhood of I in M;
- $\widetilde{X}(M)$ lies close to $X(M) \cup \bigcup_{p\in I} Y(p,\overline{\mathbb{D}})$;
- $\widetilde{X}(p)$ lies close to the curve $Y(p, b\mathbb{D})$ for every $p \in I$.

A similar result holds in $\mathbb{R}^n$ for $n > 3$ if the discs $F(p,\cdot\,)$ $(p \in I)$ are linear of the form $F(p,\xi) = f(p,\xi)\mathbf{u} + g(p,\xi)\mathbf{v}$, where $\mathbf{u},\mathbf{v} \in \mathbb{R}^n$ is an orthonormal pair and $f + \imath g$ is holomorphic in ζ.

Riemann-Hilbert problem for null discs

We outline the proof in the special case when

- M is the closed disc $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1$,
- $\vartheta \in \mathcal{A}_*$ is a null vector,
- $$\mu \colon b\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\} \to \mathbb{R}_+ := [0, +\infty)$$

  is a continuous function (the size function), and
- $$Y \colon b\mathbb{D} \times \overline{\mathbb{D}} \to \mathbb{C}^3, \quad Y(z, \xi) = X(z) + \mu(z)\, \xi\, \vartheta.$$

Geometry of the null quadric in $\mathbb{C}^3$

$$\mathcal{A} = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1^2 + z_2^2 + z_3^2 = 0\}.$$

- $\mathcal{A}$ is a complex cone with vertex at 0; $\mathcal{A}_* = \mathcal{A} \setminus \{0\}$ is smooth.
- $\mathcal{A}_*$ is a holomorphic fiber bundle with fiber $\mathbb{C}_*$ over the curve $\Lambda = \{[z_1 : z_2 : z_3] \colon z_1^2 + z_2^2 + z_3^2 = 0\} \subset \mathbb{CP}^2$. We have $\Lambda \cong \mathbb{CP}^1$.
- The spinor representation:

$$\pi \colon \mathbb{C}^2 \to \mathcal{A}, \quad \pi(u, v) = \big(u^2 - v^2, \mathrm{i}(u^2 + v^2), 2uv\big).$$

  The map $\pi \colon \mathbb{C}_*^2 \to \mathcal{A}_*$ is a nonramified two-sheeted covering.
- $\mathcal{A}_*$ is an Oka manifold; in particular, maps $M \to \mathcal{A}_*$ satisfy the Runge and Mergelyan approximation properties.

Riemann-Hilbert problem for null discs – Proof

$$
\begin{aligned}
X' &= \pi(u,v) = \big(u^2 - v^2, \mathrm{i}(u^2+v^2), 2uv\big) : \overline{\mathbb{D}} \to \mathcal{A}_* \\
\vartheta &= \pi(p,q) = \big(p^2 - q^2, \mathrm{i}(p^2+q^2), 2pq\big) \in \mathcal{A}_* \\
\eta &= \sqrt{\mu} : b\mathbb{D} \to \mathbb{R}_+ \quad \text{(square root of the size function)} \\
\eta(z) &\approx \tilde{\eta}(z) = \sum_{j=1}^{N} A_j z^{j-m} \quad \text{(rational approximation)} \\
u_n(z) &= u(z) + \sqrt{2n+1}\,\tilde{\eta}(z)\, z^n p \quad (n > m,\ u_n(0) = u(0)) \\
v_n(z) &= v(z) + \sqrt{2n+1}\,\tilde{\eta}(z)\, z^n q \quad (v_n(0) = v(0)) \\
\Phi_n &= \pi(u_n, v_n) = \big(u_n^2 - v_n^2, \mathrm{i}(u_n^2+v_n^2), 2u_n v_n\big) : \overline{\mathbb{D}} \to \mathcal{A}_* \\
\widetilde{X}_n(z) &= X(0) + \int_0^z \Phi_n(\xi)\, d\xi, \qquad z \in \overline{\mathbb{D}}.
\end{aligned}
$$

It follows that $\widetilde{X}_n(z) \approx X(z) + z^{2n+1}\mu(z)\vartheta$. Take $\widetilde{X} = \widetilde{X}_n$ for big n.

Complete minimal surfaces with Jordan boundaries

The following theorem also hinges upon the Riemann-Hilbert method.

Theorem (Alarcón, Drinovec, F., López, Proc. LMS 2015)

Assume that M is a **compact bordered Riemann surface.**
Every conformal minimal immersion $X_0 \colon M \to \mathbb{R}^n$ $(n \geq 3)$ can be approximated, uniformly on M, by continuous maps $X \colon M \to \mathbb{R}^n$ such that $X \colon \mathring{M} \to \mathbb{R}^n$ is a **complete conformal minimal immersion** *and $X(bM) \subset \mathbb{R}^n$ is a union of Jordan curves.*
If $n \geq 5$ then $X \colon M \to \mathbb{R}^n$ can be chosen a topological embedding.

This result falls within the scope of the classical **Calabi–Yau problem** (**Calabi 1965, Yau 2000**). Some pioneering results:

1996 Nadirashvili: A complete bounded minimal disc in $\mathbb{R}^3$.

2007 Martín, Nadirashvili: A complete minimal disc with Jordan boundary. The proof is not fully convincing.

Colding, Minicozzi: The Calabi-Yau conjectures hold for embedded surfaces of finite topological type. Ann. of Math. (2008)

The Gauss map of a minimal surface

The *complex Gauss map* of a conformal minimal immersion $X = (X_1, X_2, X_3)\colon M \to \mathbb{R}^3$ is the holomorphic map

$$g_X = \frac{\partial X_3}{\partial X_1 - \mathrm{i}\,\partial X_2} = \frac{\partial X_2 - \mathrm{i}\,\partial X_1}{\mathrm{i}\,\partial X_3} : M \longrightarrow \mathbb{CP}^1.$$

The function g_X is the stereographic projection to $\mathbb{CP}^1$ of the real Gauss map $N = (N_1, N_2, N_3)\colon M \to \mathbb{S}^2 \subset \mathbb{R}^3$:

$$g_X = \frac{N_1 + \mathrm{i} N_2}{1 - N_3} : M \longrightarrow \mathbb{C} \cup \{\infty\} = \mathbb{CP}^1.$$

Theorem (Alarcón, Lopez., F.; J. Geom. Anal. 2017)

Let M be an open Riemann surface. Every holomorphic map $g\colon M \to \mathbb{CP}^1$ (i.e., every meromorphic function on M) is the complex Gauss map of a conformal minimal immersion $X\colon M \to \mathbb{R}^3$.

The generalised Gauss map

Let $X = (X_1, \ldots, X_n) \colon M \to \mathbb{R}^n$ be a conformal minimal immersion.

Since the 1-form $\partial X = (\partial X_1, \ldots, \partial X_n)$ is holomorphic and nowhere vanishing, it determines the Kodaira type holomorphic map

$$G_X \colon M \to \mathbb{CP}^{n-1}, \quad G_X(p) = [\partial X_1(p) \colon \cdots \colon \partial X_n(p)] \quad (p \in M),$$

called the **generalised Gauss map of X**. It is of great importance in the theory of minimal surfaces, especially when $n = 3$.

Since $\sum_{j=1}^n (\partial X_j)^2 = 0$, G_X assumes values in the complex hyperquadric

$$Q^{n-2} = \{[z_1 : \ldots : z_n] \in \mathbb{CP}^{n-1} : z_1^2 + \cdots + z_n^2 = 0\} = \pi(\mathcal{A}_*^{n-1}),$$

where $\pi \colon \mathbb{C}_*^n \to \mathbb{CP}^{n-1}$ denotes the canonical projection.

Every map $M \to Q^{n-2}$ is the generalized Gauss map

Theorem (Alarcón, Lopez., F.; J. Geom. Anal. 2017)

Let M be an open Riemann surface and $n \geq 3$.

For every holomorphic map $\mathscr{G}\colon M \to Q^{n-2} \subset \mathbb{CP}^{n-1}$ there exists a conformal minimal immersion $X\colon M \to \mathbb{R}^n$ with $G_X = \mathscr{G}$.

If $\mathscr{G}(M)$ is not contained in a proper projective subspace of $\mathbb{CP}^{n-1}$, then X can be chosen an embedding if $n \geq 5$, and it can be chosen an immersion with simple double points if $n = 4$.

The conformal minimal immersions in the above theorem cannot be complete or proper in general. In fact:

- 1988 **Fujimoto 1988, 1990** The Gauss map $G : M \to \mathbb{CP}^1$ of a nonflat complete minimal surface in $\mathbb{R}^3$ can omit at most 4 values, where the upper bound 4 is best possible.
- 1991 **Min Ru** If $X : M \to \mathbb{R}^n$ is a complete nonflat minimal surface then its Gauss map $G_X : M \to \mathbb{CP}^{n-1}$ can omit at most $n(n+1)/2$ hyperplanes in general position.