Stochastic thermodynamics: an emerging, evolving field

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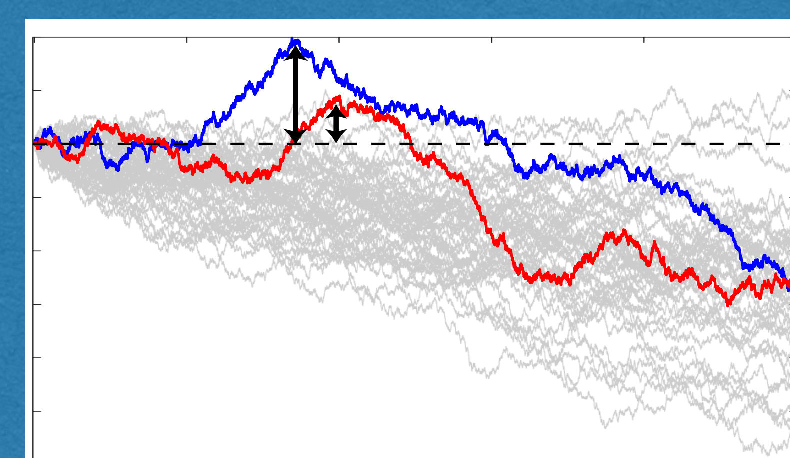
(Trieste, Italy)

United Nations
Educational, Scientific and
Cultural Organization

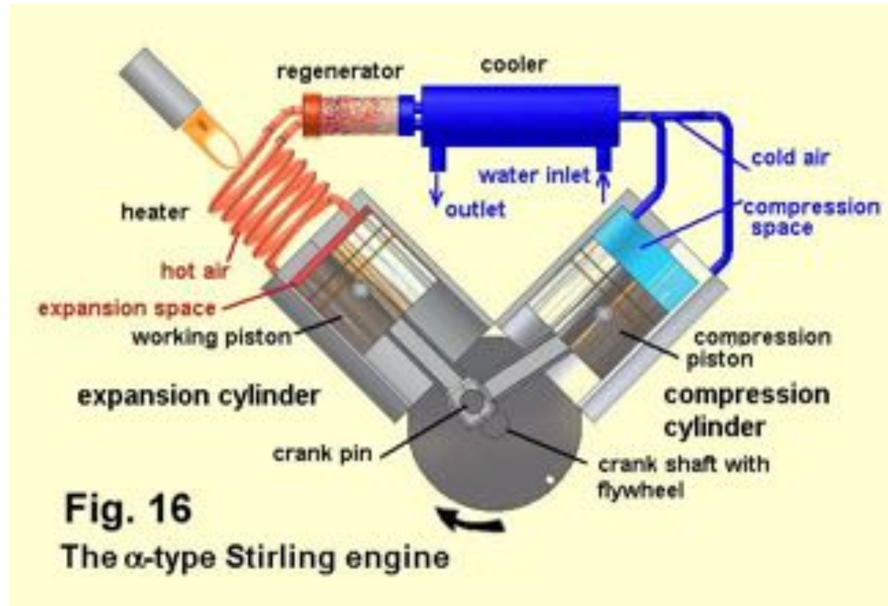
Workshop on Martingales
in Finance and Physics

Trieste, 24 May 2019




Stochastic thermodynamics

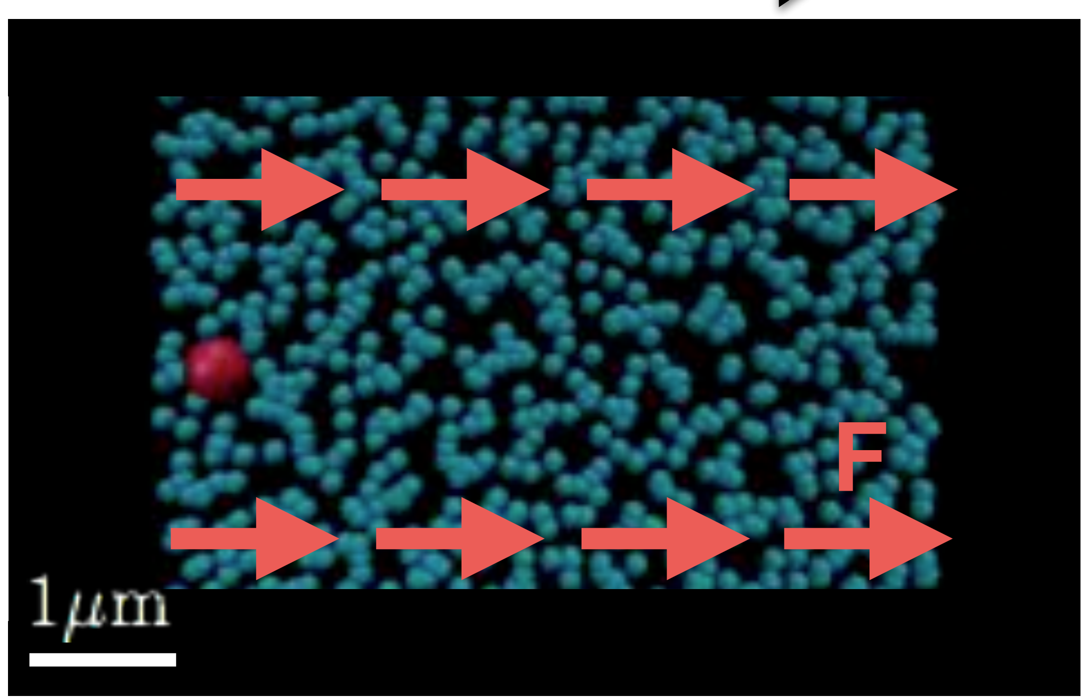
Standard Thermodynamics: macroscopic systems



Heat engine

Stochastic Thermodynamics: mesoscopic systems

Average tendency



Drifted colloidal particle

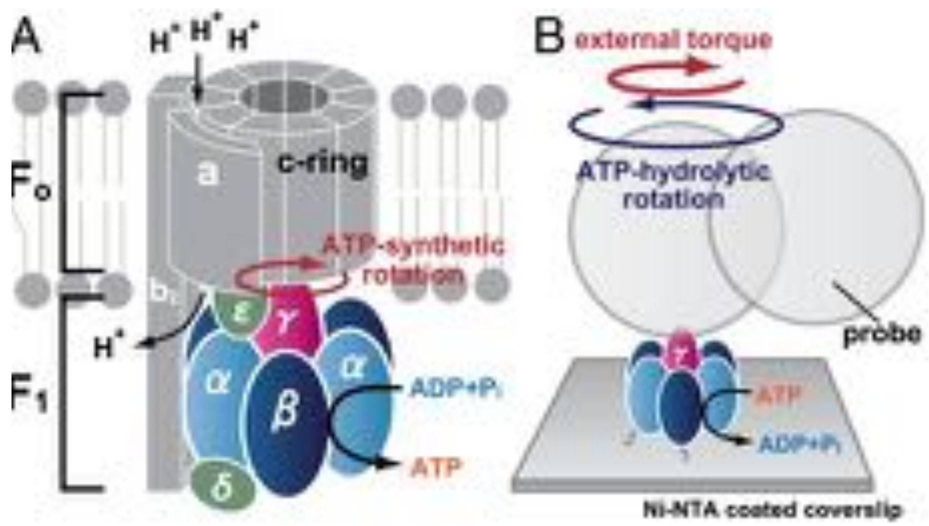
Rare events

Stochastic thermodynamics

An emerging, evolving field studying nonequilibrium fluctuations of mesoscopic systems

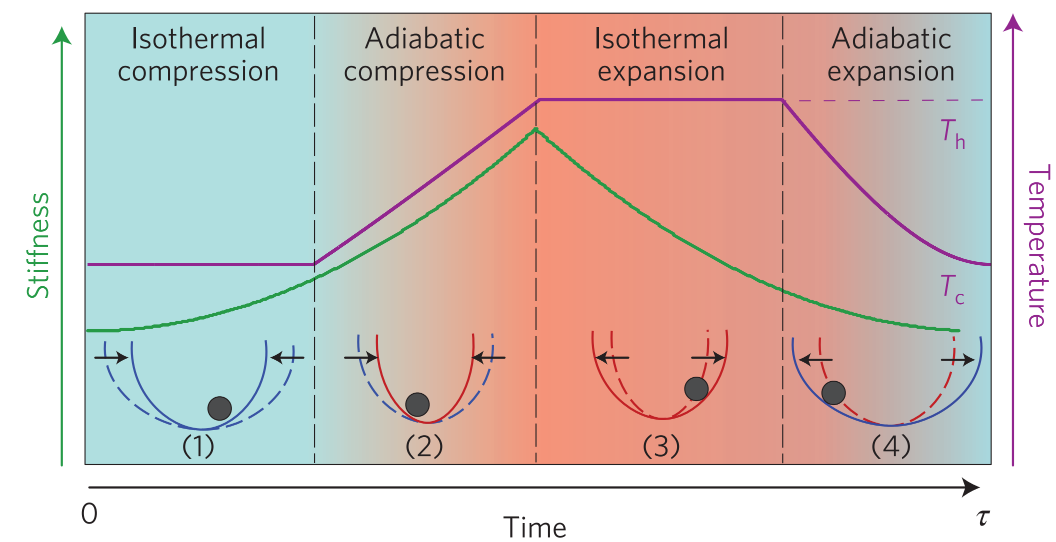
Molecular motors

(S Toyabe et al., PNAS 108, 17951 (2011))



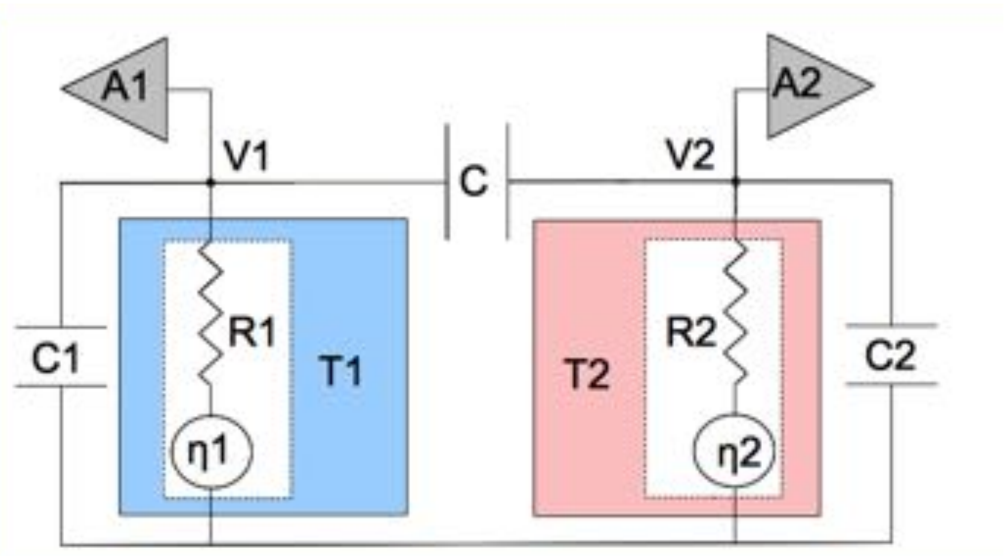
Colloidal heat engines

IA Martinez et al., Soft Matter 13, 22 (2017)



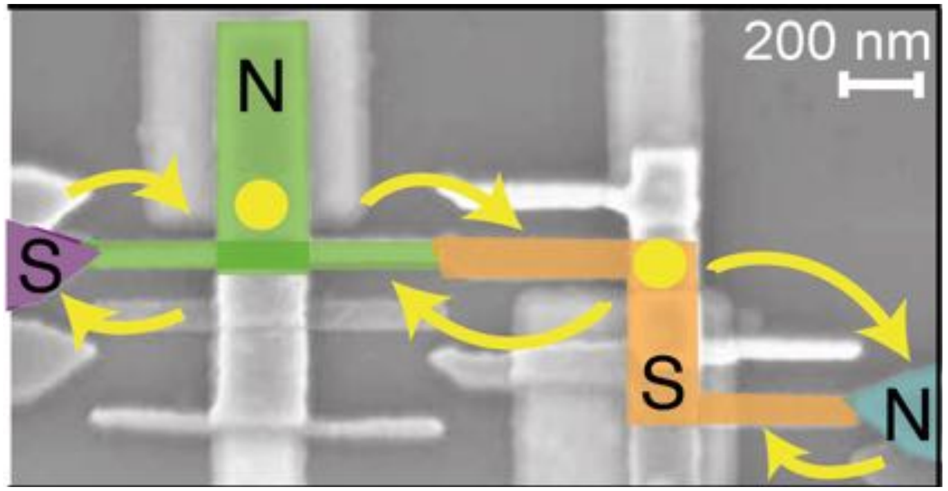
Electric circuits

(S Ciliberto et al., PRL 110, 180601 (2013))



Quantum dots

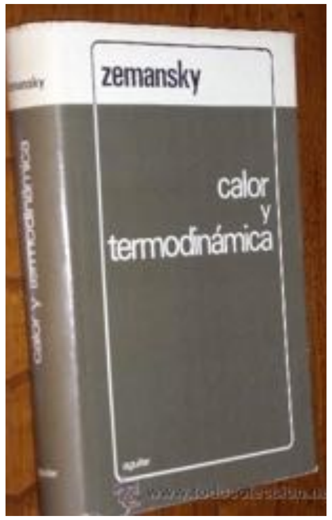
(S. Singh et al., PRB (2019))



Plot of this talk

- Brief review of classical “textbook” thermodynamics
- Stochastic thermodynamics: first and second laws
- Fluctuation theorems
- Maxwell’s demons and feedback control
- The latest: Uncertainty relations and Martingales

Plot of this talk



- **Brief review of classical "textbook" thermodynamics**
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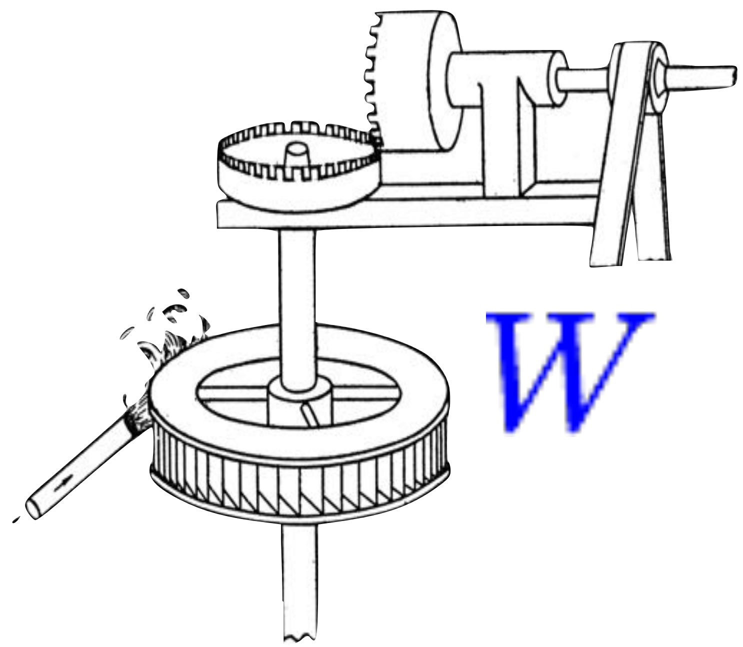
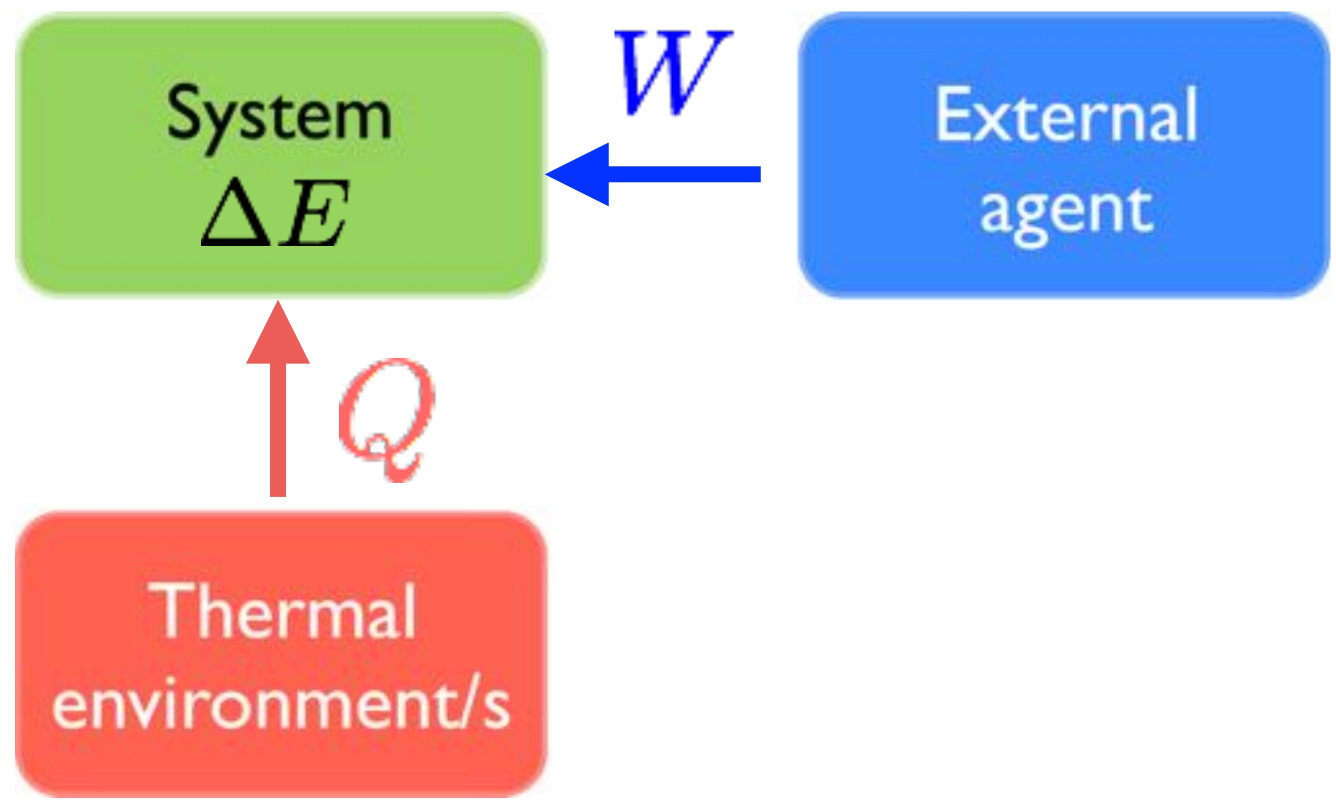
First Law of Thermodynamics

Heat

Work



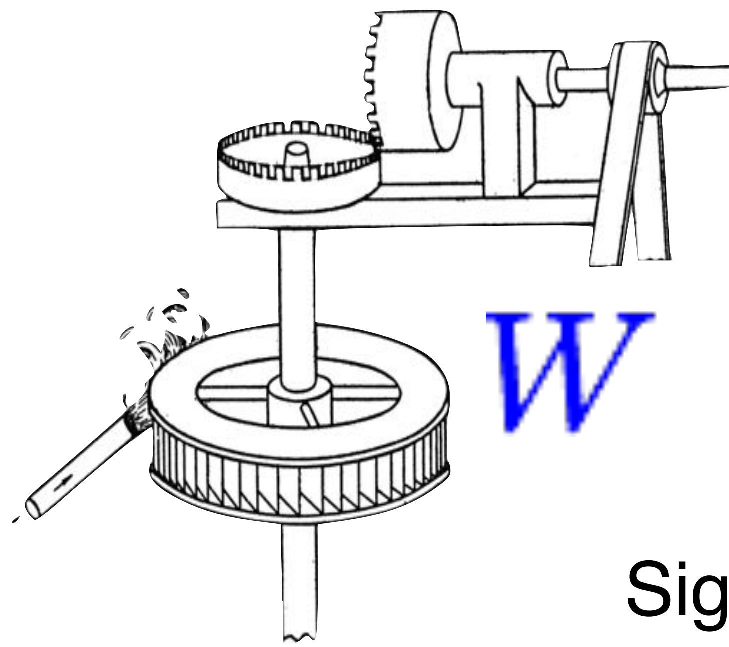
$$\Delta E = Q + W$$



First Law of Thermodynamics

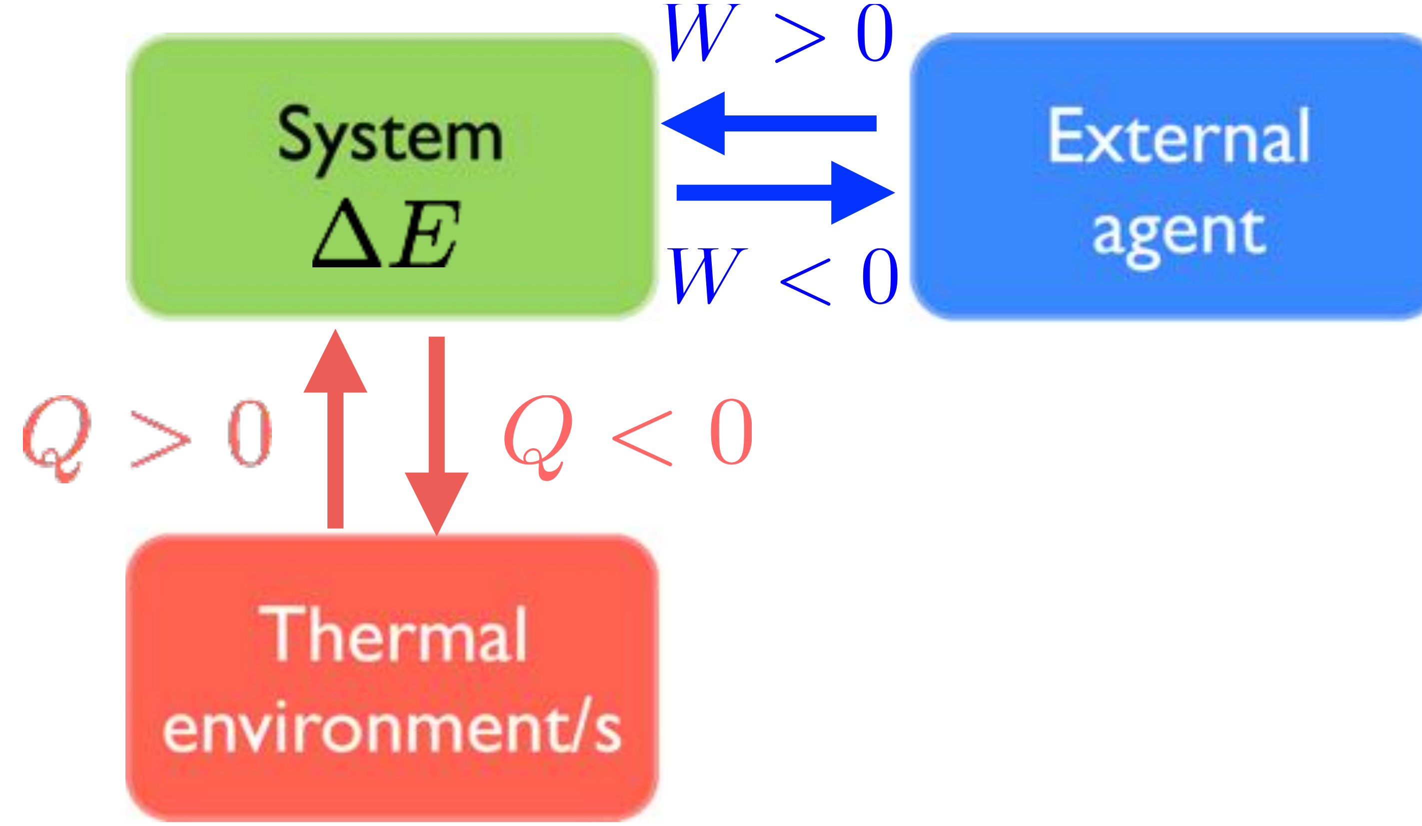
Heat

Work



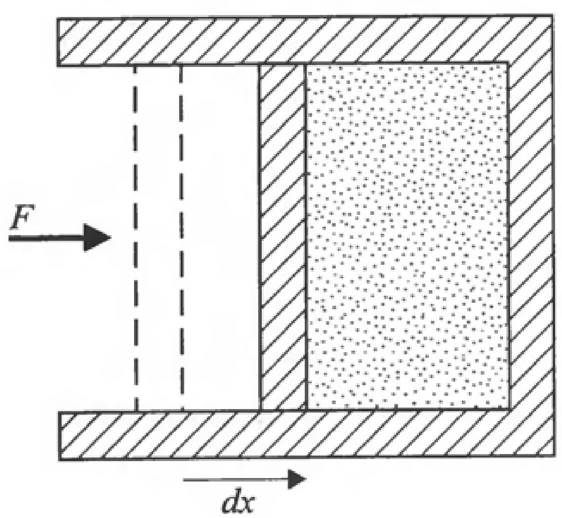
$$\Delta E = Q + W$$

Sign convention:

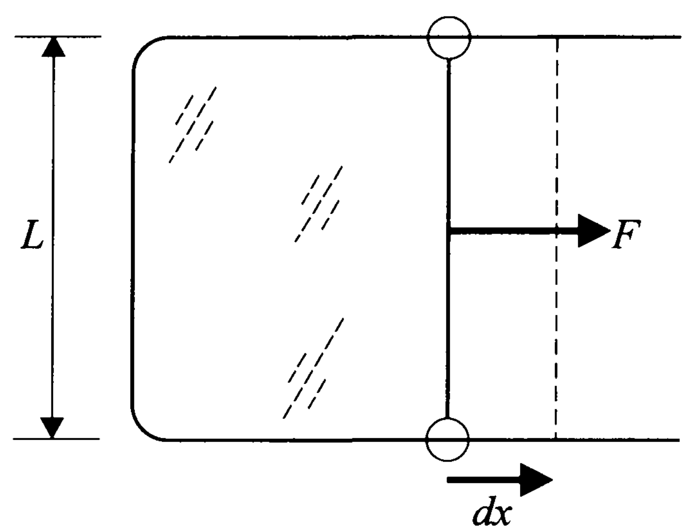


First Law of Thermodynamics

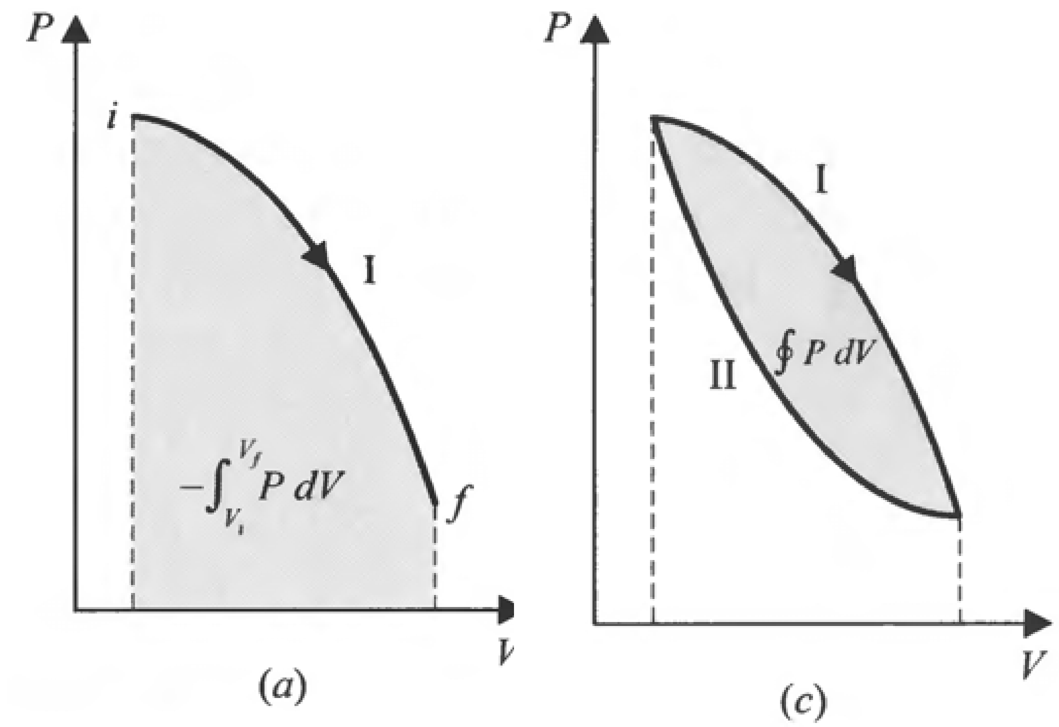
$$dE = \delta Q + \delta W$$ First Law for infinitesimal changes



ideal gas



soap film



Energy: state function
Heat and work: path-dependent

Work of simple systems

Simple system	Intensive coordinate (generalized force)	Extensive coordinate (generalized displacement)	Work, J
Hydrostatic system	P, in Pa	V, in m^3	$-P\,dV$
Wire	$\mathcal{F}$, in N	L, in m	$\mathcal{F}\,dL$
Surface film	γ, in N/m	A, in m^2	$\gamma\,dA$
Electrochemical cell	$\mathcal{E}$, in V	Z, in C	$\mathcal{E}\,dZ$
Dielectric solid	E, in V/m	$\mathcal{P}$, in C·m	$E\,d\mathcal{P}$
Paramagnetic solid	$\mu_0\mathcal{H}$, in N/A·m	$\mathcal{M}$, in A·m^2	$\mu_0\mathcal{H}\,d\mathcal{M}$

generalized displacement

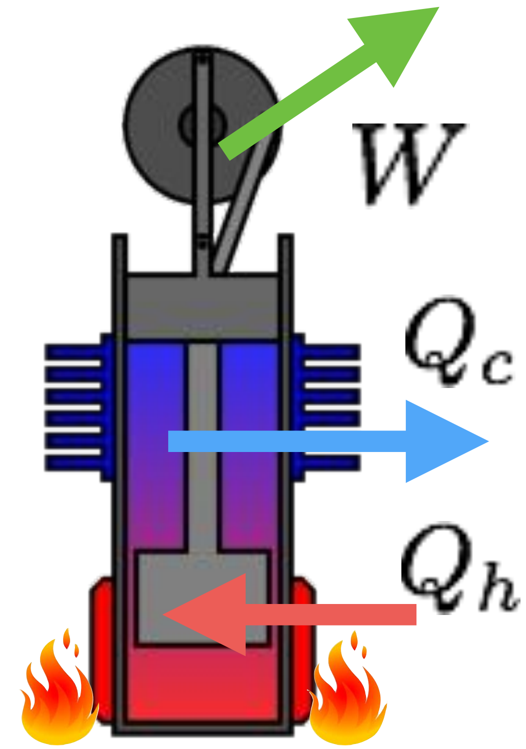
$$\delta W = F_\lambda d\lambda$$

generalized force

Zemansky, Dittman, "Heat and thermodynamics" (1997)

First Law for Heat engines

A system interacting with two thermal baths

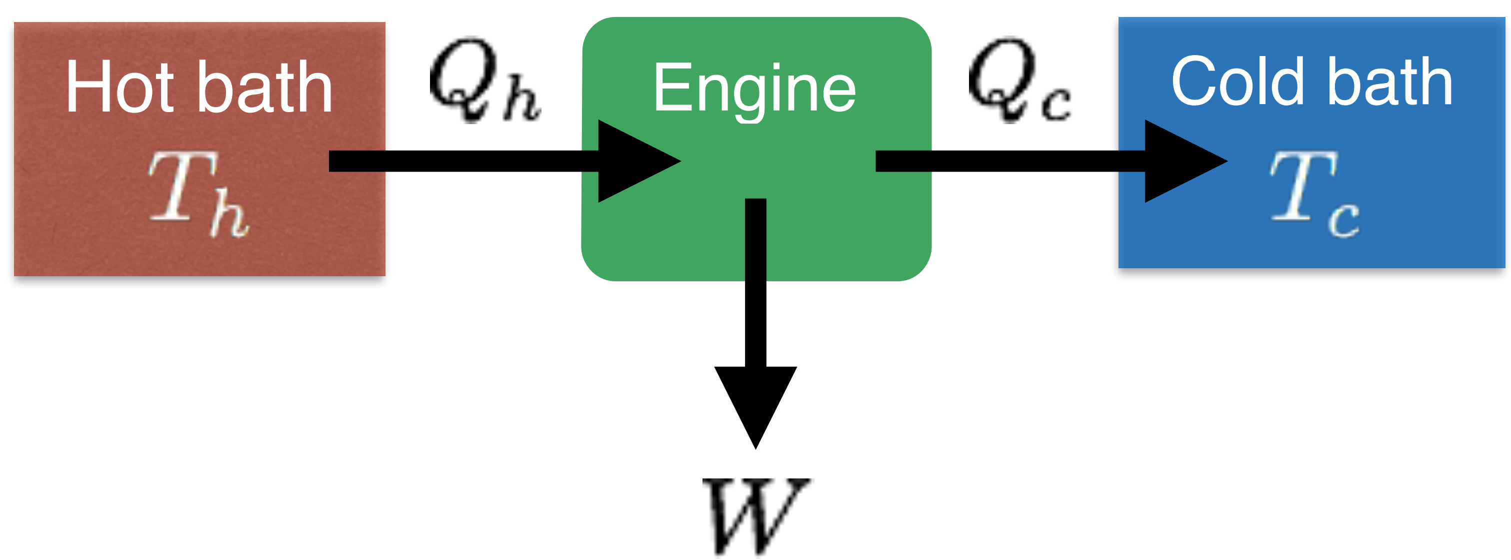


First Law

$$\Delta E = Q_h + Q_c + W = 0$$

Extracted Work

$$-W = Q_h + Q_c$$



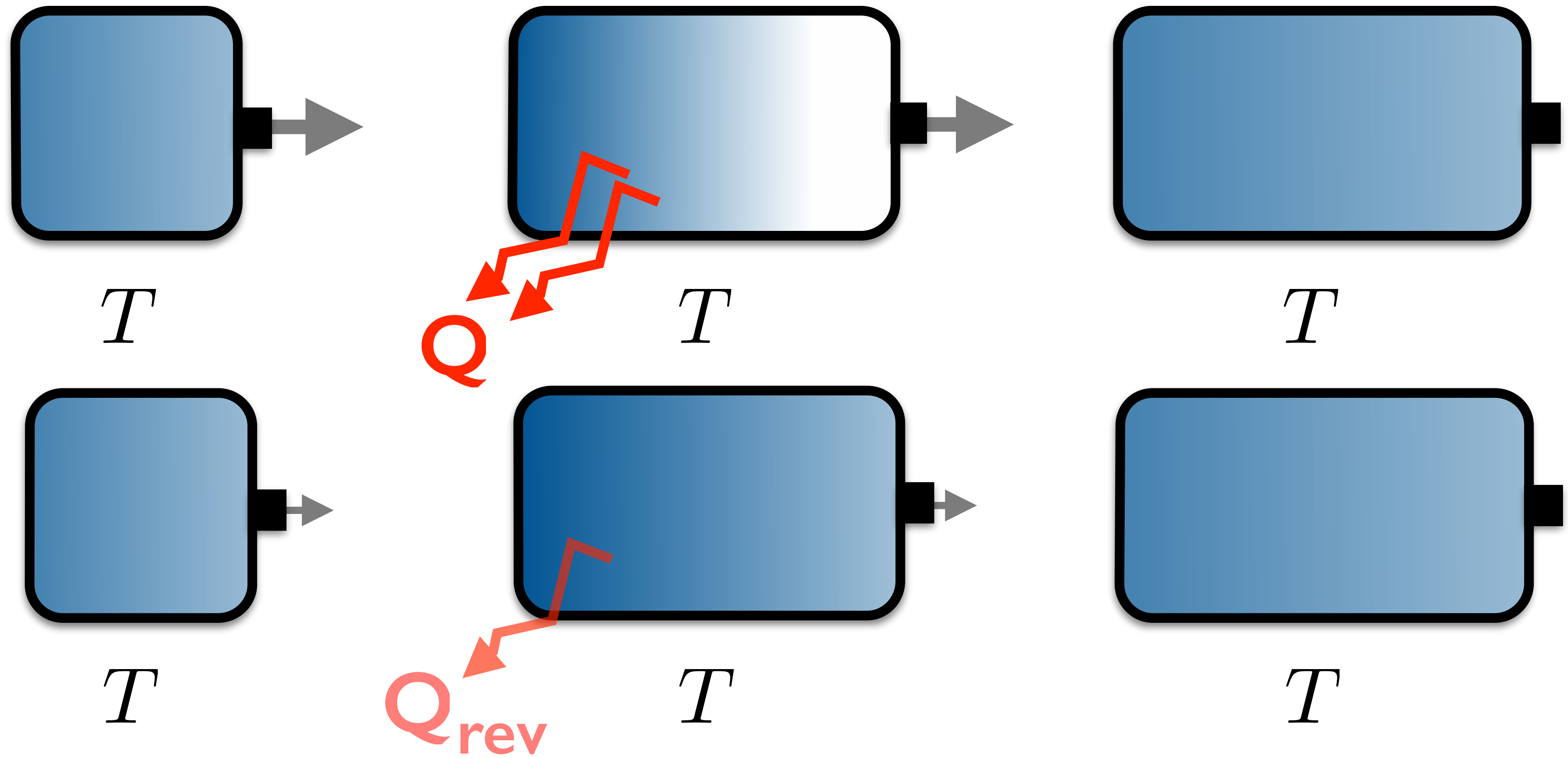
Second Law of Thermodynamics

$$S_{\text{tot}} > 0 \qquad\qquad S_{\text{tot}} < 0$$

Second Law of Thermodynamics

Sadi Carnot

Rudolf Clausius



For any **nonequilibrium** process

$$\frac{Q}{T} \leq \frac{Q_{\mathrm{rev}}}{T} \qquad \frac{Q_{\mathrm{rev}}}{T} = S_{\mathrm{fin}} - S_{\mathrm{ini}} = \underbrace{\Delta S}_{\mathrm{entropy}}$$

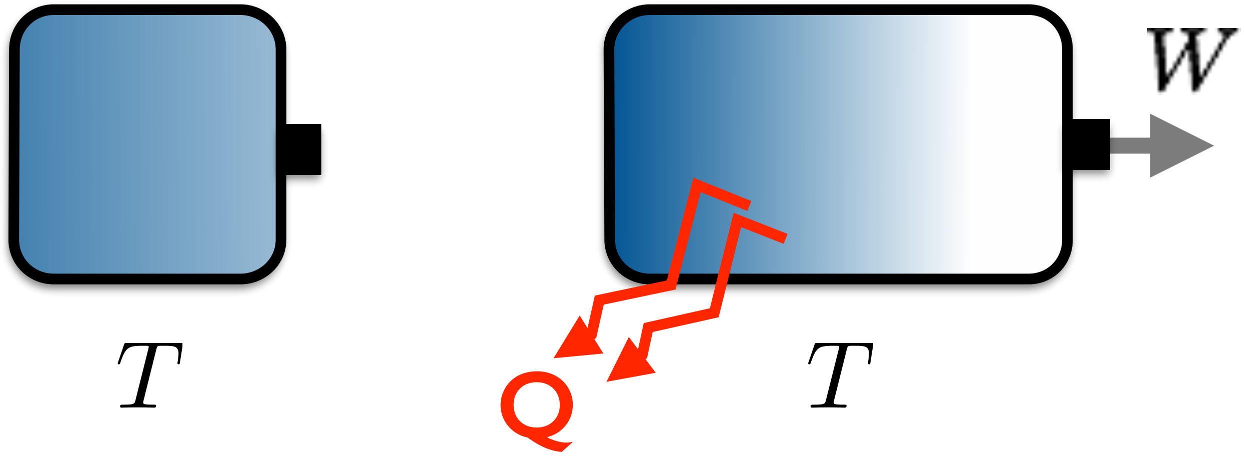
Entropy production

$$\frac{Q_{\mathrm{rev}}}{T} - \frac{Q}{T} = \underbrace{\Delta S + S_{\mathrm{env}}}_{\equiv S_{\mathrm{tot}}} \geq 0$$

$$S_{\mathrm{tot}} \geq 0$$

Second Law of Thermodynamics

Isothermal processes



Entropy production $S_{\text{tot}} = \Delta S - (Q/T)$

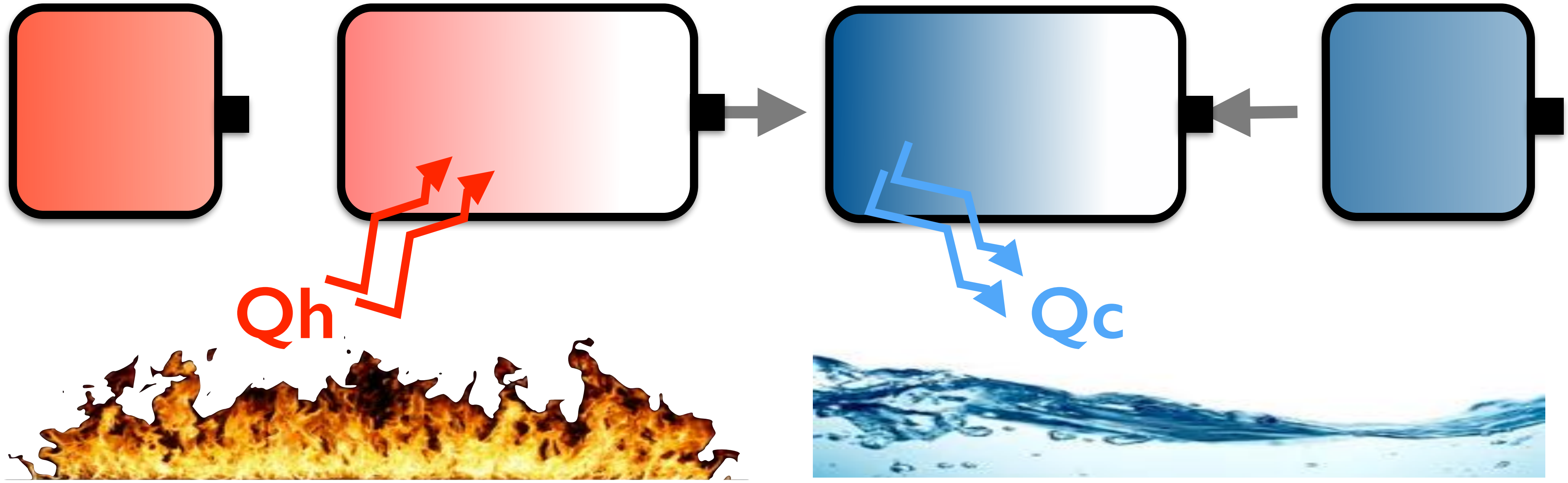
$F = E - TS$ $\quad = (\Delta E - \Delta F - Q)/T$

$\Delta E = Q + W$ $\quad = (W - \Delta F)/T$

$S_{\text{tot}} \geq 0$ $\quad W \geq \Delta F$

Second Law of Thermodynamics

Non-isothermal Heat engines

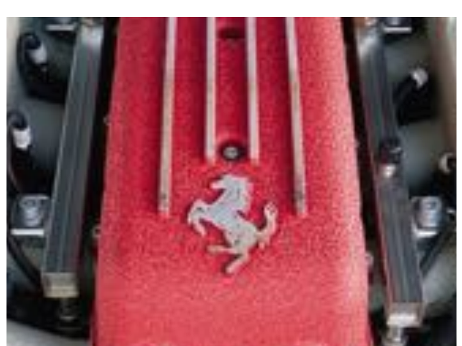


Entropy production in a cycle

$$S_{\rm tot} = \underbrace{\Delta S}_{=0} - Q_h/T_h - Q_c/T_c \geq 0$$

$$0 = Q_h + Q_c + W \qquad -\frac{W}{Q_h} \leq 1 - \frac{T_c}{T_h}$$

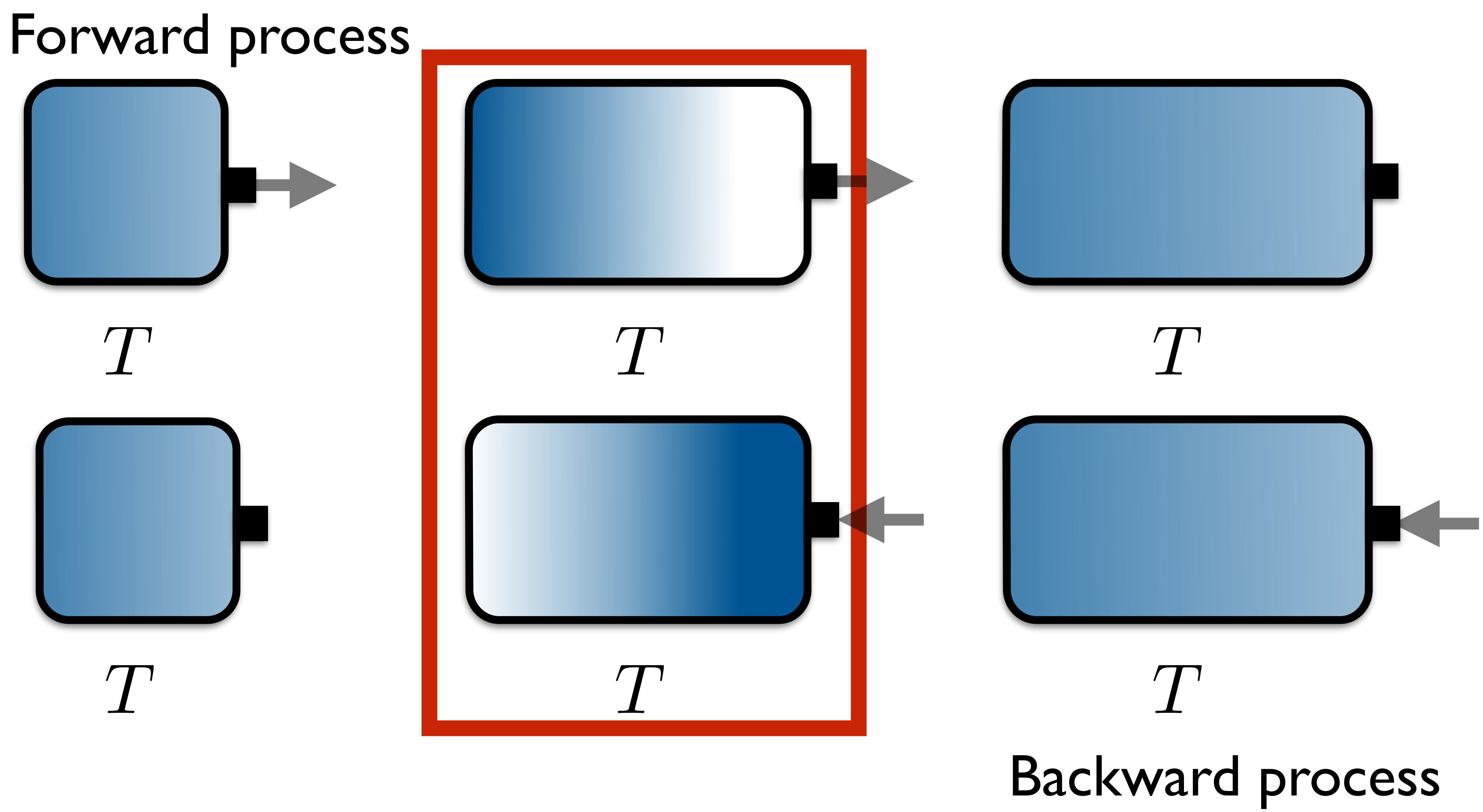
Efficiency of Ferrari



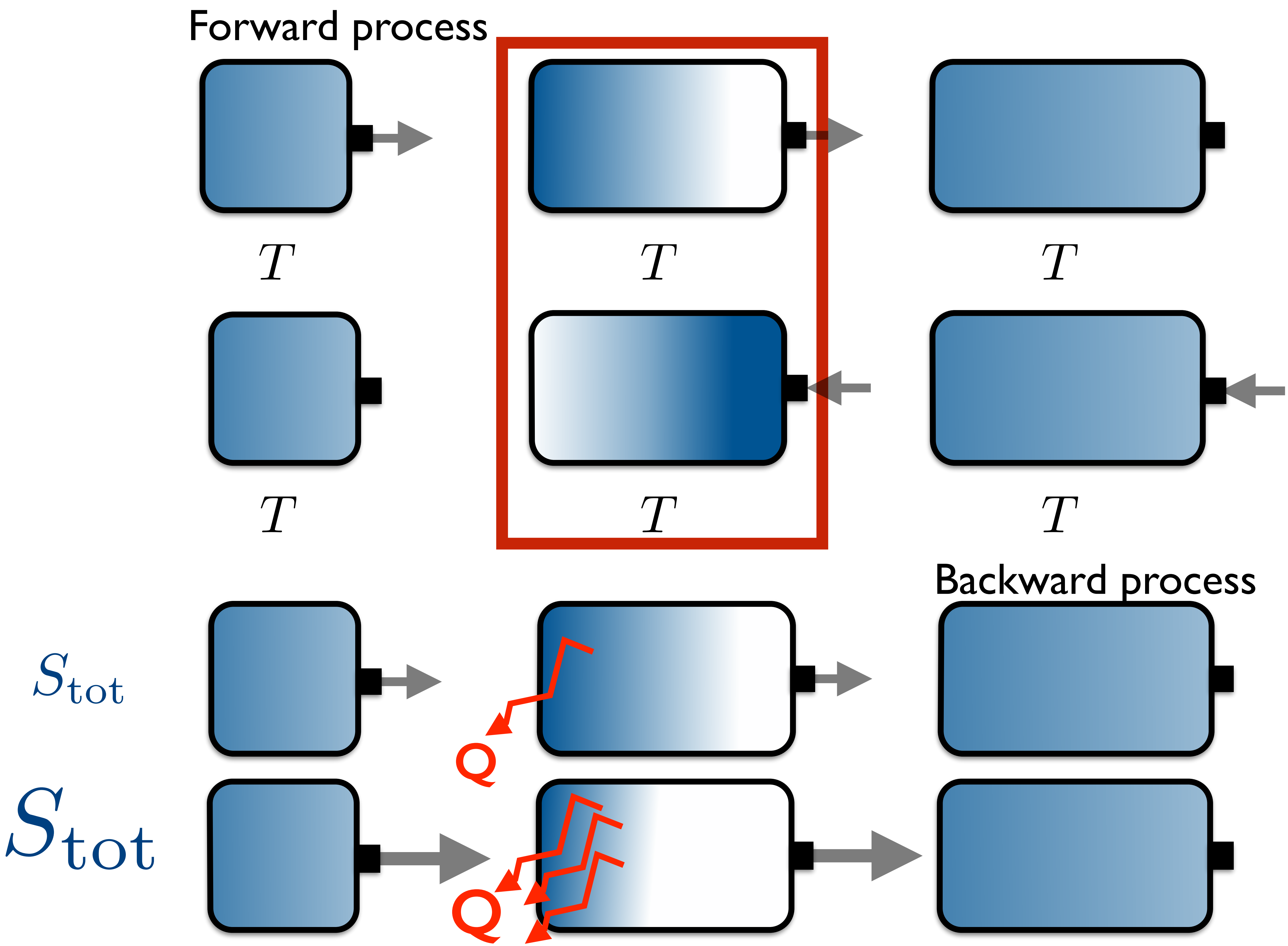
$$\eta \leq \eta_C$$

Carnot efficiency

Entropy production and irreversibility



Entropy production and irreversibility

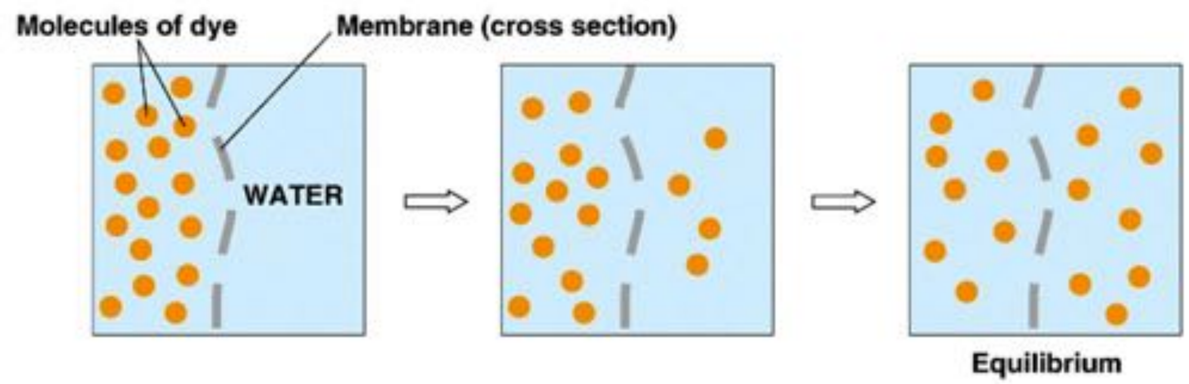


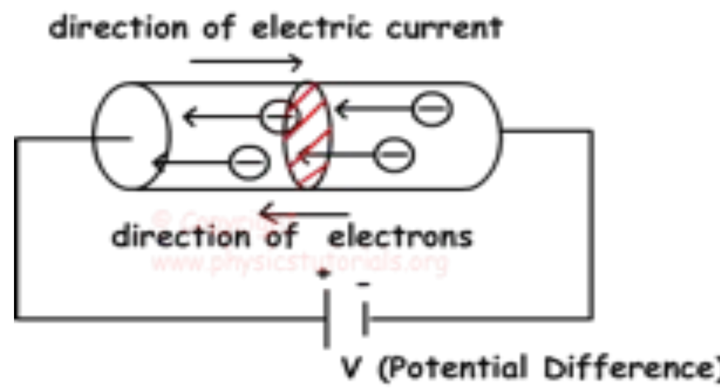
Entropy production and irreversibility

Linear irreversible thermodynamics (near equilibrium)

Ilya Prigogine

Lars Onsager





Force	**Flux**
∇C	J_{part}
V_e	I_e

Rate of entropy production

$$\frac{\mathrm{d}S_{\mathrm{tot}}}{\mathrm{d}t} = \lim_{t\to\infty} \frac{S_{\mathrm{tot}}(t)}{t}$$

$$F_\mu = \sum_\nu L_{\mu\nu} J_\nu \qquad (L_{\mu\nu} = L_{\nu\mu})$$

Onsager reciprocity

$$\frac{\mathrm{d}S_{\mathrm{tot}}}{\mathrm{d}t} \propto \sum_\mu F_\mu J_\mu = \sum_{\mu,\nu} L_{\mu,\nu} J_\mu J_\nu \geq 0$$

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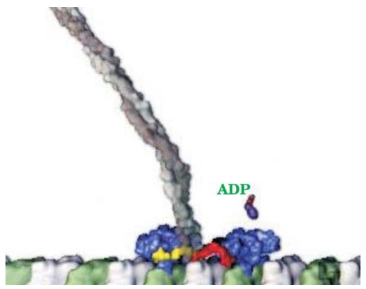
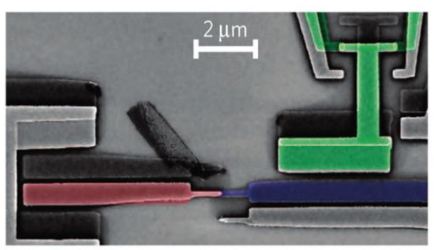
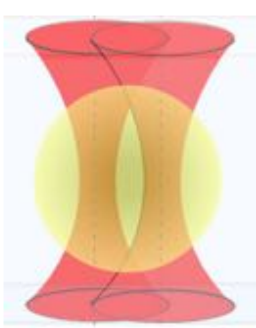
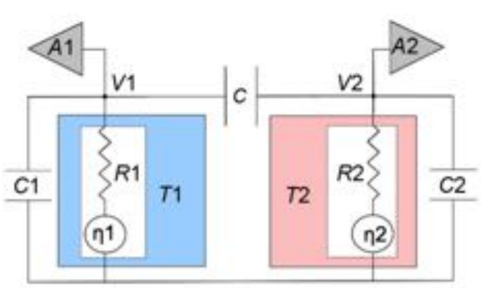
Stochastic thermodynamics

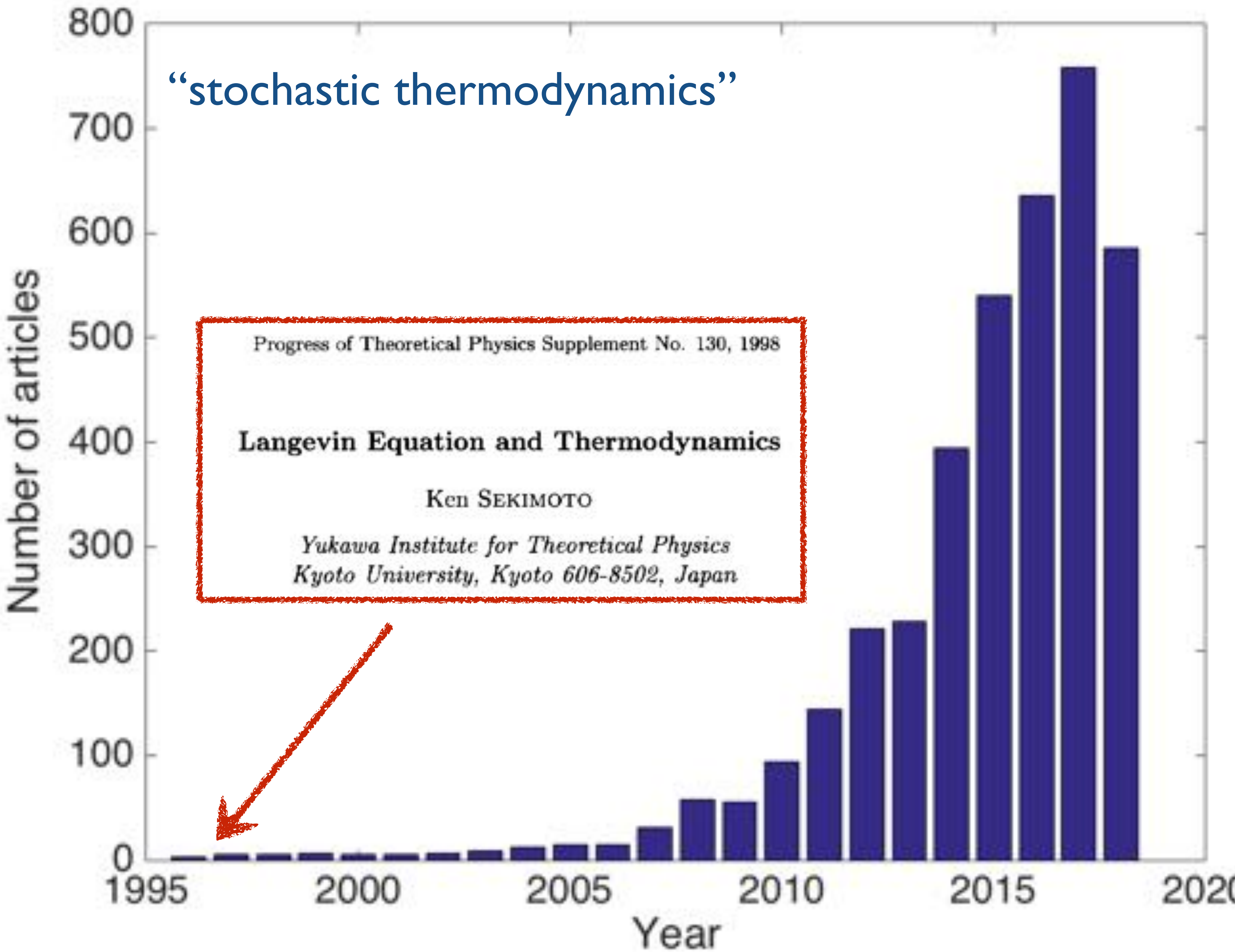
Currently, an **emerging field** within statistical physics
(theory and experiment)

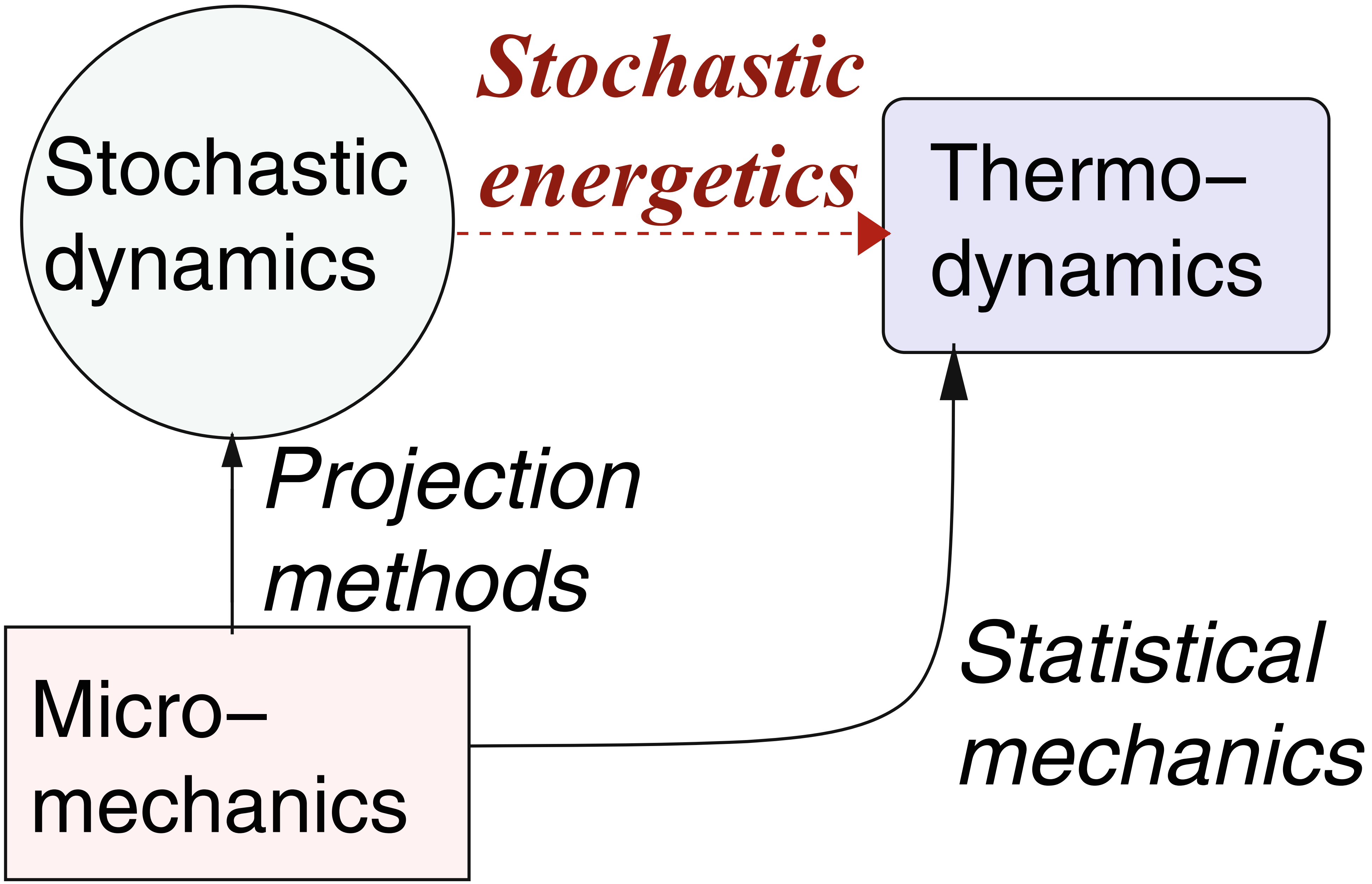






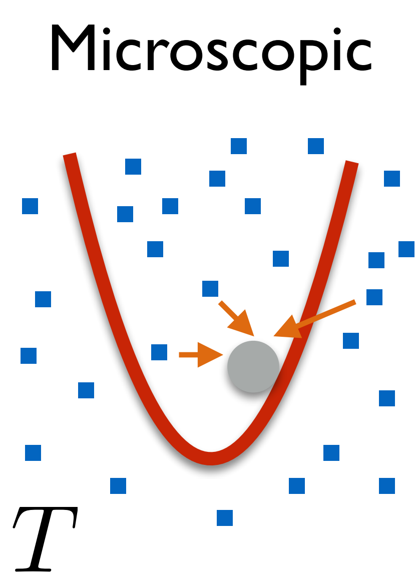


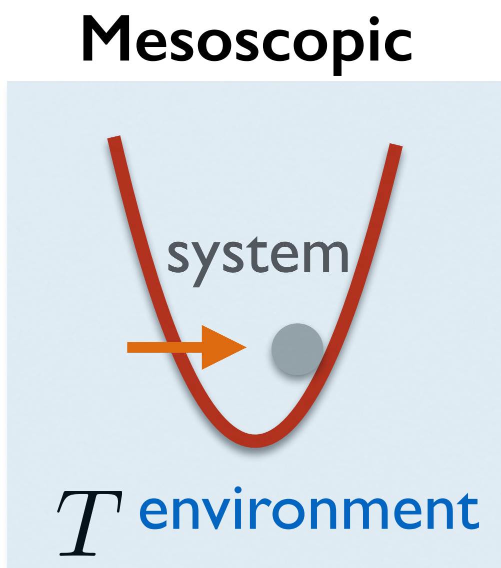


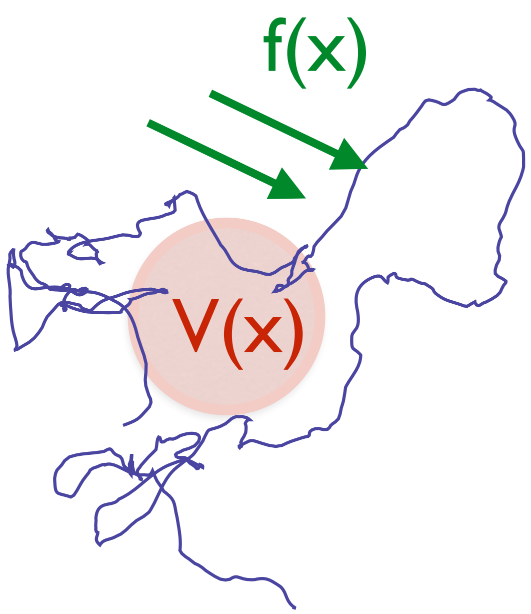


K. Sekimoto, "Stochastic energetics" (Springer, 2010)

Langevin equation







Langevin equation

$$m\ddot{x}(t) = -\gamma\dot{x}(t) - V'(x(t)) + f(x(t);t) + \xi(t)$$

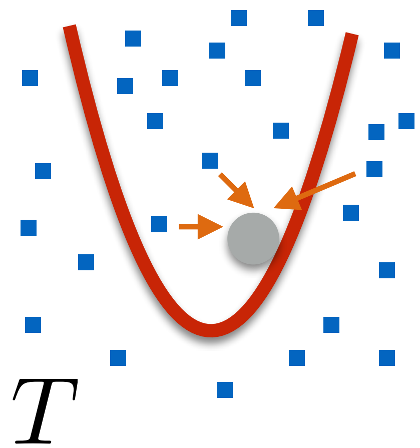
Gaussian white noise $\langle \xi(t) \rangle = 0$

$$\langle \xi(t)\xi(0) \rangle = 2k_{\mathrm{B}}T\gamma\delta(t)$$

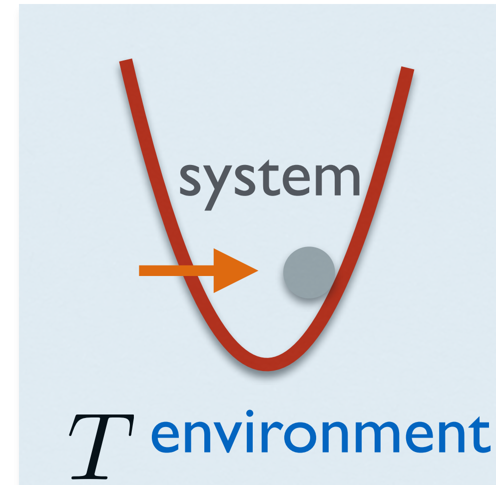
P. Langevin, Compt. Rendus **146**, 530 (1908)

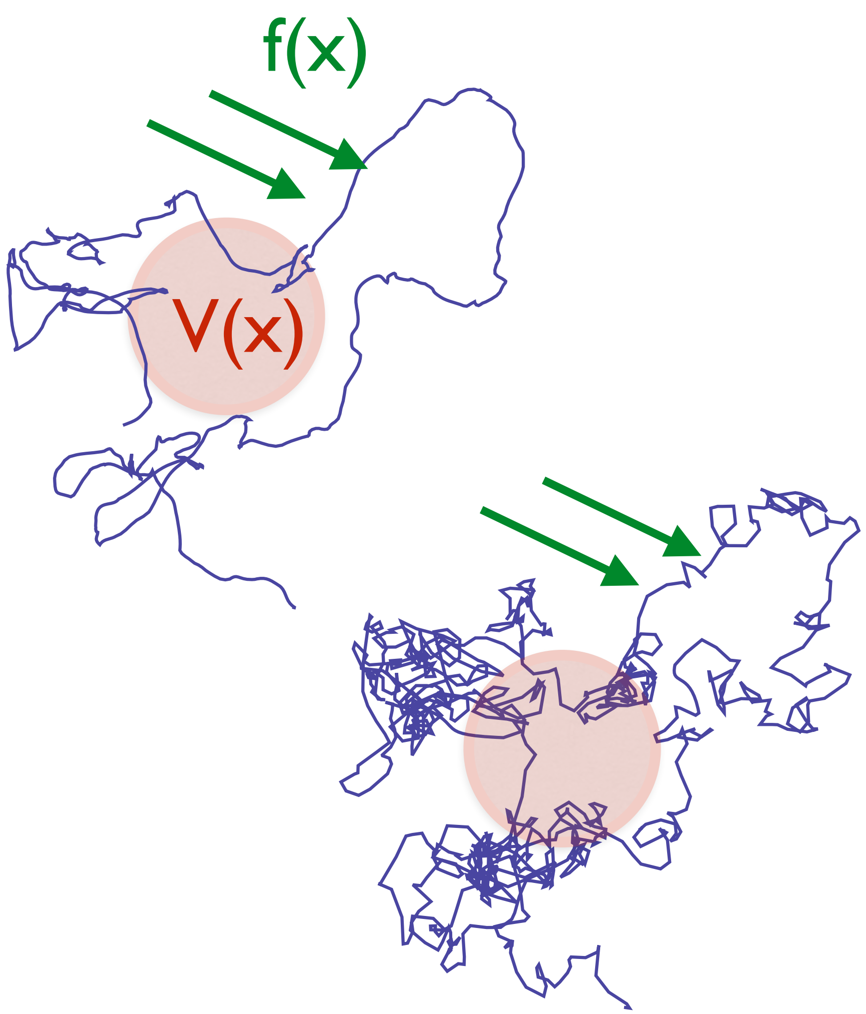
Langevin equation

Microscopic



Mesoscopic





Langevin equation

$$m\ddot{x}(t) = -\gamma\dot{x}(t) - V'(x(t)) + f(x(t);t) + \xi(t)$$

Gaussian white noise $\langle \xi(t) \rangle = 0$

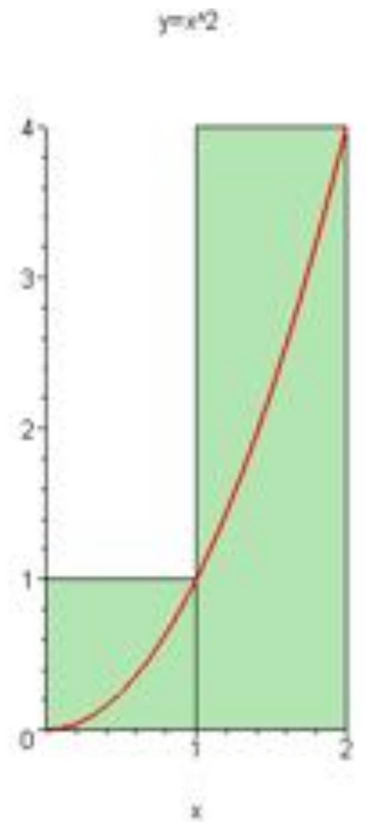
$$\langle \xi(t)\xi(0) \rangle = 2k_{\mathrm{B}}T\gamma\delta(t)$$

Overdamped Langevin equation $(\Delta t \gg m/\gamma)$

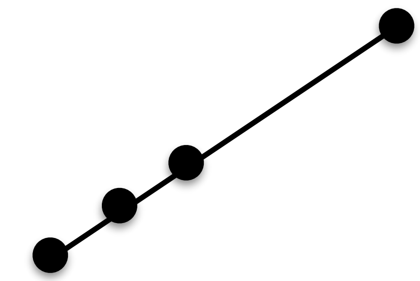
$$\gamma\dot{x}(t) = -V'(x(t)) + f(x(t);t) + \xi(t)$$

P. Langevin, Compt. Rendus **146**, 530 (1908)

Primer on stochastic calculus



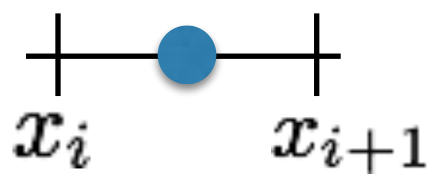
Standard calculus $a = x_0 < x_1 < x_2 < \cdots < x_n = b$



$$I = \int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

$$\Delta x_i = x_i - x_{i-1}$$

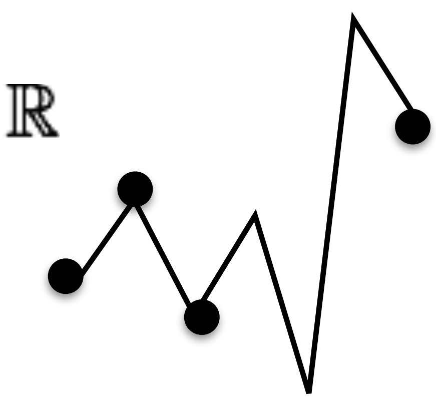
$$x_i^* \in [x_i, x_{i+1}]$$



Midpoint rule

$$x_i^* = (x_i + x_{i+1})/2$$

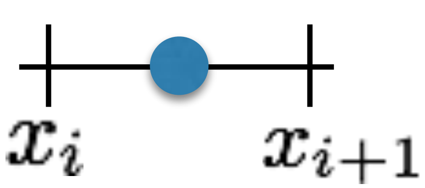
Stochastic calculus: integration over a trajectory $x_0, x_1, x_2, \ldots, x_n \in \mathbb{R}$



$$I = \int_{x_0}^{x_n} f(x(s))\mathrm{d}x(s) = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

$$x_i^* \in [x_i, x_{i+1}]$$

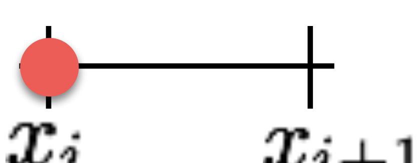
Stratonovich calculus (midpoint rule)



$$x_i^* = (x_i + x_{i+1})/2$$

$$\int_{x_0}^{x_n} f(x(s)) \circ \mathrm{d}x(s)$$

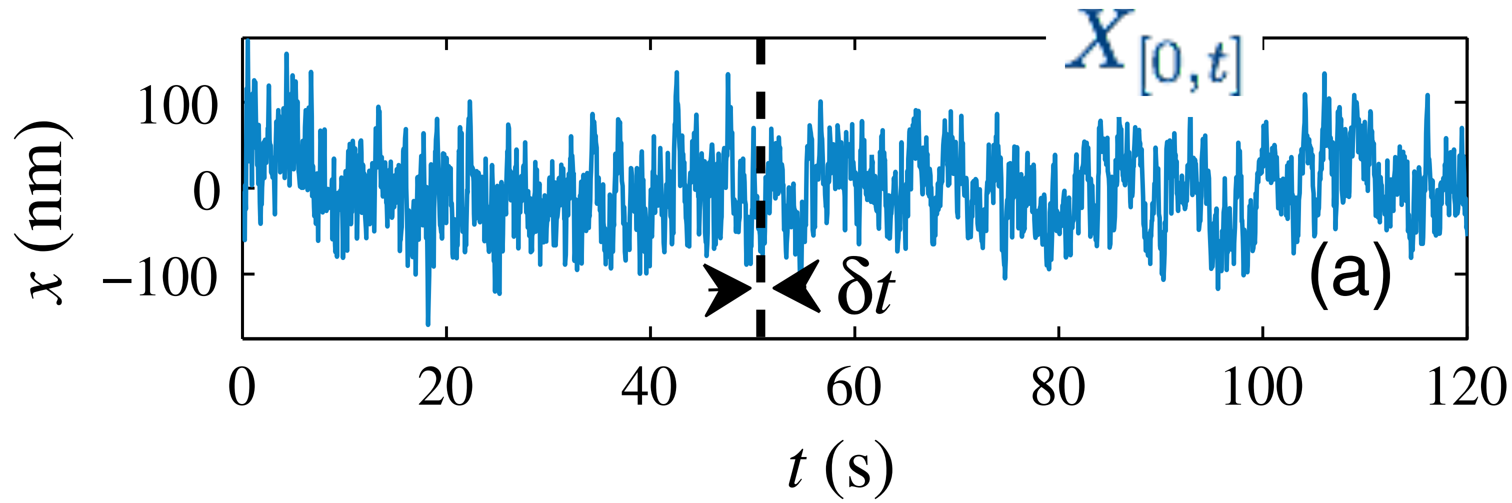
Ito calculus



$$x_i^* = x_i$$

$$\int_{x_0}^{x_n} f(x(s)) \cdot \mathrm{d}x(s)$$

Stochastic heat



How much heat Q(t) is exchanged between the particle and the bath in a **single** trajectory?

$$0 = -\gamma\dot{x}(t) - V'(x(t)) + f(x(t);t) + \xi(t)$$

Heat absorbed by the x degree of freedom in [t,t+dt]

$$\delta Q(t) = [-\gamma\dot{x}(t) + \xi(t)] \circ \mathrm{d}x(t)$$

Particle's instataneous displacement

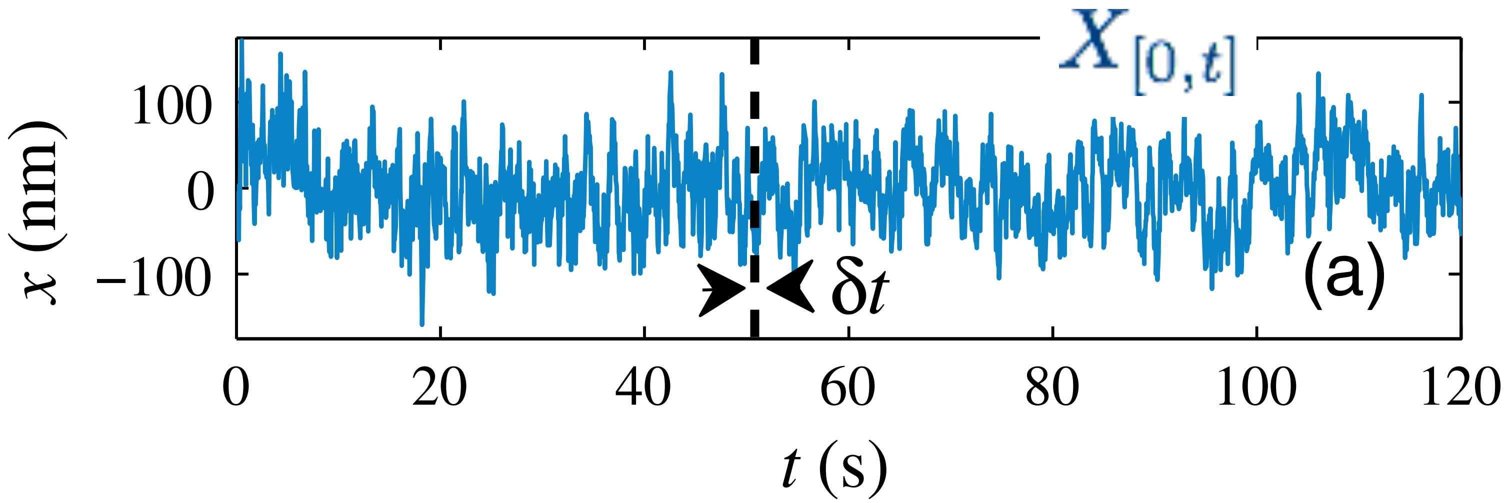
Environment's force

$$= [V'(x(t)) - f(x(t);t)] \circ \mathrm{d}x(t)$$

K. Sekimoto, Prog. Theor. Phys. Suppl. 130, 17 (1998)

$\circ$ = **Stratonovich** product

Stochastic heat



How much heat Q(t) is exchanged between the particle and the bath in a **single** trajectory?

$$0 = -\gamma\dot{x}(t) - V'(x(t)) + f(x(t);t) + \xi(t)$$

Heat absorbed by the x degree of freedom in [0,t]

$$Q(t) \equiv Q[X_{[0,t]}] = \int_{x(0)}^{x(t)} \left[\frac{\partial V(x(s))}{\partial x} + f(x(s);s)\right] \circ \mathrm{d}x(s)$$

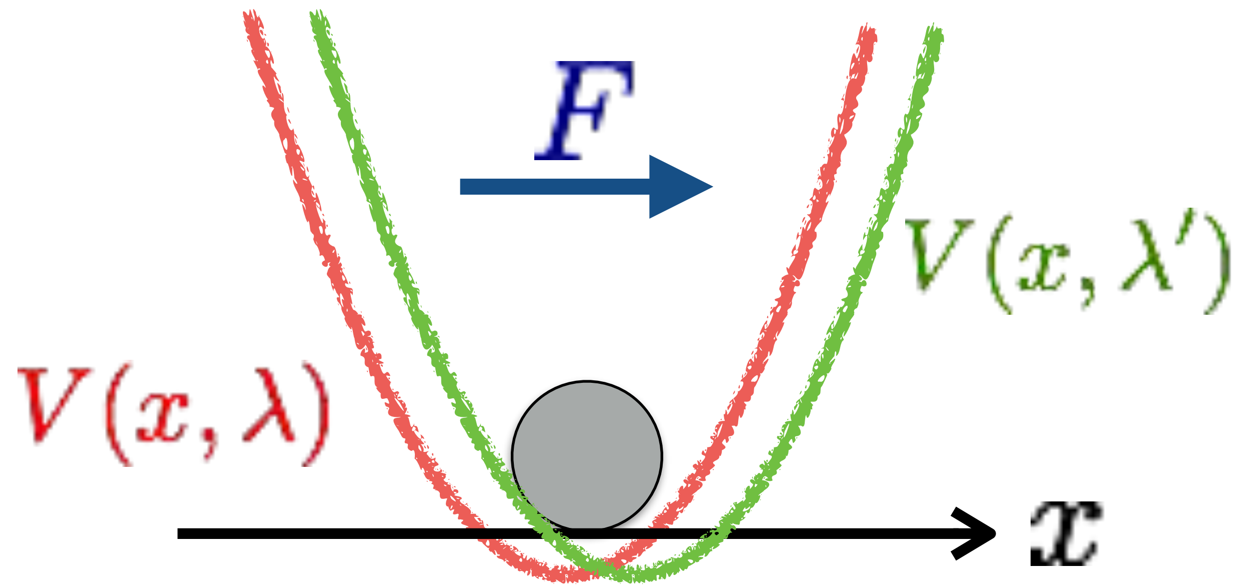
a **functional of the stochastic trajectory** $X_{[0,t]}$

[not a state function!]

K. Sekimoto, Prog. Theor. Phys. Suppl. 130, 17 (1998)

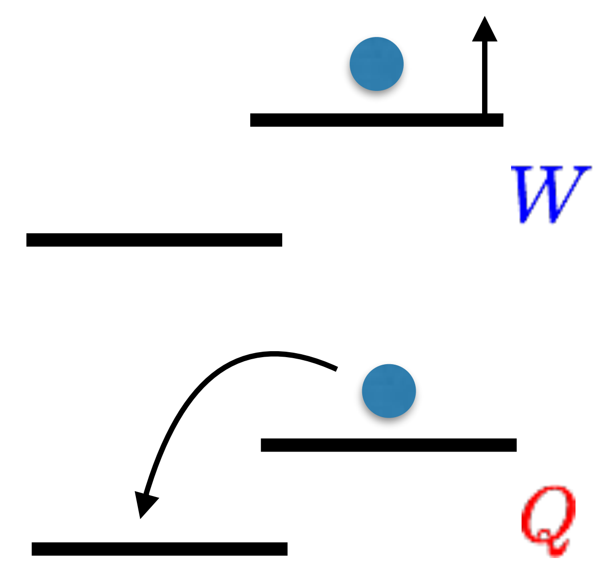
First law of stochastic thermodynamics

Work exerted on the particle in [t,t+dt]



$$\delta W(t) = \partial_\lambda V(x(t))\mathrm{d}\lambda + f(x(t);t)\circ \mathrm{d}x(t)$$

$$+ \quad \delta Q(t) = [V'(x(t)) - f(x(t);t)]\circ \mathrm{d}x(t)$$



$$\delta W(t) + \delta Q(t) = \partial_\lambda V(x(t))\mathrm{d}\lambda + \partial_x V(x(t))\circ \mathrm{d}x(t) = \mathrm{d}V(t)$$

$$\boxed{\delta W(t) + \delta Q(t) = \mathrm{d}V(t)}$$

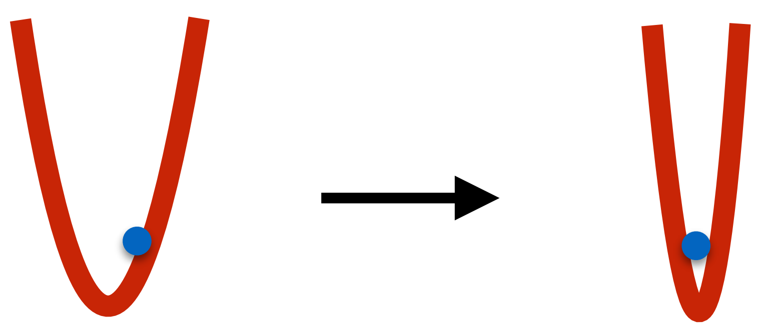
K. Sekimoto, Prog. Theor. Phys. Suppl. 130, 17 (1998)

Heat and work along a trajectory

Simple case: no external, non-conservative force (f=0)

$$Q(t) = \int_{x(0)}^{x(t)} \frac{\partial V(x(s))}{\partial x} \circ \mathrm{d}x(s) \qquad W(t) = \int_{x(0)}^{x(t)} \frac{\partial V(x(s))}{\partial \lambda} \mathrm{d}\lambda$$

Colloidal **microscopic** particle in a harmonic potential

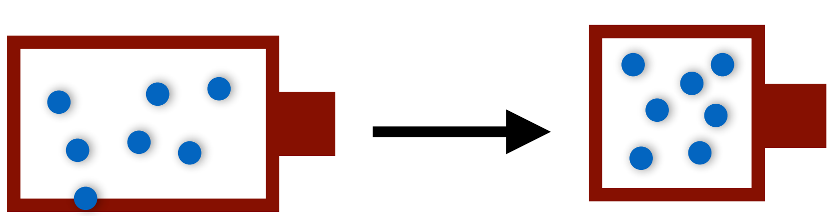


$$V(x,t) = \frac{1}{2}\kappa(t)x^2$$

$$W(t) = \int_{x(0)}^{x(t)} \boxed{\frac{x^2(s)}{2}} \mathrm{d}\kappa(s)$$

stochastic

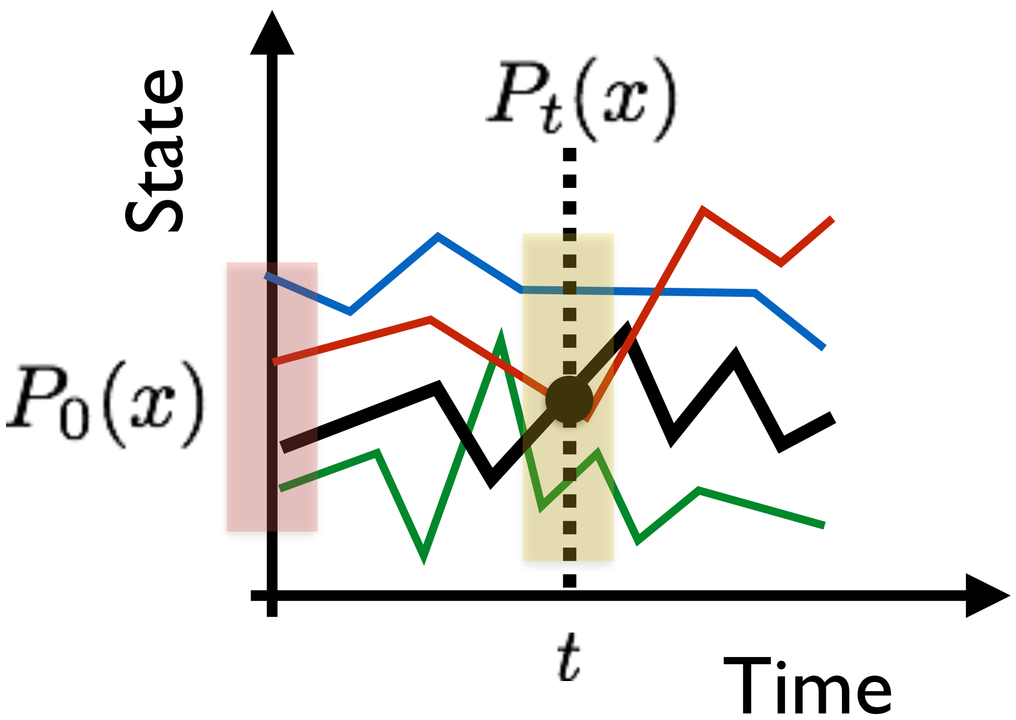
Ideal **macroscopic** gas in a movable piston



$$W(t) = -\int_{V(0)}^{V(t)} \boxed{p(s)} \mathrm{d}V(s)$$

deterministic

Stochastic entropy



Nonequilibrium system's entropy

$$S(t) \equiv -k_{\mathrm{B}} \ln P_t(x(t))$$

Average over many realizations

$$\langle S(t) \rangle / k_{\mathrm{B}} = \underbrace{-\int \mathrm{d}x P_t(x) \ln P_t(x)}_{=S_{\mathrm{GS}}}$$

System entropy change along a single trajectory

$$\Delta S(t) = k_{\mathrm{B}} \ln \frac{P_0(x(0))}{P_t(x(t))}$$

Second law?

$$\Delta S(t) - \frac{Q(t)}{T} \equiv S_{\mathrm{tot}}(t) \underbrace{\geq 0}_{???}$$

U. Seifert, Phys. Rev. Lett. **95**(4) 040602 (2005)

Second law and stochastic entropy

System's entropy change $$\Delta S(t) = k_B \ln \frac{P_0(x(0))}{P_t(x(t))}$$

Environment's entropy change $$S_e(t) = -\frac{Q(t)}{T}$$

Stochastic Entropy production

$$\Delta S(t) = \underbrace{-S_e(t)}_{\text{environment}} + \underbrace{S_{\text{tot}}(t)}_{\text{production}} \Rightarrow S_{\text{tot}}(t) = \Delta S(t) + S_e(t)$$

$$S_{\text{tot}}(t) = k_B \ln \frac{\mathcal{P}(X_{[0,t]})}{\widetilde{\mathcal{P}}(\tilde{X}_{[0,t]})}$$

trajectory

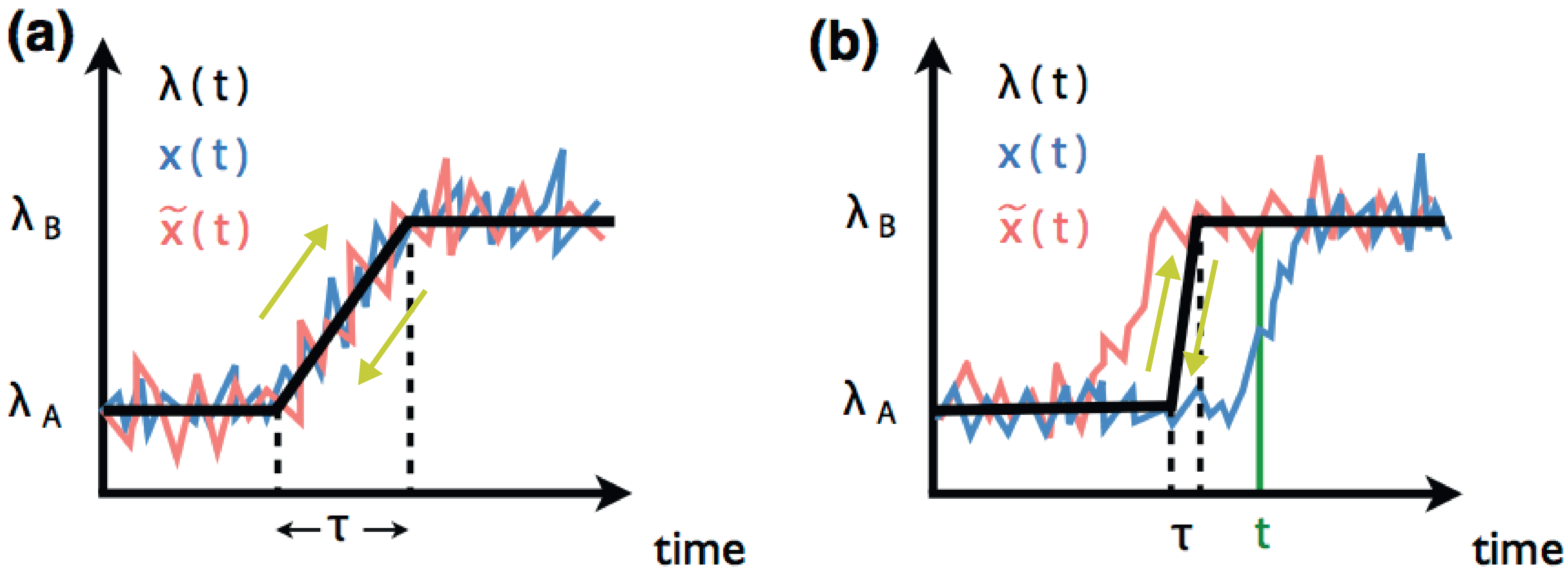
time-reversed trajectory

(***key assumption** local detailed balance: bath/s in thermal equilibrium)

U. Seifert, Phys. Rev. Lett. **95**(4) 040602 (2005)

Second law and stochastic entropy

$$\langle S_{\rm tot}(t)\rangle = k_{\rm B}\int \mathcal{D}X_{[0,t]}\,\mathcal{P}\big(X_{[0,t]}\big)\ln\frac{\mathcal{P}(X_{[0,t]})}{\widetilde{\mathcal{P}}(\tilde{X}_{[0,t]})} = k_{\rm B}\,\underbrace{D\Big[\mathcal{P}\big(X_{[0,t]}\big)\|\widetilde{\mathcal{P}}\big(\tilde{X}_{[0,t]}\big)\Big]}_{\text{KL divergence}} \geq 0$$



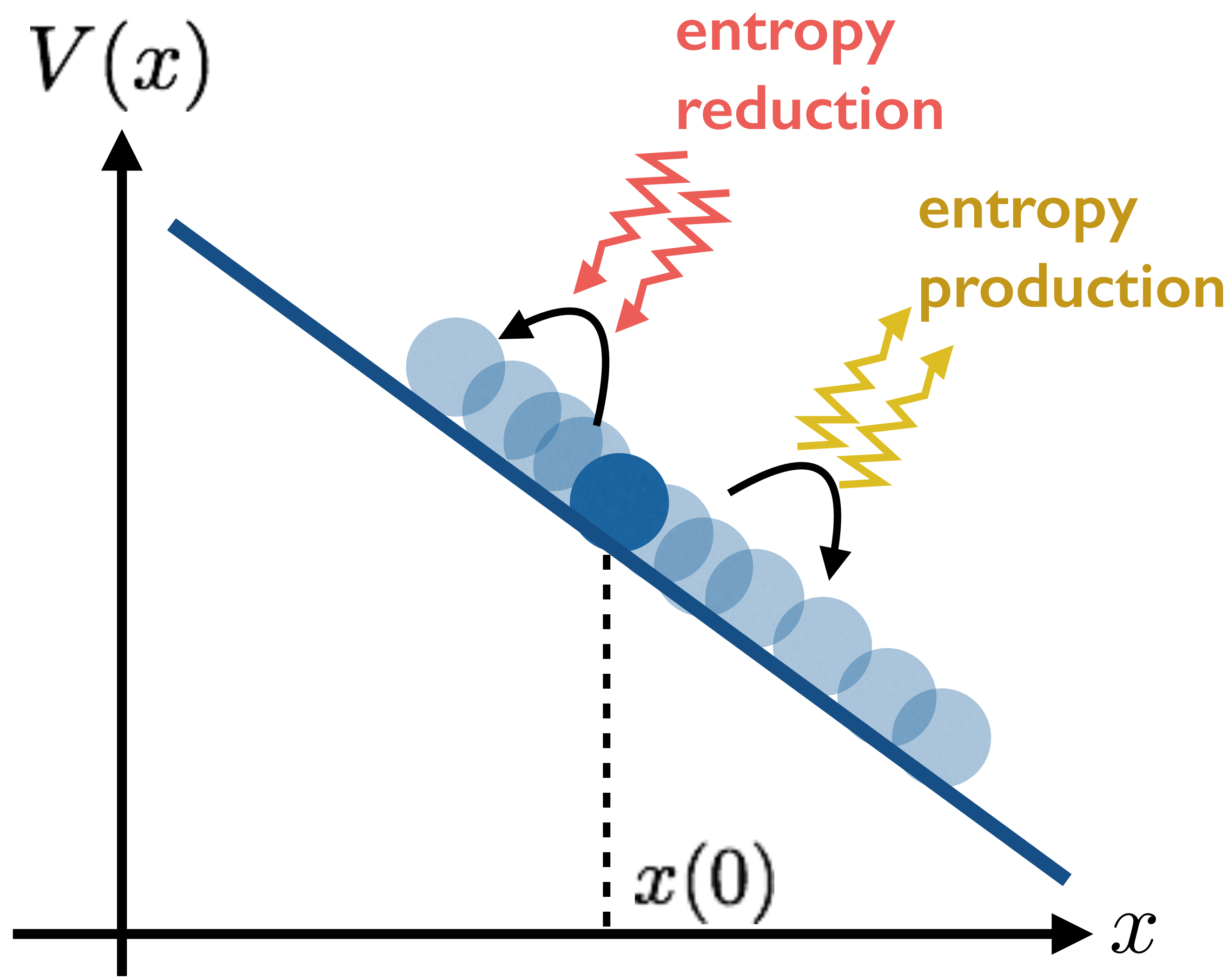
$\langle S_{\rm tot}(t)\rangle \simeq 0$ $\qquad$ $\langle S_{\rm tot}(t)\rangle \gg 0$

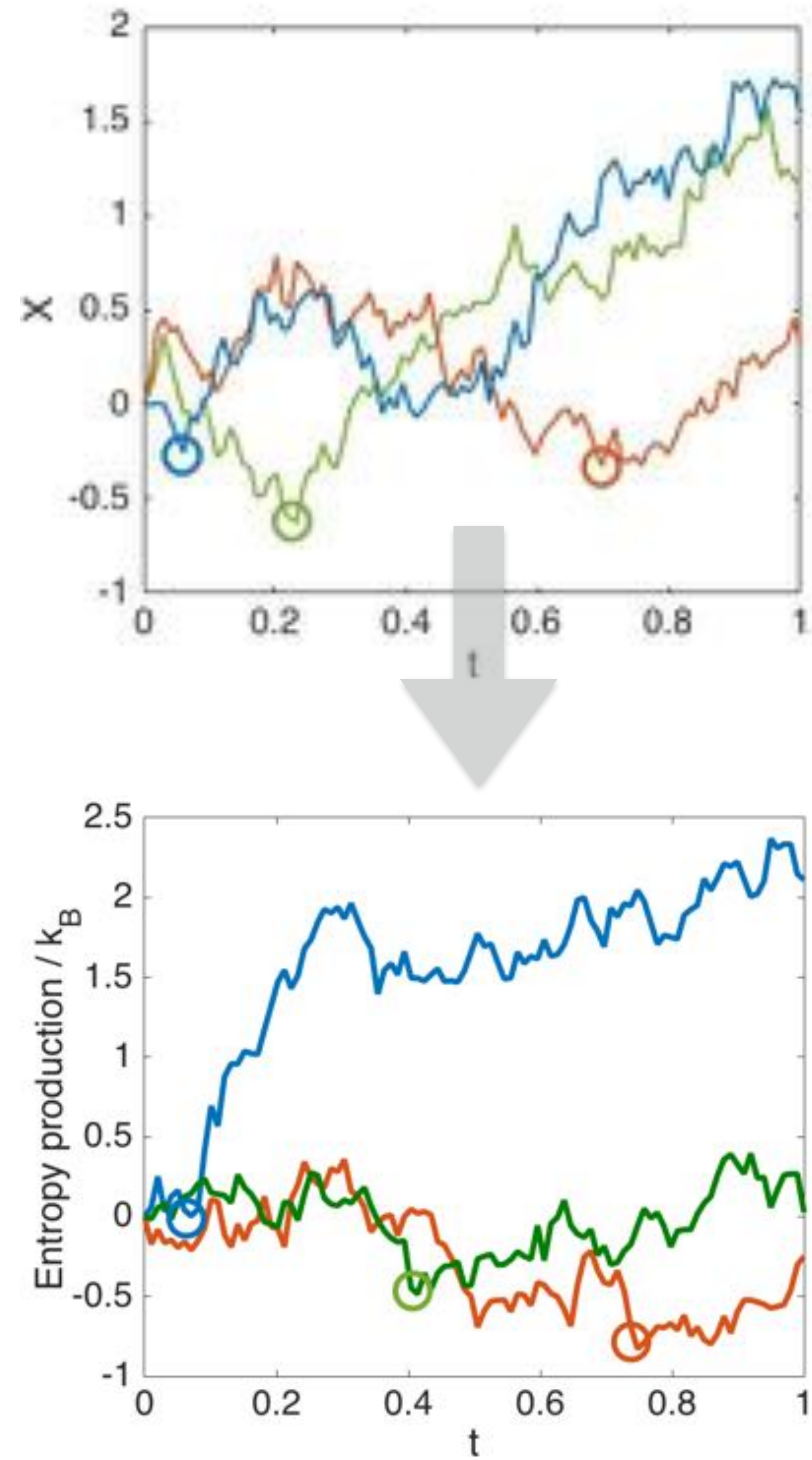
However, stochastic entropy production $S_{\rm tot}(t)$ can be negative when a rare trajectory occurs !

Negative stochastic entropy production

$$\dot{x}(t) = v + \sqrt{2D}\xi(t)$$

$$\Rightarrow S_{\rm tot}(t) = \frac{k_{\rm B}v}{D}[x(t) - x(0)]$$

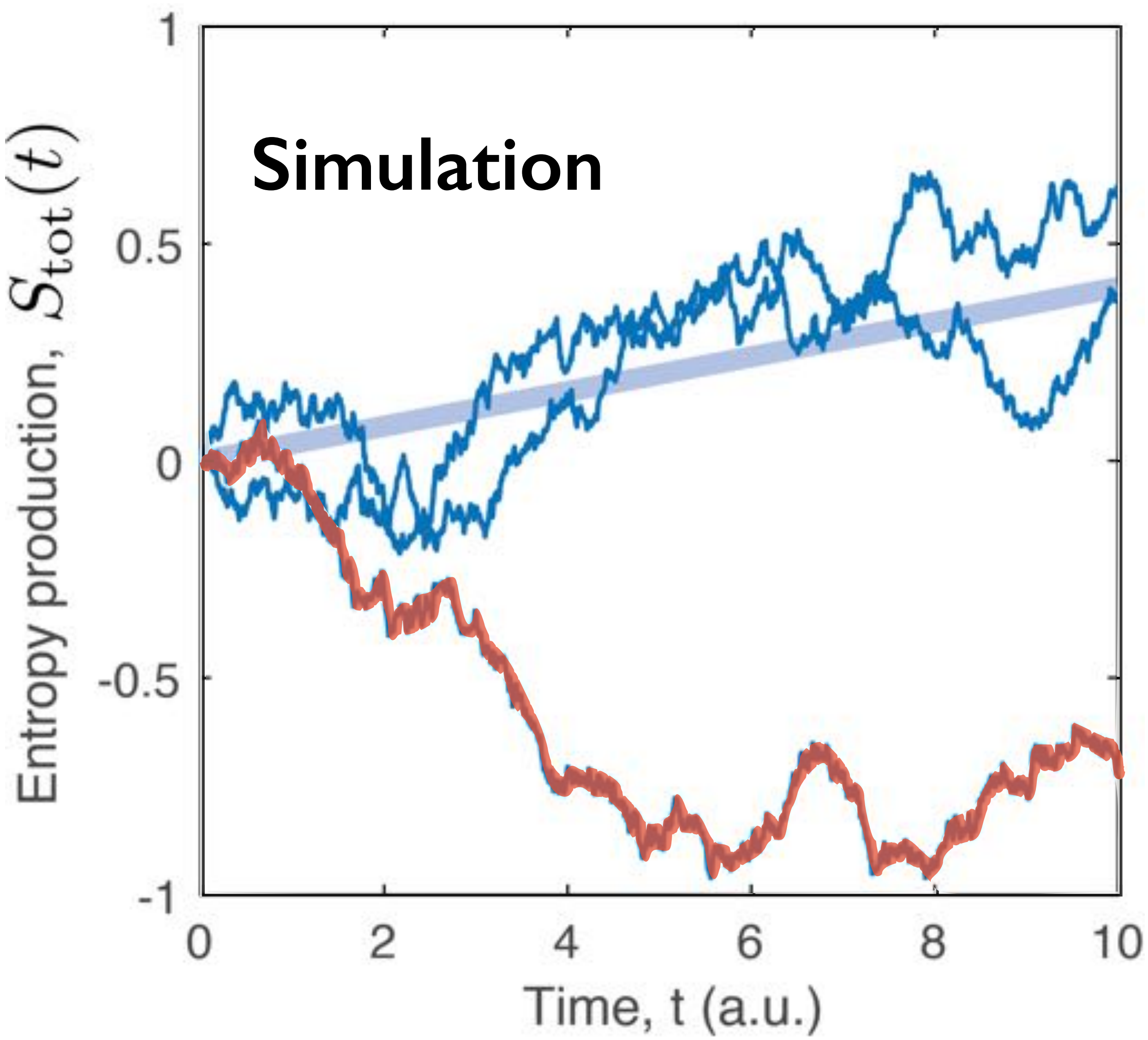


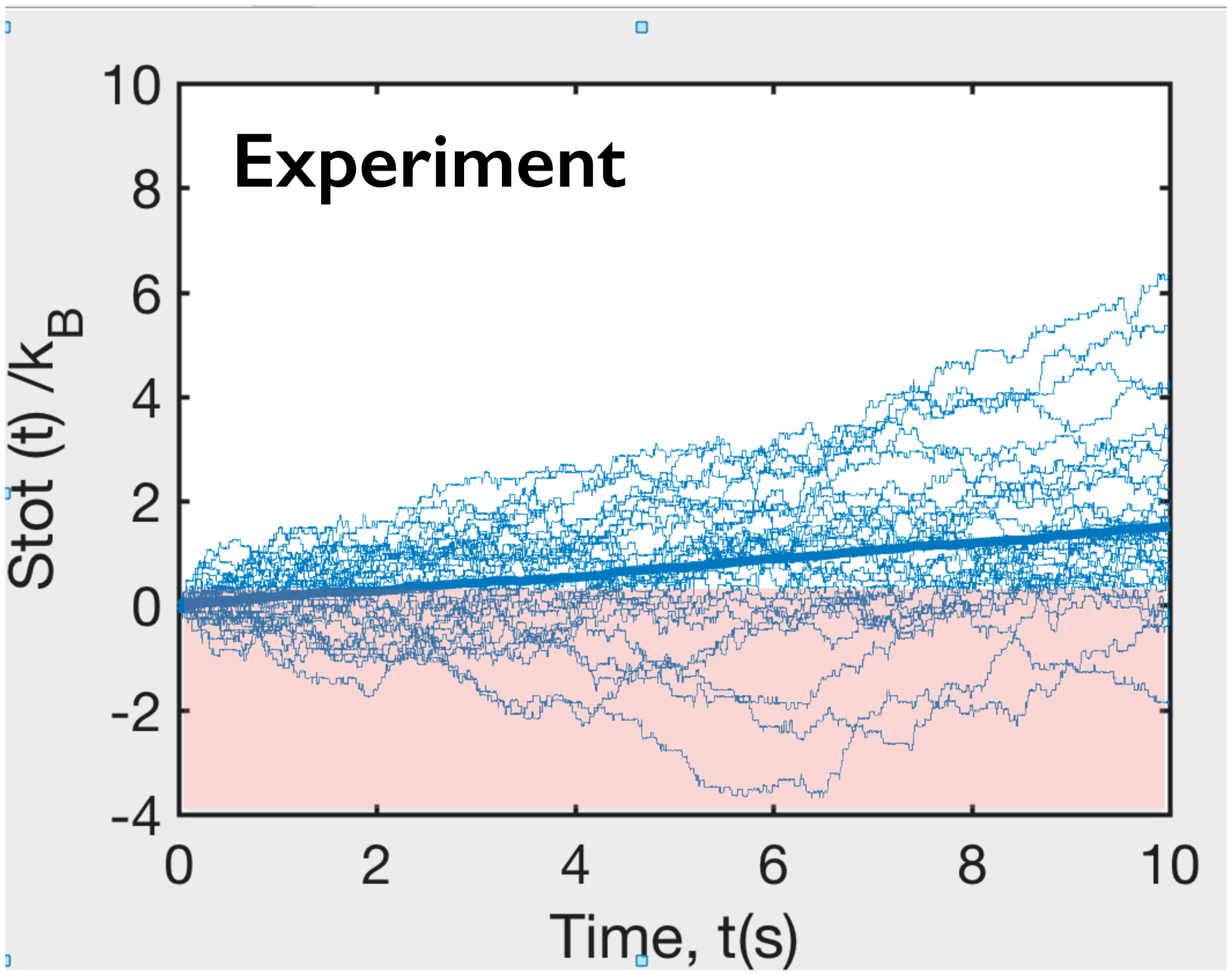


Stochastic entropy production

Equilibrium : $S_{\text{tot}}(t) = 0 \Rightarrow \langle S_{\text{tot}}(t) \rangle = 0$

Nonequilibria: $\langle S_{\text{tot}}(t) \rangle \geq 0$



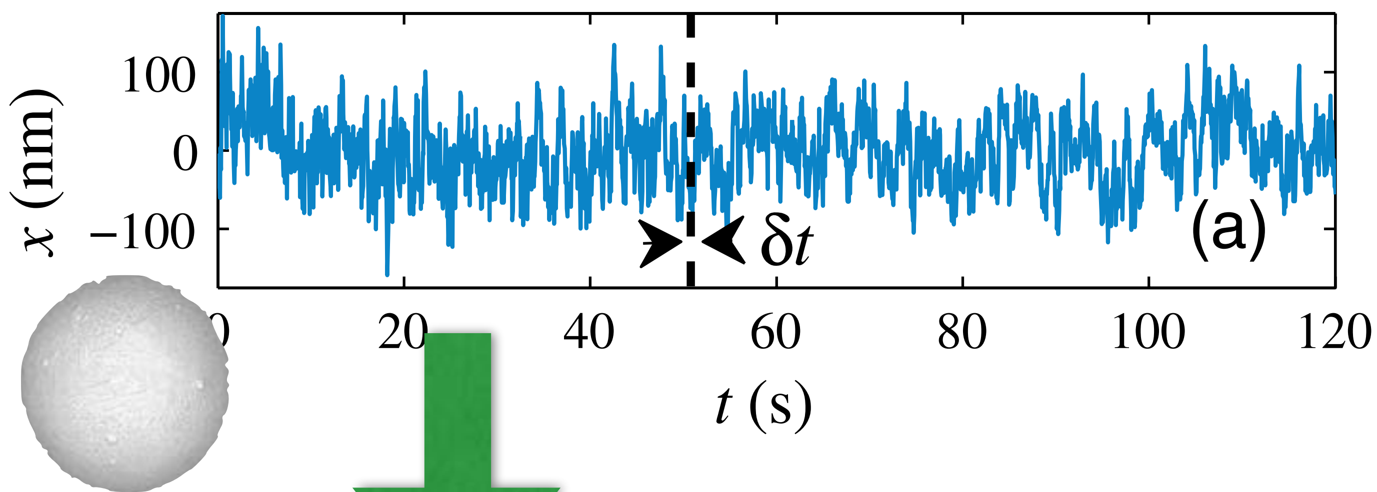


Shilpi Singh (Pekola Lab)

Relation to financial concepts

Dynamical trajectory

Particle's position

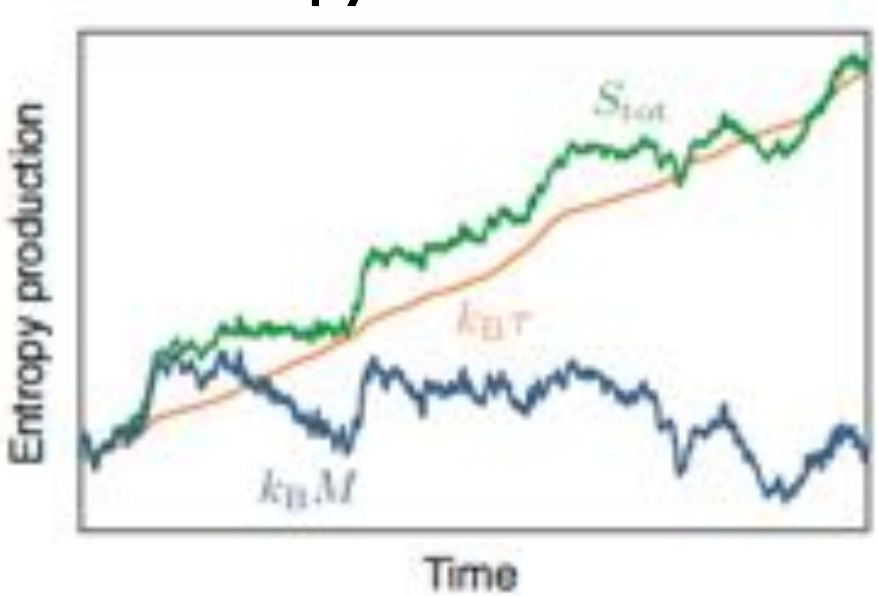


Financial Asset

Price of Gold

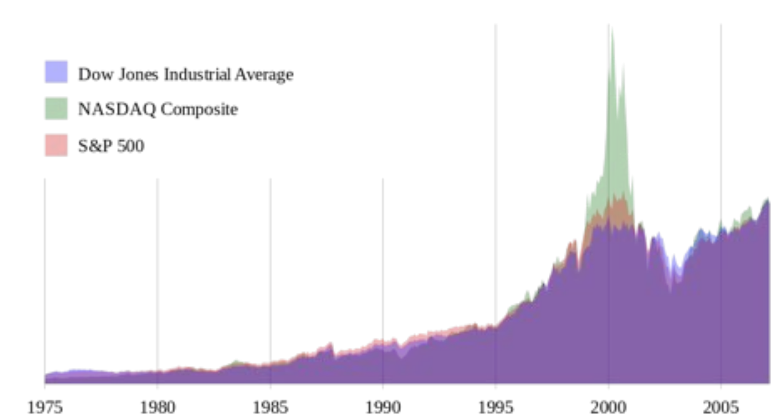
Thermodynamic quantities

Entropy, heat, work…



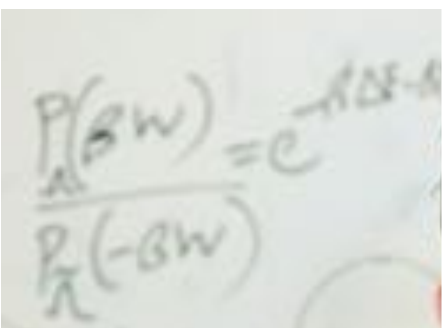
Financial Derivatives?

Market index?



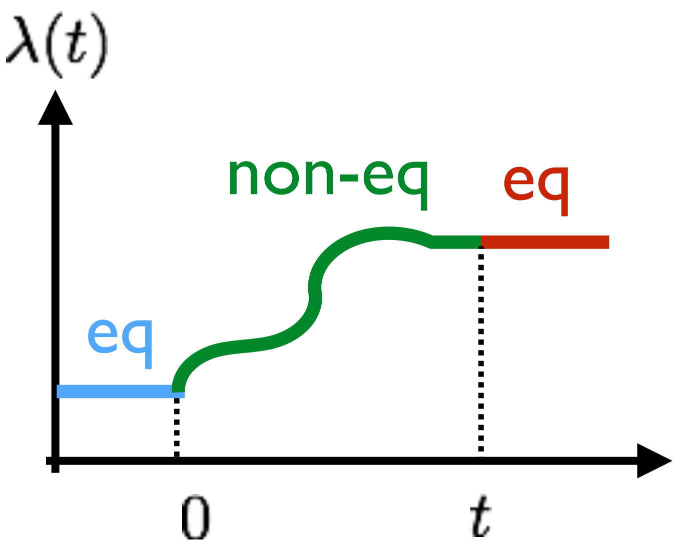
Plot of this talk

- Brief review of classical “textbook” thermodynamics
- Stochastic thermodynamics: first and second laws

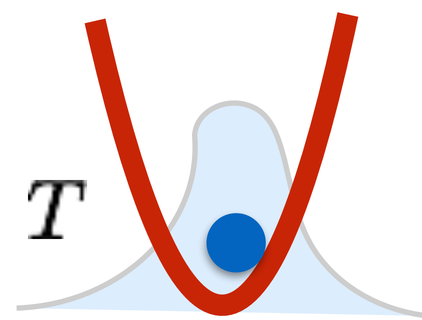


- **Fluctuation theorems**
- Maxwell’s demons and feedback control
- The latest: Uncertainty relations and Martingales

Jarzynski's equality

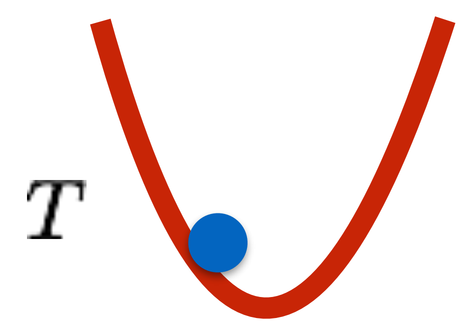


Initial state
equilibrium

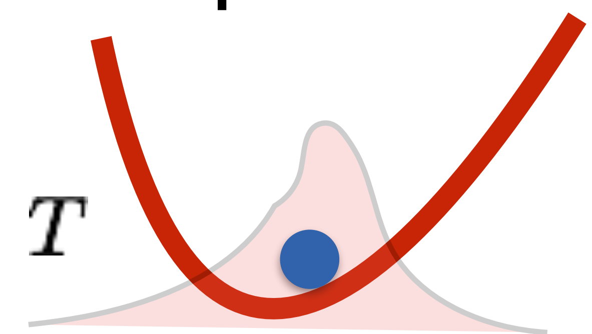


$$P_0(x) = \frac{e^{-H(x;0)/k_\mathrm{B}T}}{Z(0)}$$

Non-equilibrium
process



Final state
equilibrium



$$P_t(x) = \frac{e^{-H(x;t)/k_\mathrm{B}T}}{Z(t)}$$

Non-equilibrium equality for any protocol **arbitrarily far from equilibrium**

$$\langle e^{-\beta W(t)} \rangle = e^{-\beta \Delta F}$$

where W(t) is the **nonequilibrium** work $W(t) = \int_{x(0)}^{x(t)} \frac{\partial V(x(s))}{\partial \lambda} \mathrm{d}\lambda$

and the **equilibrium** free energy change $\Delta F = -k_\mathrm{B} T \ln(Z_\mathrm{fin}/Z_\mathrm{ini})$

C. Jarzynski, Phys. Rev. Lett. **78**, 269 (1997)

Jarzynski's equality

Non-equilibrium equality for any protocol **arbitrarily far from equilibrium**

$$\langle e^{-\beta[W(t)-\Delta F]}\rangle = 1$$

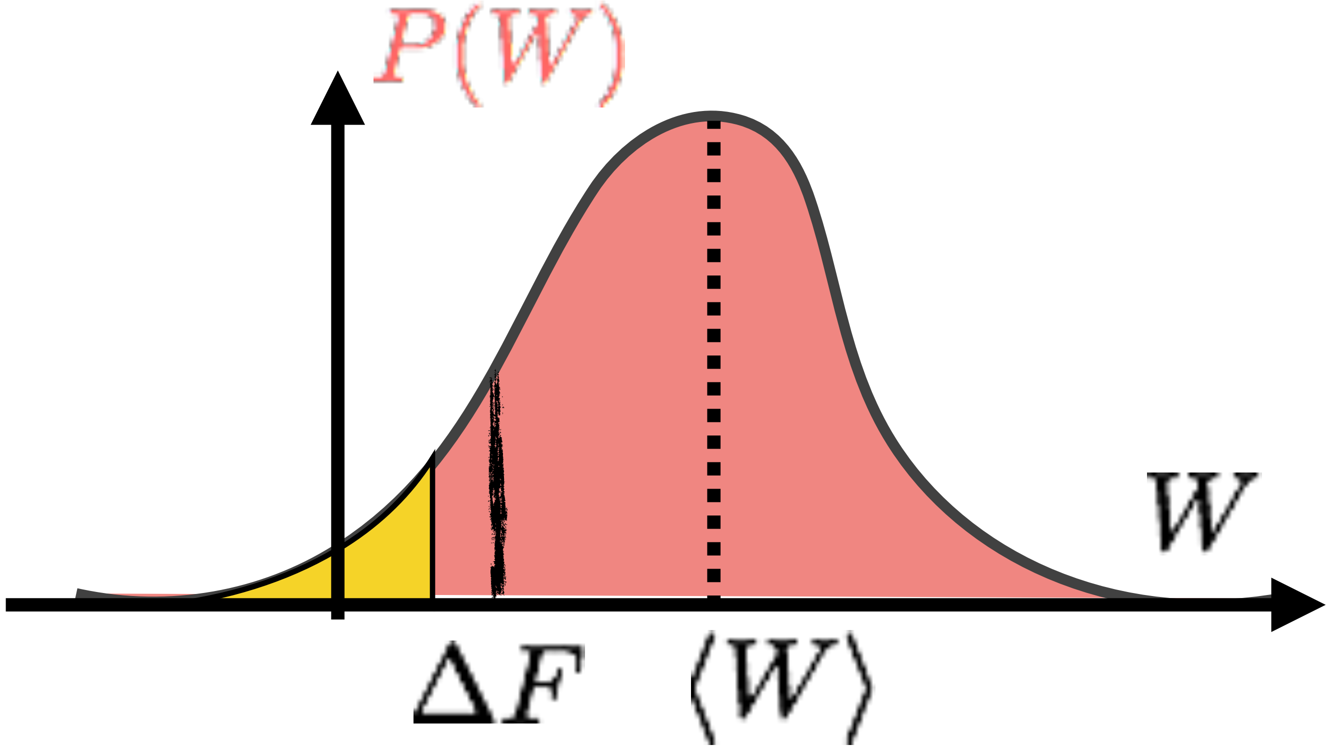
Jensen's inequality $\langle e^{x}\rangle \geq e^{\langle x\rangle}$

$$\langle W(t)\rangle \geq \Delta F$$

Probability for "second law violations"

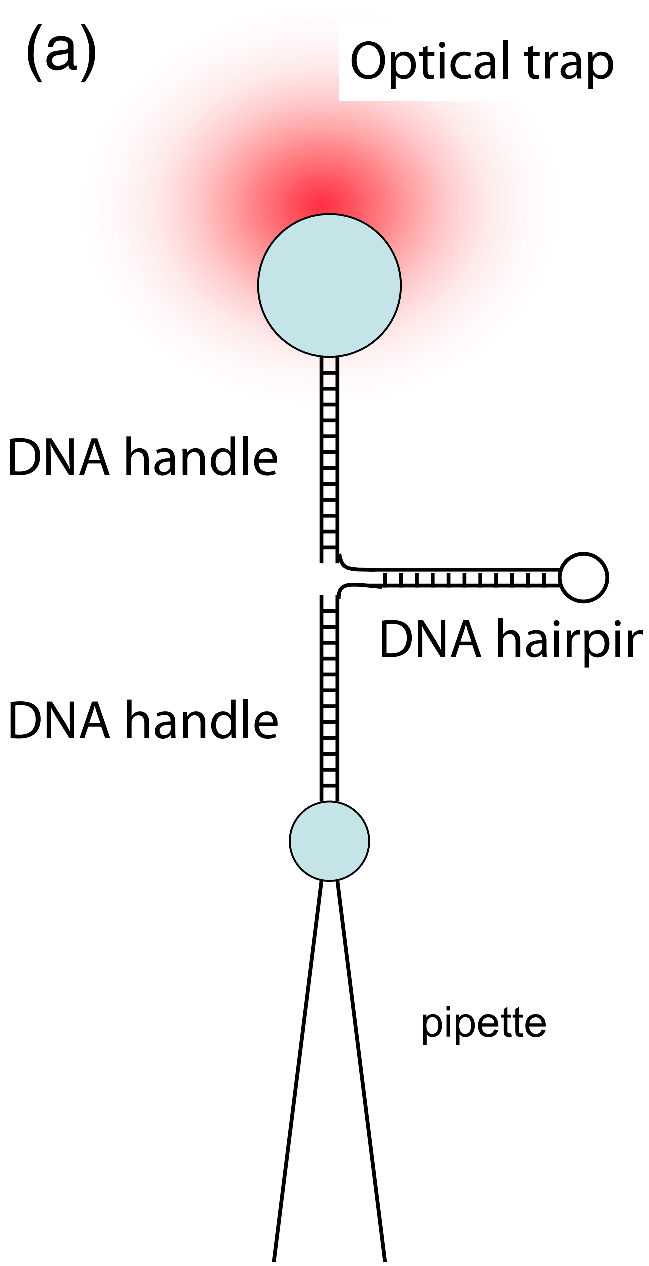
$$\Pr\big(W(t) - \Delta F \leq -\xi\big) \leq e^{-\xi/k_{\mathrm{B}}T}$$

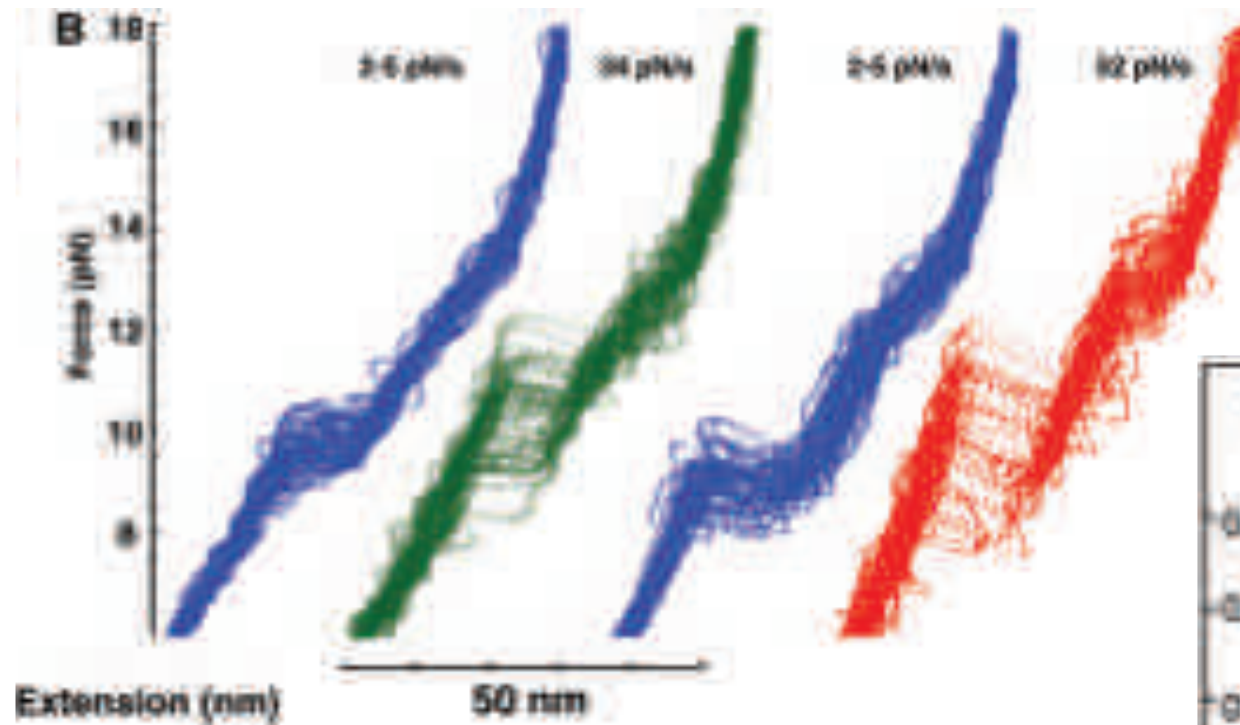
4 October 2019
Jarzynski @ ICTP Colloquium

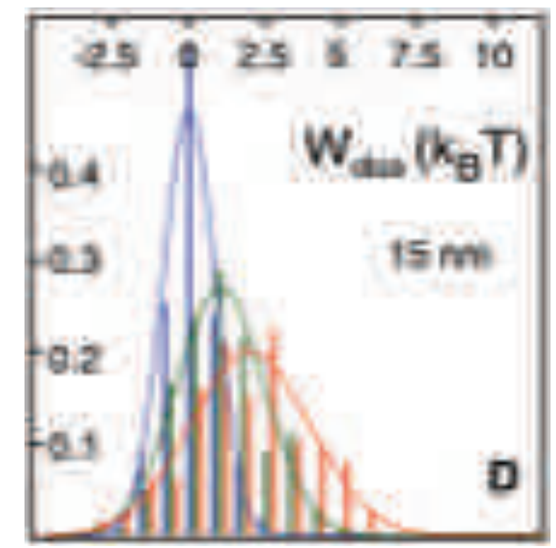


C. Jarzynski, Phys. Rev. Lett. **78**, 269 (1997)

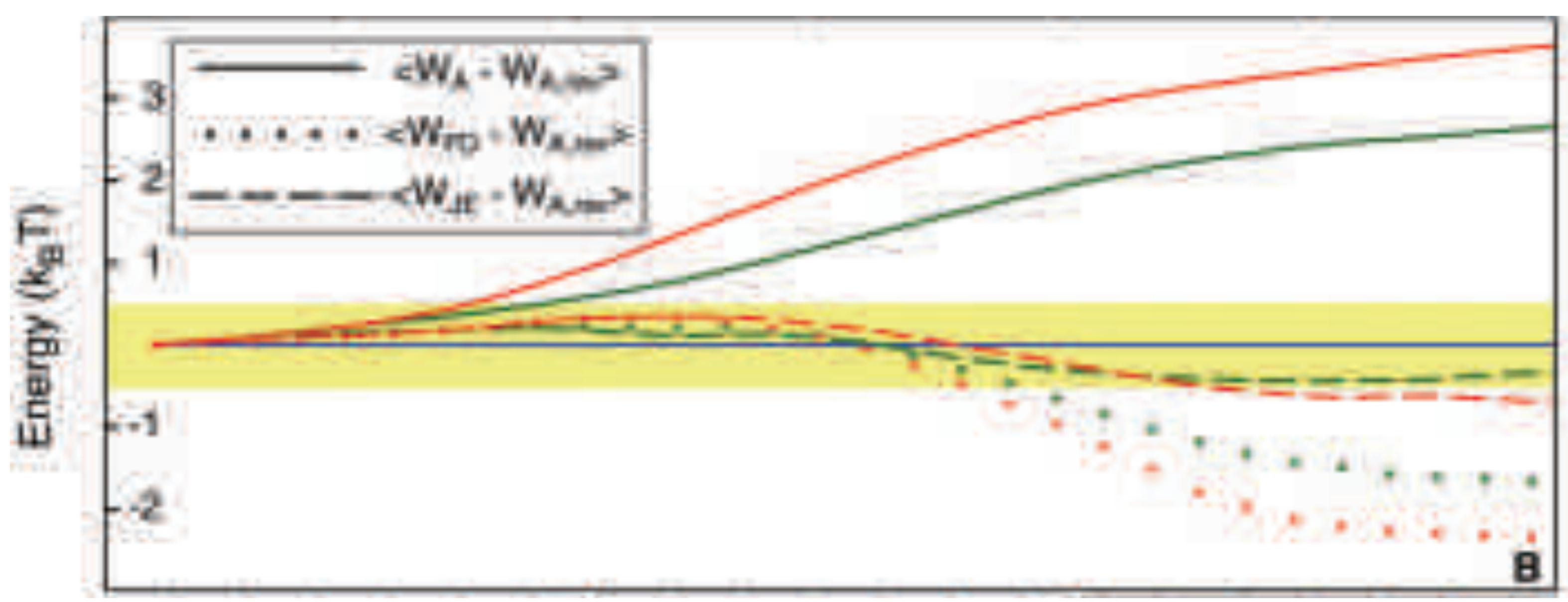
Jarzynski's equality





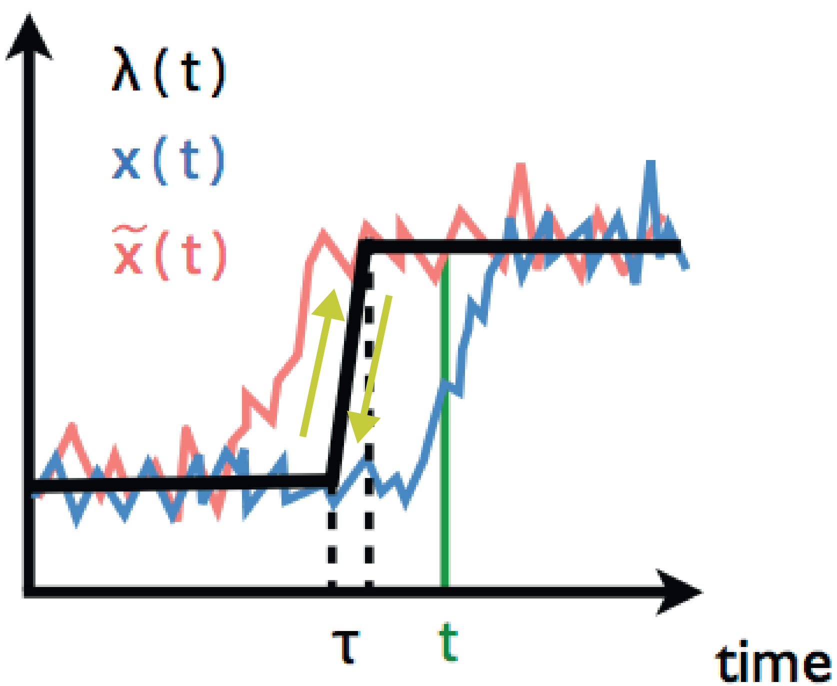


$$\Delta F = -k_{\mathrm{B}}T \ln\langle e^{-\beta W}\rangle$$



J Liphardt, S Dumont, SB Smith, I Tinoco, C Bustamante, Science 296 (5574), 1832 (2002)

Crooks theorem

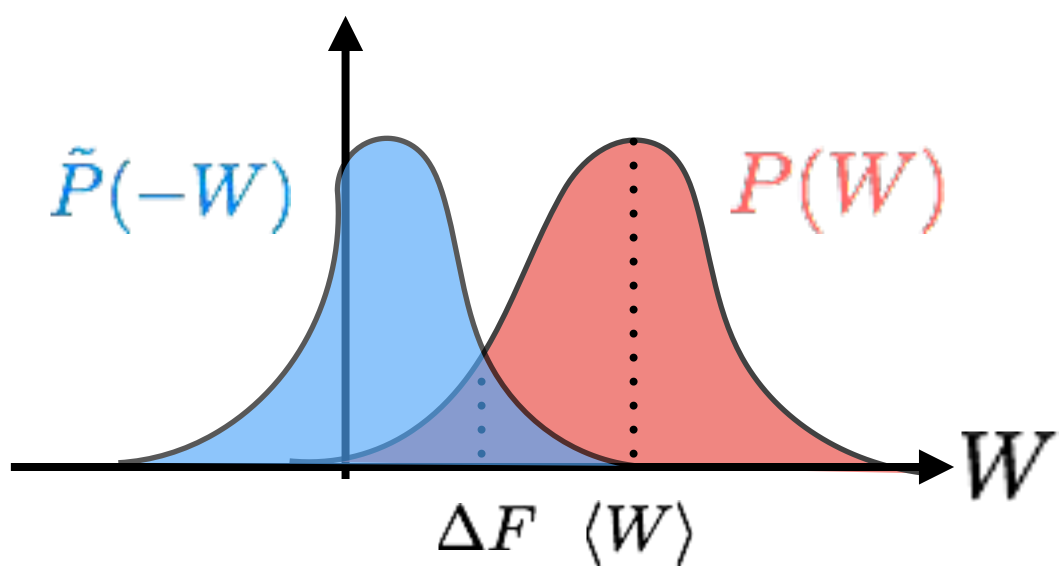


Forward process $\lambda(\tau) \qquad \tau \in [0, t]$

work distribution $P_t(W) = P(W(t) = W)$

Backward process $\tilde{\lambda}(\tau) = \lambda(t - \tau) \qquad \tau \in [0, t]$

work distribution $\tilde{P}_t(W) = \tilde{P}(W(t) = W)$



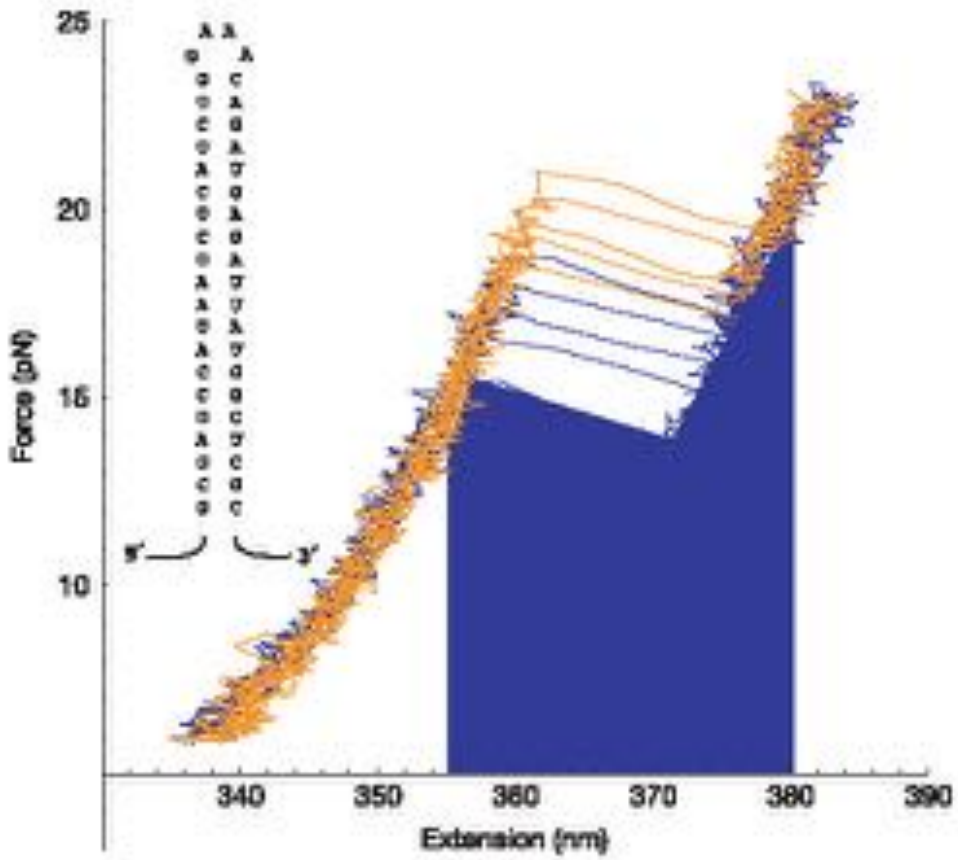
Asymmetry relation between forward and backward work distributions:

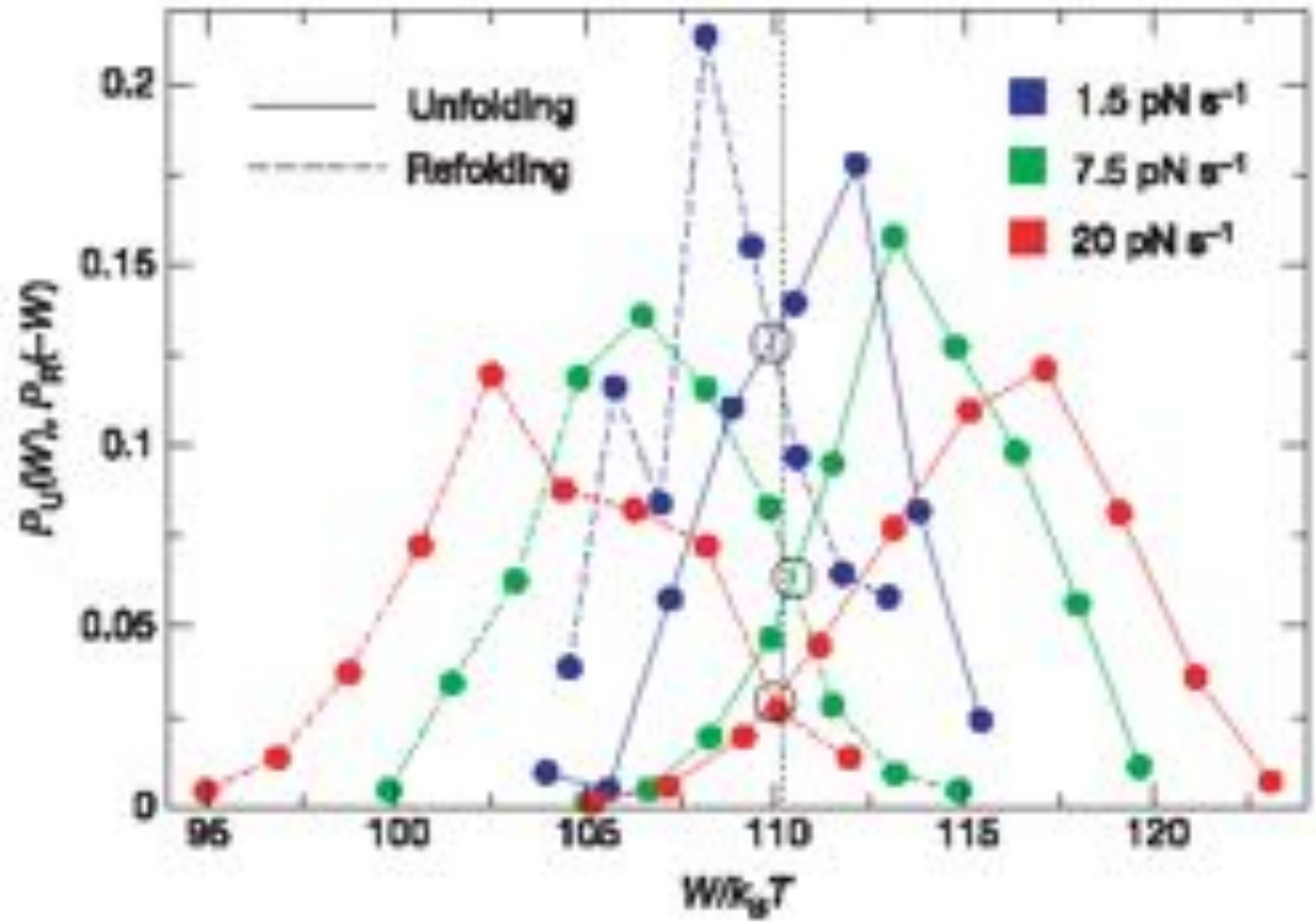
$$\frac{P_t(W)}{\tilde{P}_t(-W)} = e^{-\beta(W - \Delta F)}$$

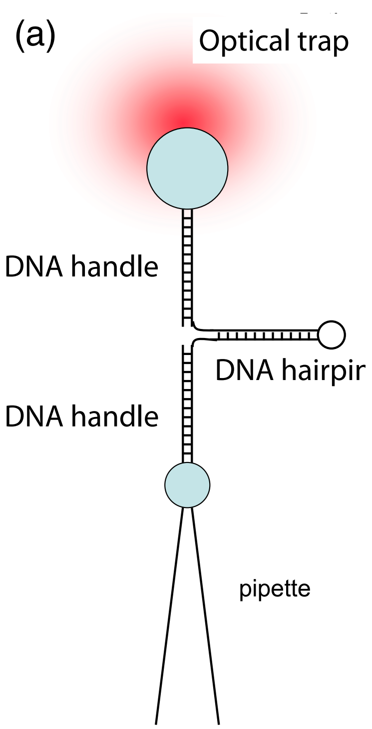
$$\Rightarrow \langle e^{-\beta W(t)} \rangle = e^{-\beta \Delta F}$$

G. E. Crooks, Phys. Rev. E **60**, 2721 (2000)

Crooks theorem

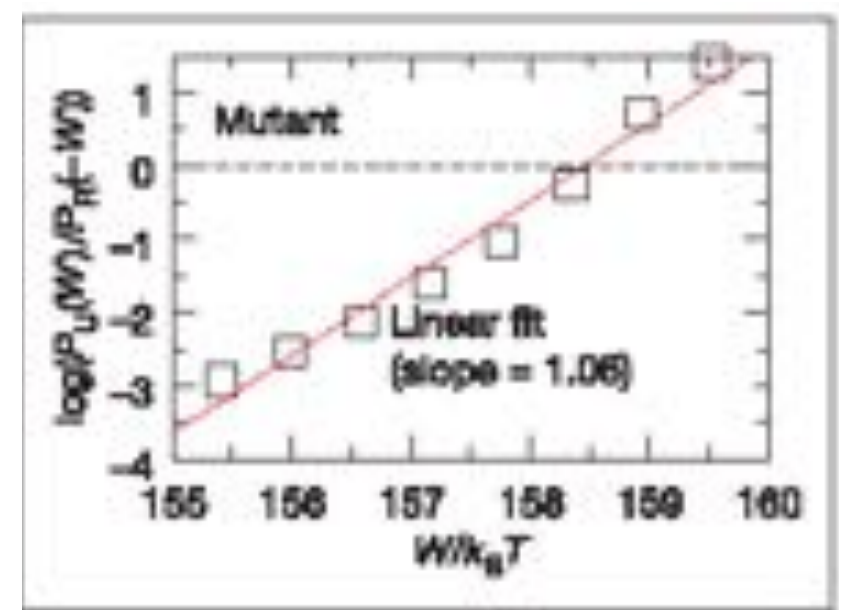






$$\Delta G_0^{\mathrm{Crooks}} = 37.2 \pm 1 \mathrm{kcalmol}^{-1}$$

$$\Delta G_0^{\mathrm{mfold}} = 38\,\mathrm{kcalmol}^{-1}$$

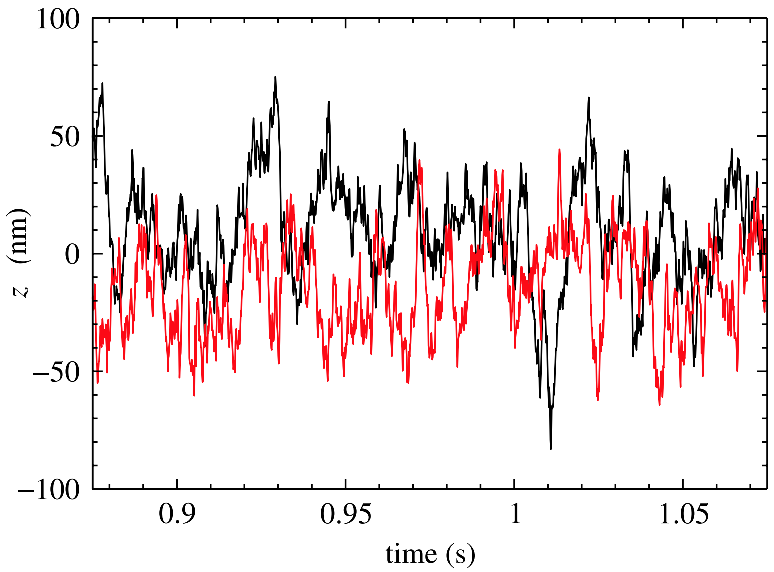


D. Collin, F. Ritort, et. al, Nature **437**, 231 (2005)

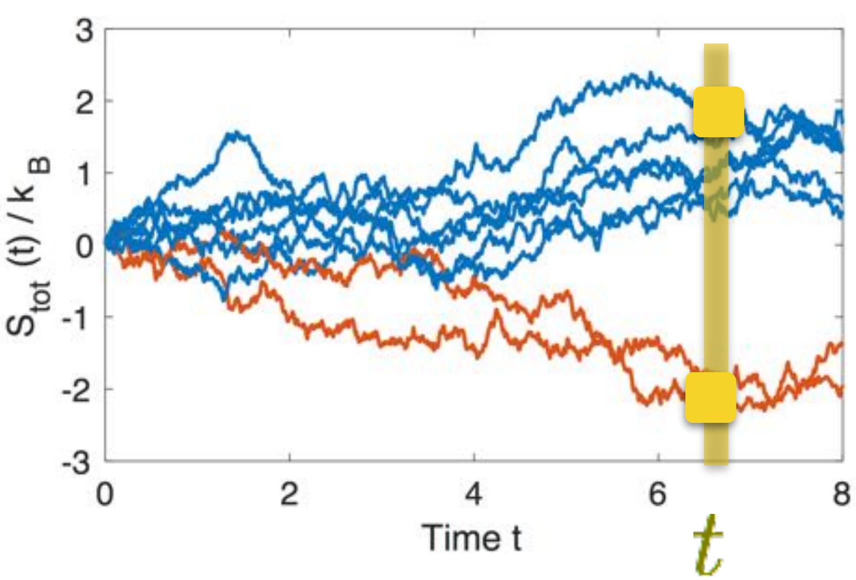
Detailed and integral fluctuation theorem

Entropy production along **a stochastic trajectory** of finite time duration in a generic non-equilibrium (stationary or non-stationary) process

- Diffusion processes e.g. Langevin
- Discrete systems e.g. continuous-time Markov processes
- Non-Markovian stochastic processes



$$S_{\rm tot}(t) = k_{\rm B} \ln \frac{\mathcal{P}(X_{[0,t]})}{\widetilde{\mathcal{P}}(\tilde{X}_{[0,t]})}$$



$P_t(S) = P(S_{\rm tot}(t) = S)$

$P_t(-S) = P(S_{\rm tot}(t) = -S)$

$$\frac{P_t(S)}{P_t(-S)} = e^{S/k_{\rm B}}$$

detailed fluctuation theorem

$$\langle e^{-S_{\rm tot}(t)/k_{\rm B}} \rangle = 1$$

integral fluctuation theorem

$$\frac{P_t(W)}{\tilde{P}_t(-W)} = e^{-\beta(W-\Delta F)}$$

$$\langle e^{-\beta[W(t)-\Delta F]} \rangle = 1$$

U. Seifert, Phys. Rev. Lett. **95**(4) 040602 (2005)

Detailed and integral fluctuation theorem

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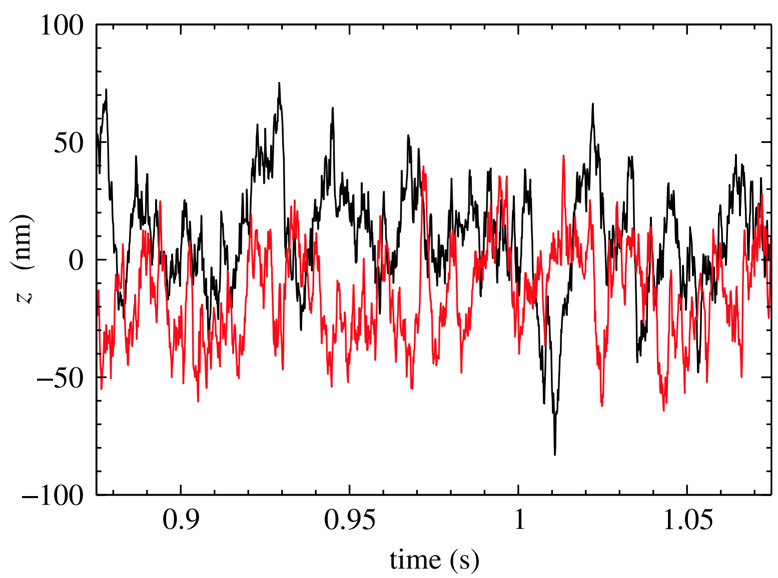
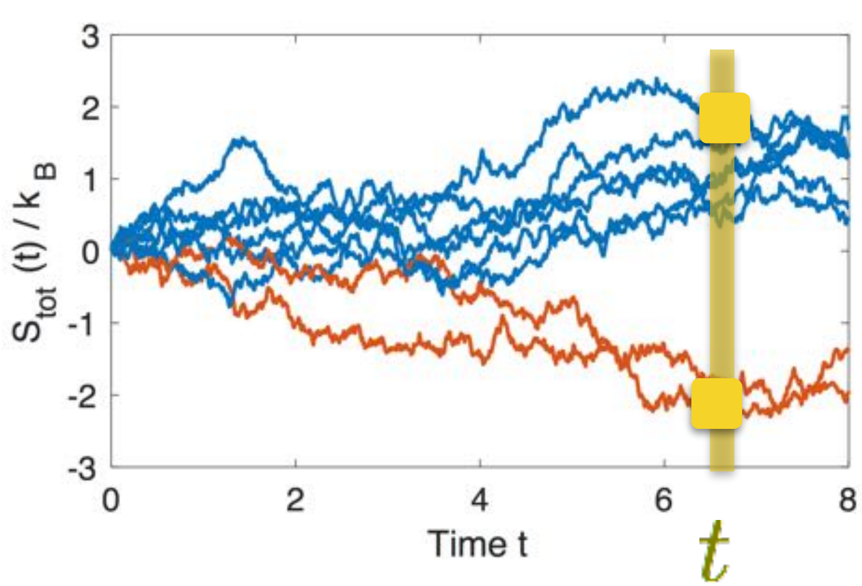


$$S_{\mathrm{tot}}(t) = k_{\mathrm{B}} \ln \left. \frac{\mathrm{d}\mathcal{P}}{\mathrm{d}(\tilde{\mathcal{P}} \circ \Theta)} \right|_{\mathcal{F}_t} (\omega)$$


$$\frac{\mathrm{d}P}{\mathrm{d}Q}$$

Radon - Nykodim derivative



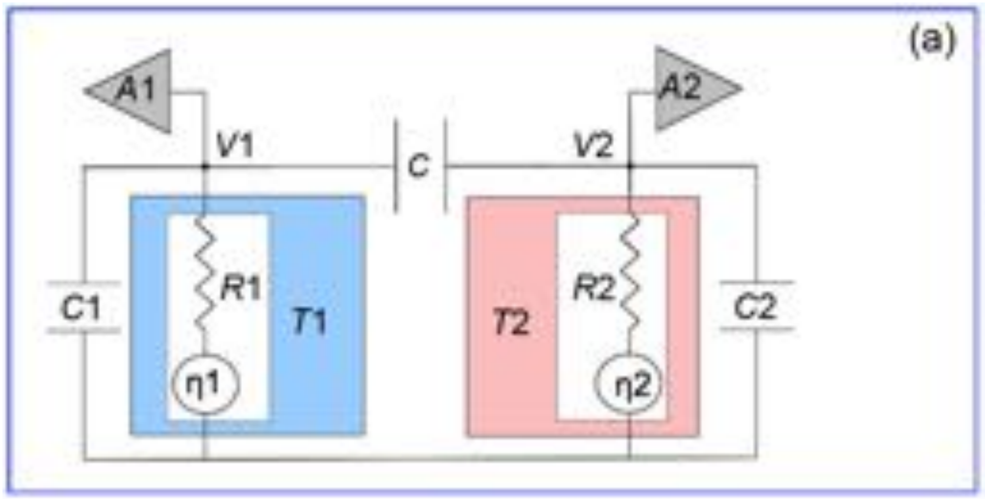
$$\left\langle \frac{\mathrm{d}Q}{\mathrm{d}P} \right\rangle_P = \left\langle e^{-S_{\mathrm{tot}}(t)/k_{\mathrm{B}}} \right\rangle = 1$$

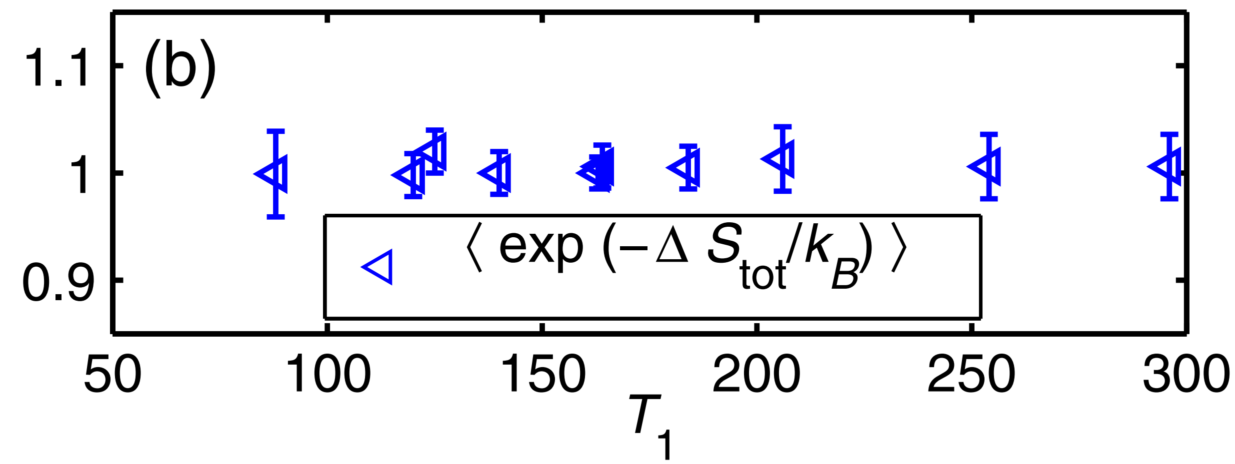
U. Seifert, Phys. Rev. Lett. **95**(4) 040602 (2005)

Smell of Martingales…..

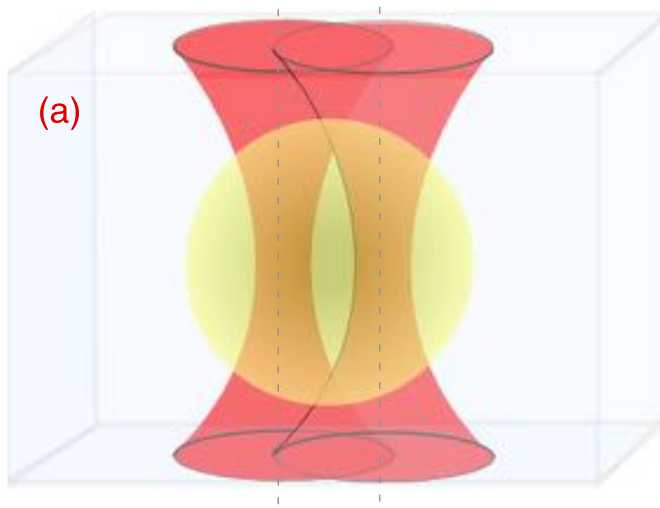
Detailed and integral fluctuation theorem

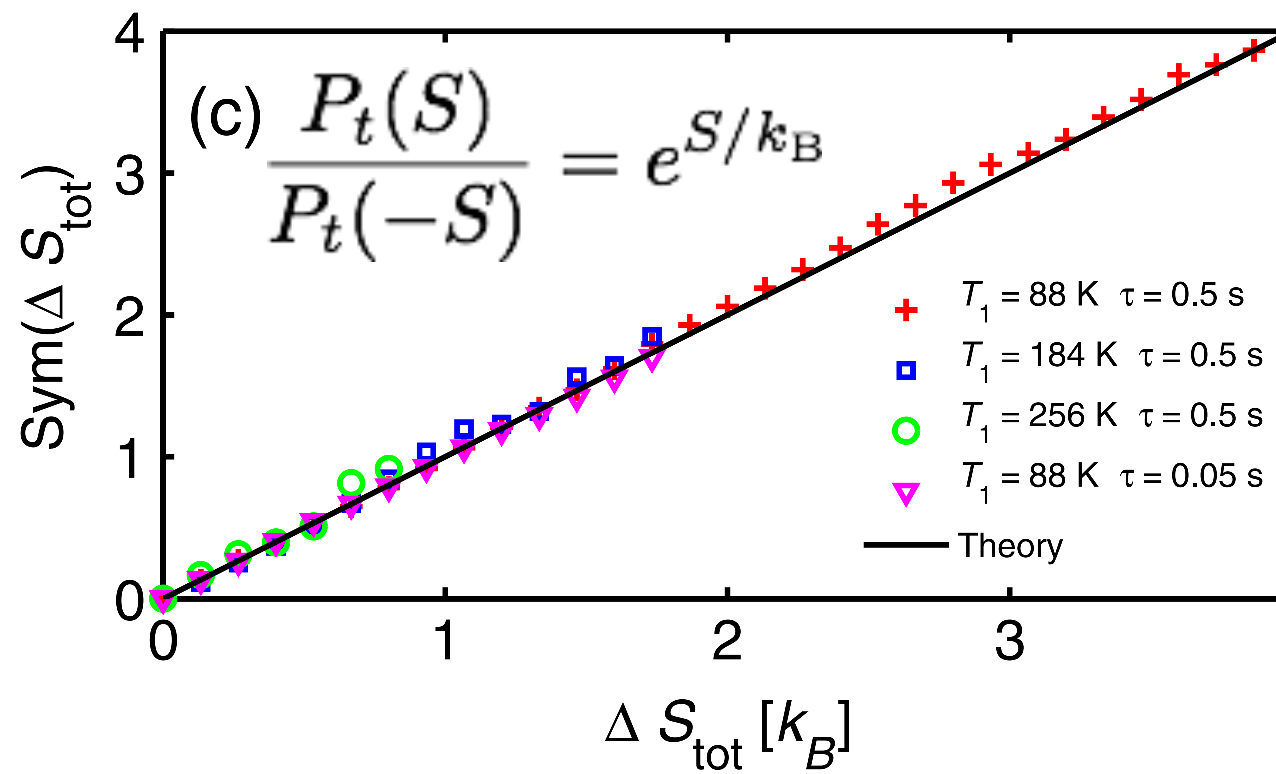
Experimental tests: electrical circuits, colloidal particles, nanoelectronic devices…

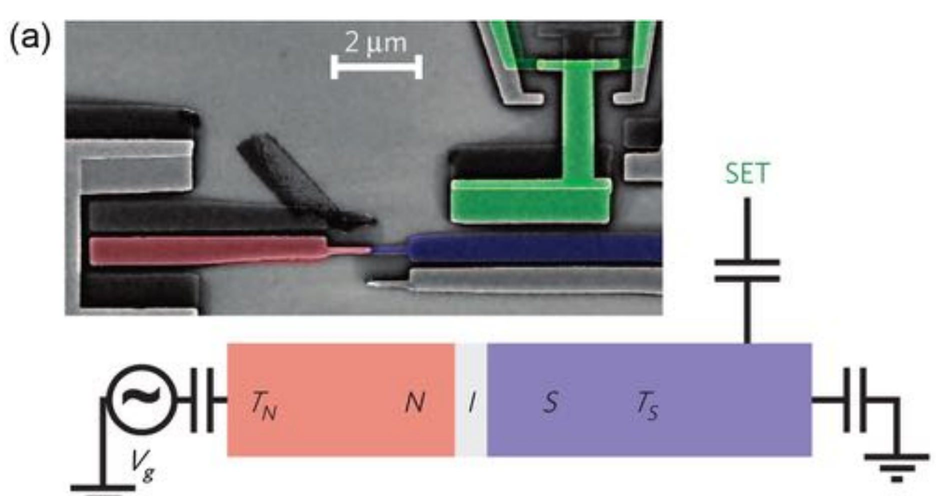




$$\langle e^{-S_{\rm tot}(t)/k_{\rm B}} \rangle = 1$$



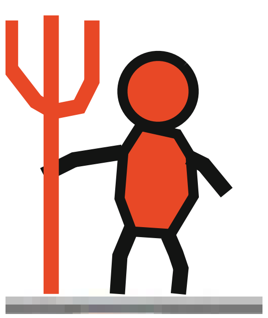




Reviews: S. Ciliberto, PRX **7,** 021051 (2017); Martinez.Soft Matter '17

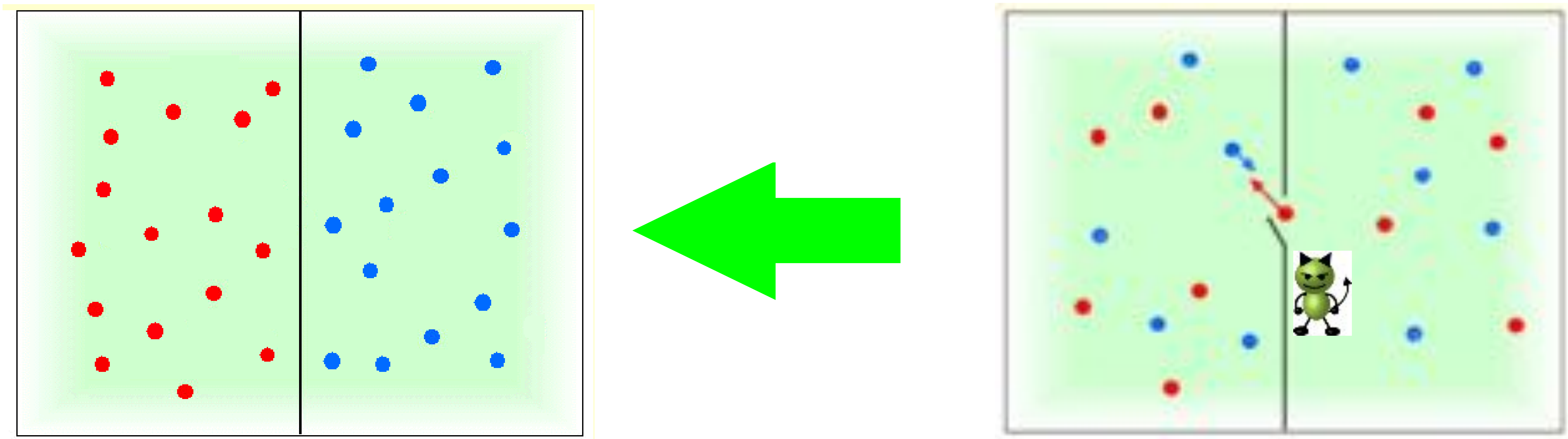
Plot of this talk

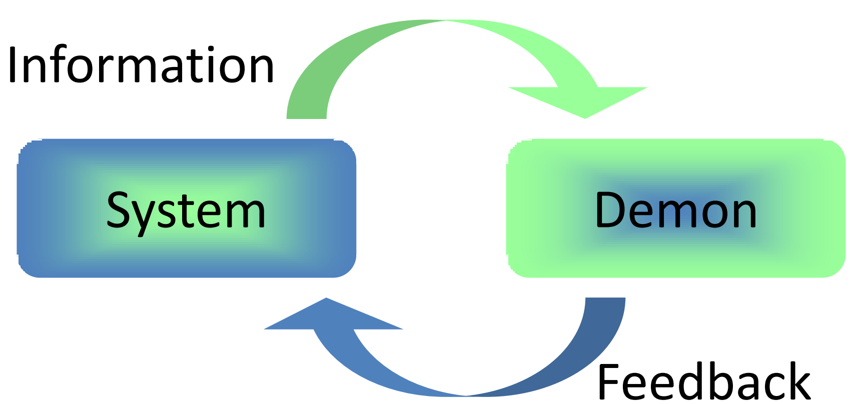
- Brief review of classical "textbook" thermodynamics
- Stochastic thermodynamics: first and second laws
- Fluctuation theorems



- **Maxwell's demons and feedback control**
- The latest: Uncertainty relations and Martingales

Maxwell's demon





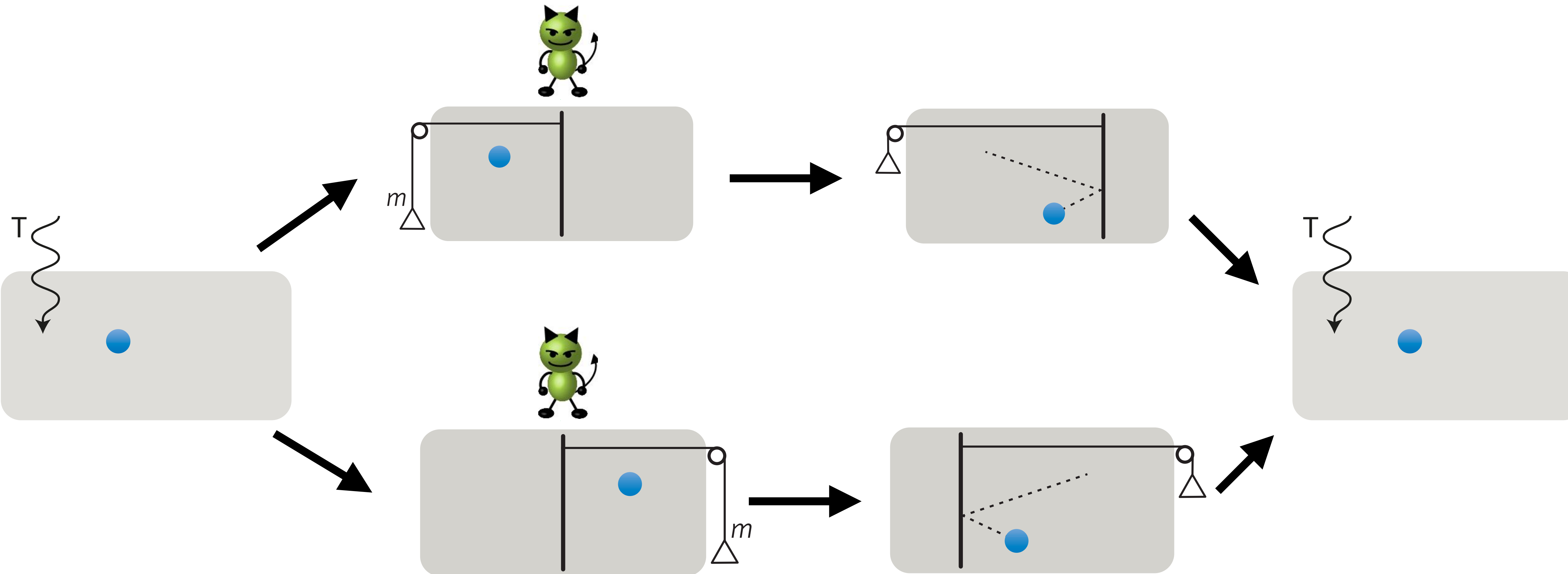
Information processing at the level of thermal fluctuations

✓ Foundation of the second law of thermodynamics

✓ Application to nanomachines and nanodevices

Review: J. M. R. Parrondo, J. M. Horowitz, & T. Sagawa, *Nature Physics* **11**, 131-139 (2015)

Szilard engine



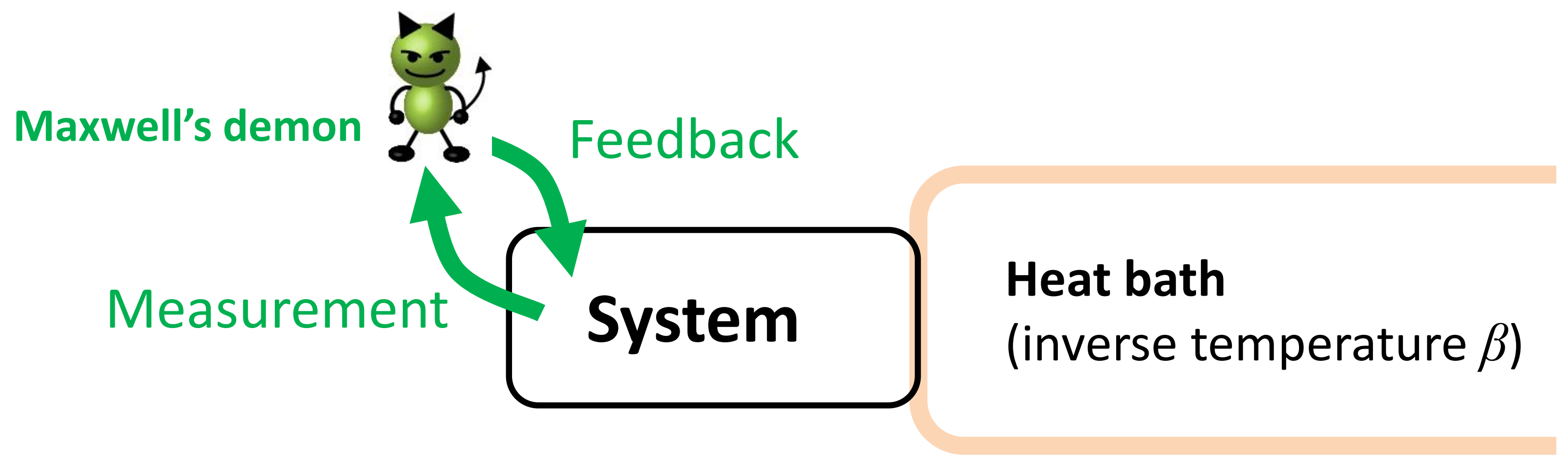
$$\langle W \rangle = -\int_{V/2}^{V} p\mathrm{d}V = -\int_{V/2}^{V} \frac{k_{\mathrm{B}}T}{V}\mathrm{d}V = -k_{\mathrm{B}}T\ln 2$$

$$\cancel{\langle W \rangle - \Delta F \geq 0}$$

$$\langle W \rangle - \Delta F \geq -k_{\mathrm{B}}TH(p)$$

$$H(p) = -\sum_{i=1,2} p_i \ln p_i = \ln 2$$

Sagawa-Ueda fluctuation theorems



***Stochastic* mutual information:** $i(x:y) \equiv \ln \dfrac{P(x,y)}{P(x)P(y)}$

x: system's state
y: demon's outcome

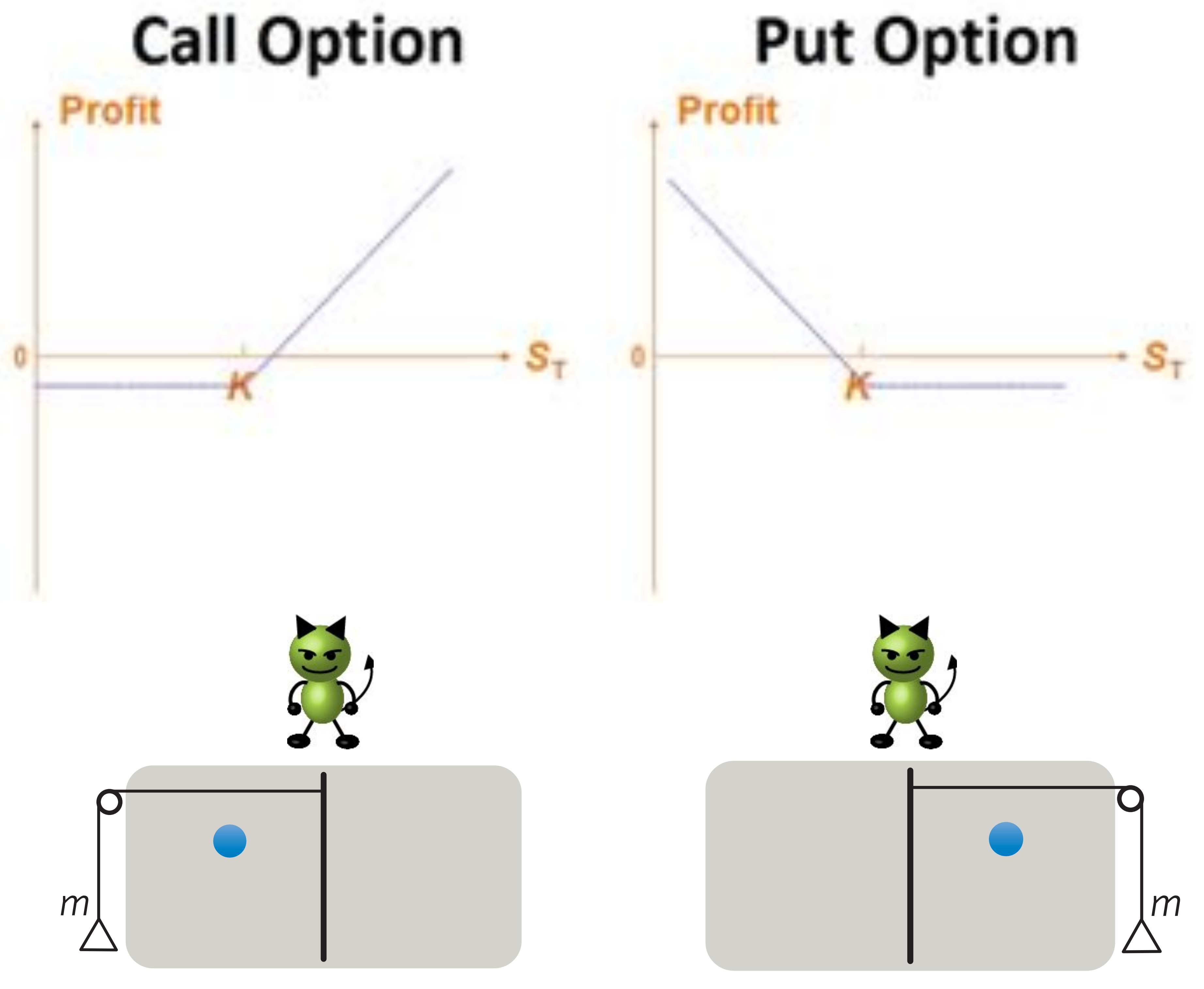
Generalized fluctuation theorem: $\langle e^{-(\Sigma_{\mathrm{tot}}+i)} \rangle = 1$

$$\Sigma_{\mathrm{tot}} \equiv S_{\mathrm{tot}}/k_{\mathrm{B}}$$

generalized second law: $\langle S_{\mathrm{tot}}(t) \rangle \geq -k_{\mathrm{B}} I(t)$

T. Sagawa, M. Ueda, Phys. Rev. Lett. **104,** 090602 (2010)

Finance and feedback control



Plot of this talk

- Brief review of classical “textbook” thermodynamics
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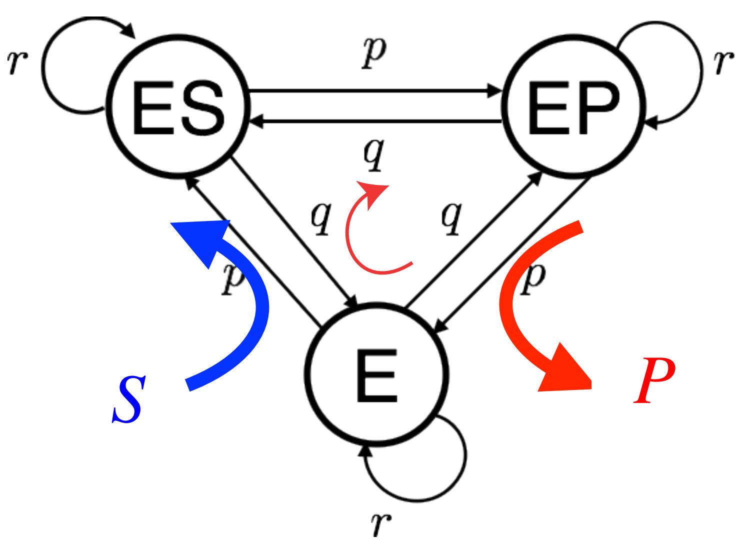
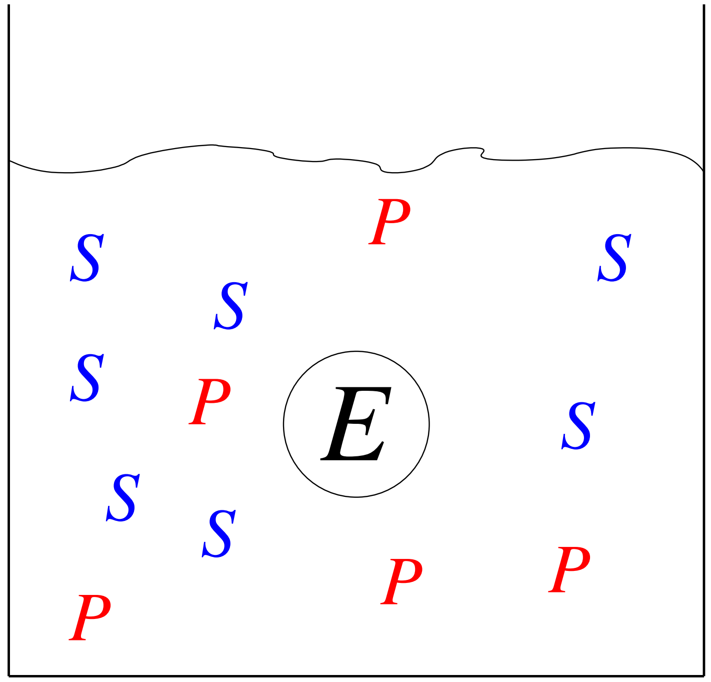
- Fluctuation theorems
- Maxwell’s demons and feedback control
- **The latest: Uncertainty relations and Martingales**



Appetizer over thermodynamic uncertainty relations

Example: Single Enzyme $$E + S \underset{k_1^-}{\overset{k_0^+}{\rightleftharpoons}} ES \underset{k_2^-}{\overset{k_1^+}{\rightleftharpoons}} EP \underset{k_0^-}{\overset{k_2^+}{\rightleftharpoons}} E + P$$



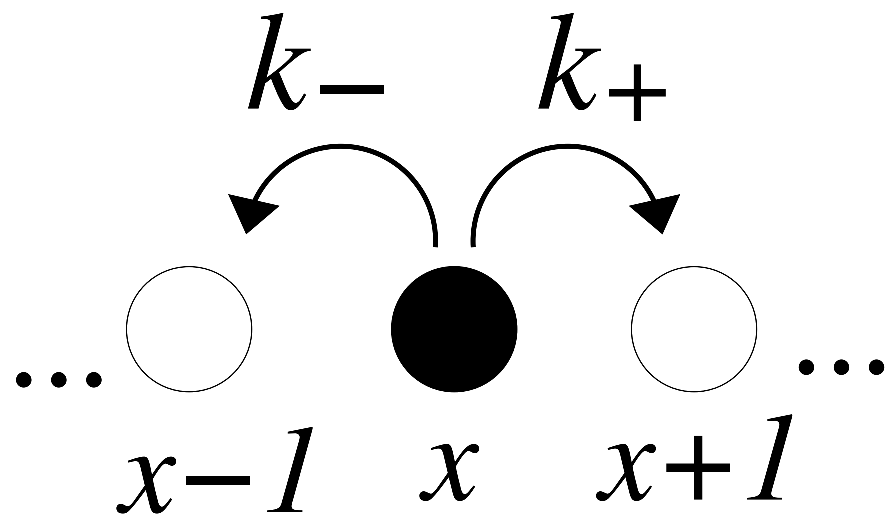


Generation of P is driven by free energy consumption

Is there a fundamental relation between the uncertainty (in the number of produced P's) and the free energy cost to sustain the bimolecular process?

A. C. Barato, U. Seifert, PRL **114**(15), 158101 (2015); J. M. Horowitz, T. R. Gingrich, PRE **96**(2), 020103 (2017)

Appetizer over thermodynamic uncertainty relations



$$\frac{d}{dt}P(x,t) = k_+ P(x-1,t) + k_- P(x+1,t) - (k_+ - k_-)P(x,t)$$

Velocity $\quad J \equiv \langle x \rangle / t = (k_+ - k_-)$

Dispersion $\quad D \equiv (\langle x^2 \rangle - \langle x \rangle^2)/(2t) = (k_+ + k_-)/2$

Relative uncertainty $\quad \epsilon^2 \equiv \dfrac{\langle x^2 \rangle - \langle x \rangle^2}{\langle x \rangle^2} = \dfrac{2D}{J^2 t} = \dfrac{k_+ + k_-}{(k_+ - k_-)^2} t^{-1}$

A. C. Barato, U. Seifert, PRL **114**(15), 158101 (2015); J. M. Horowitz, T. R. Gingrich, PRE **96**(2), 020103 (2017)

Appetizer over thermodynamic uncertainty relations

Relative uncertainty $\quad \epsilon^2 \equiv \frac{\langle x^2 \rangle - \langle x \rangle^2}{\langle x \rangle^2} = \frac{2D}{J^2 t} = \frac{k_+ + k_-}{(k_+ - k_-)^2} t^{-1}$

Thermodynamic Affinity $\Longrightarrow \mathcal{A} = (\mu_S - \mu_P)/(k_B T) = \ln(k_+/k_-)$

Rate of entropy production $\quad \sigma = \mathcal{A}(k_+ - k_-)$

Total dissipated heat (cost) $\mathcal{C} = k_B T \sigma t = \langle S_{\mathrm{tot}}(t) \rangle$

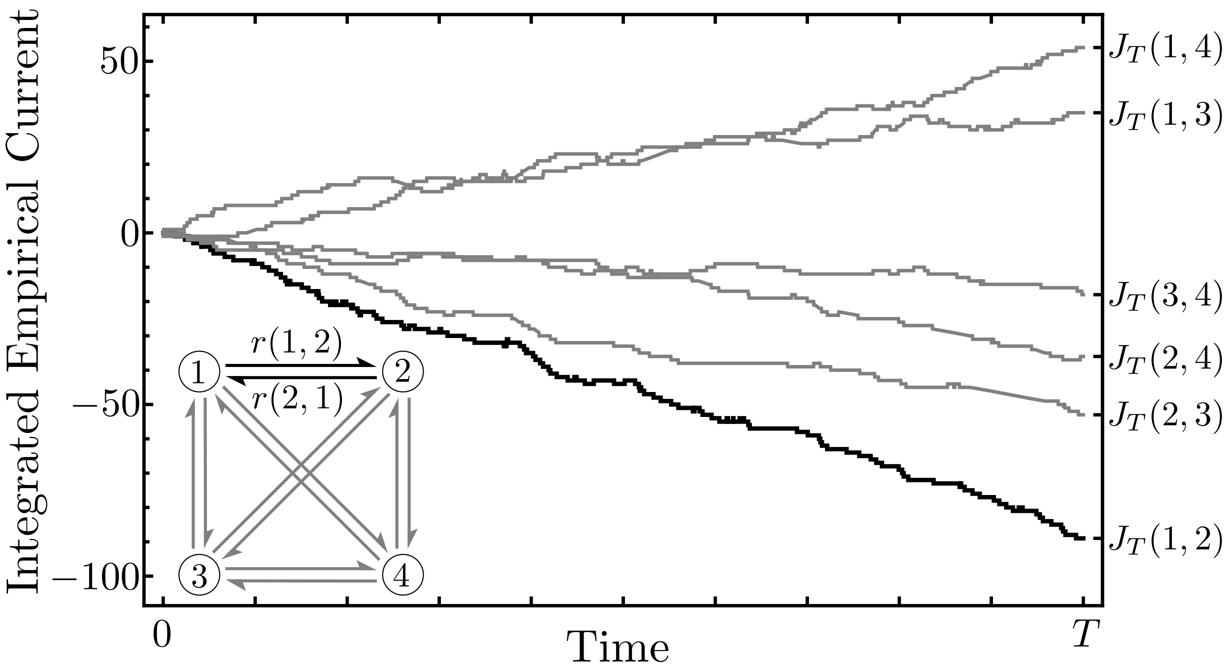
Total cost $\times$ relative uncertainty

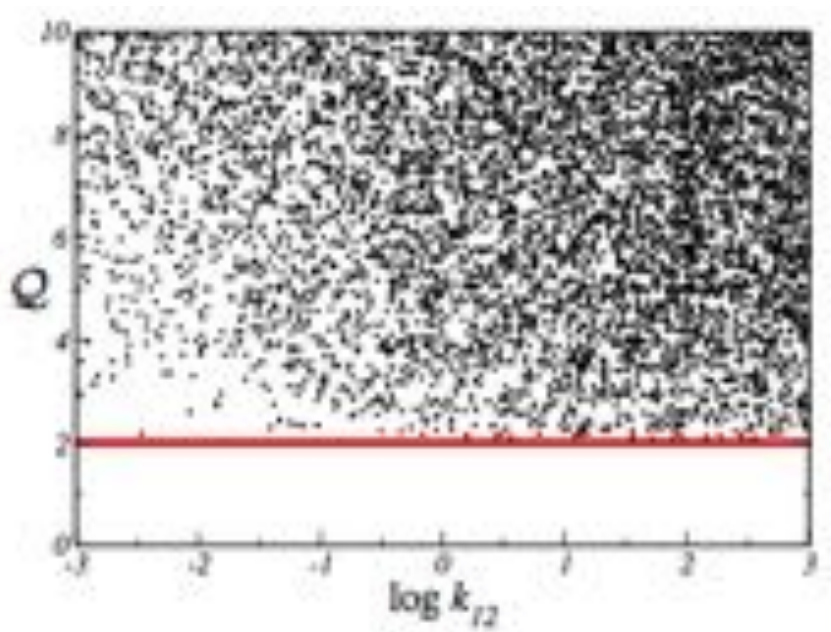
$$\mathcal{C}\epsilon^2 = k_B T \mathcal{A} \coth[\mathcal{A}/2] \geq 2 k_B T$$

A. C. Barato, U. Seifert, PRL **114**(15), 158101 (2015);

Appetizer over thermodynamic uncertainty relations

Universal cost-uncertainty tradeoff for time-integrated currents in non-equilibrium stationary Markov processes





A. C. Barato, U. Seifert, PRL **114**(15), 158101 (2015)

Continuous time

$$\frac{\mathrm{var}(J(t))}{\langle J(t)\rangle^2} \geq \frac{2k_\mathrm{B}}{\langle S(t)\rangle}$$

J. M. Horowitz, T. R. Gingrich, PRE **96**(2), 020103 (2017)

Discrete time

$$\frac{\mathrm{var}(J(t))}{\langle J(t)\rangle^2} \geq \frac{2\Delta S}{e^{\Delta S/k_\mathrm{B}} - 1}$$

K. Proesmans, C. Van den Broeck, EPL **119**(2), 20001 (2017)

Consequence of joint fluctuation theorem

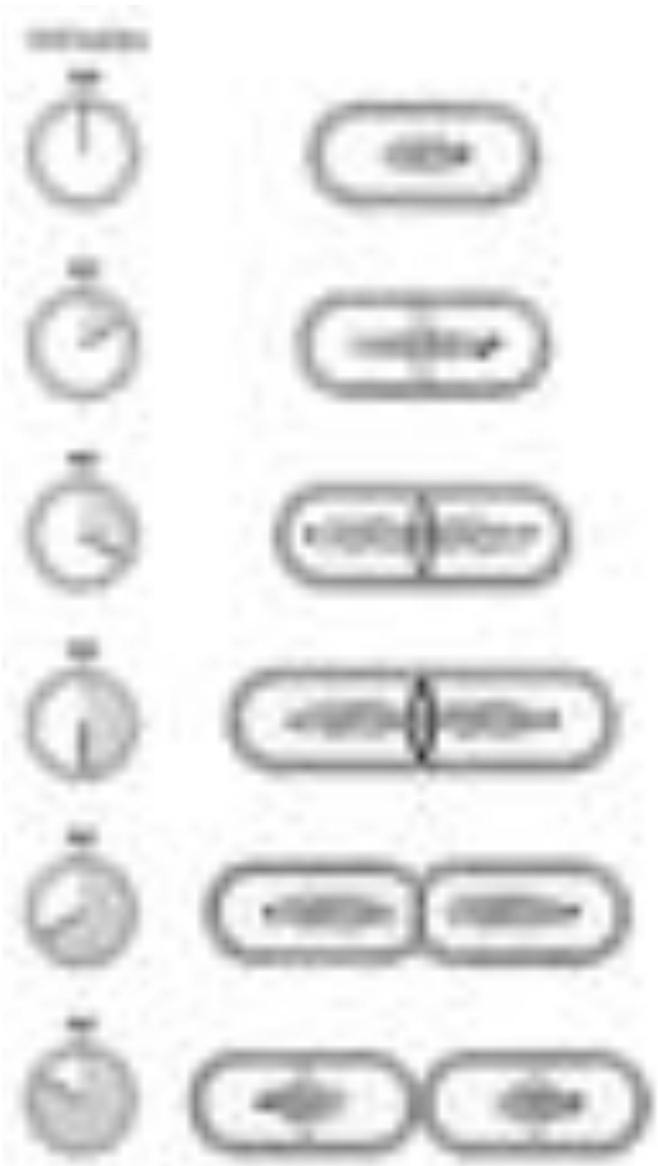
$$\frac{P_t(S_\mathrm{tot}, J)}{P_t(-S_\mathrm{tot}, -J)} = e^{S_\mathrm{tot}/k_\mathrm{B}}$$

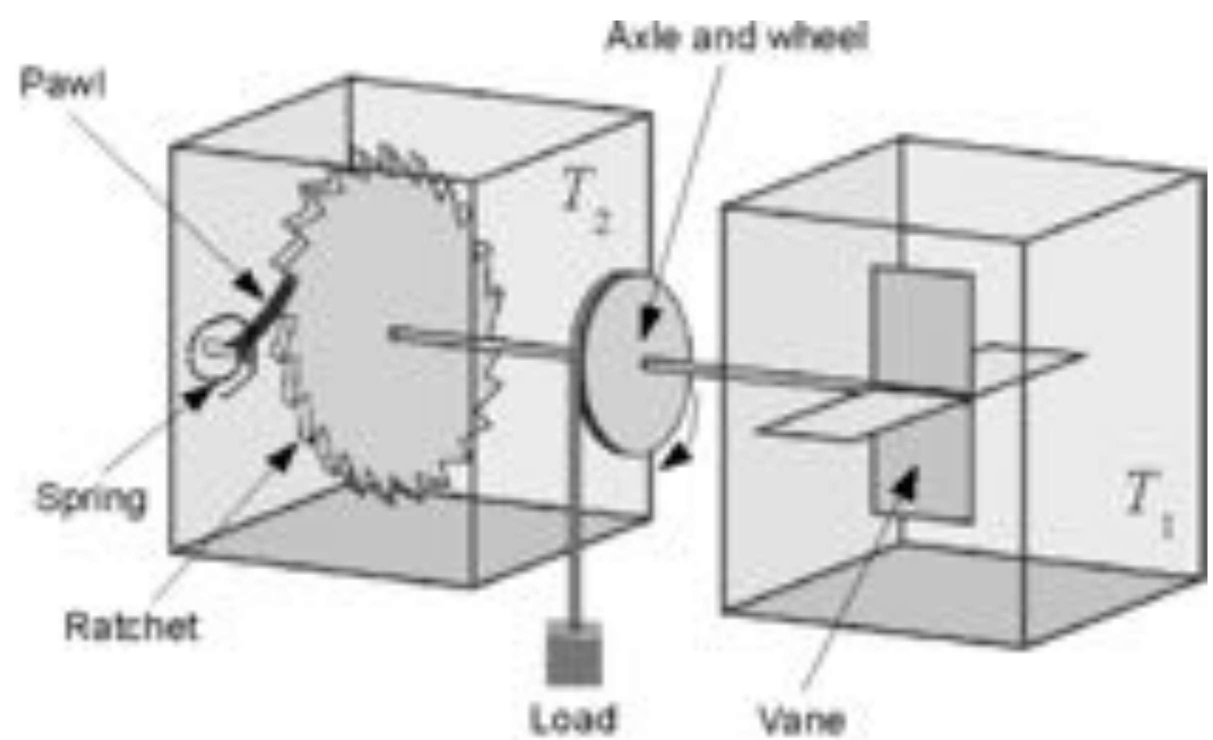
Y Hasegawa, T Van Vu arXiv:1902.06376 (2019)

Why martingales?

Most fluctuation theorems concern events that take place at a fixed time

However, most interesting stuff happens at random *stopping times*



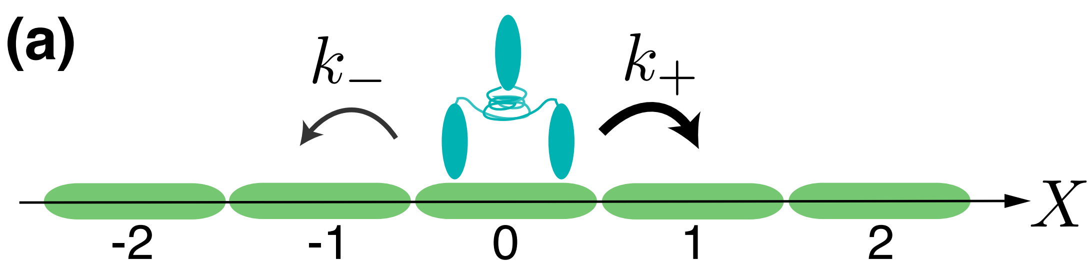


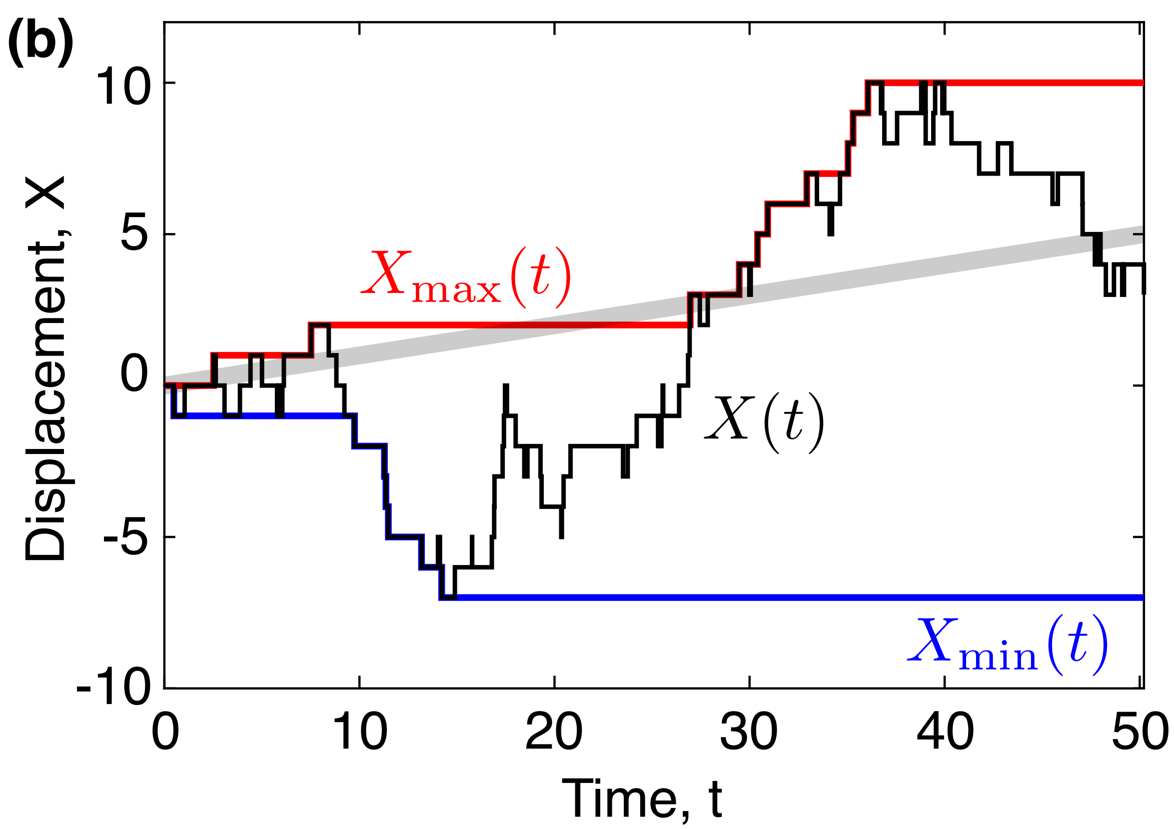
Completion of a cycle in
e.g. autonomous heat engines

Execution of cellular functions
e.g. bacterial cell cycle

Why martingales?

Extreme-value statistics
of active molecular processes



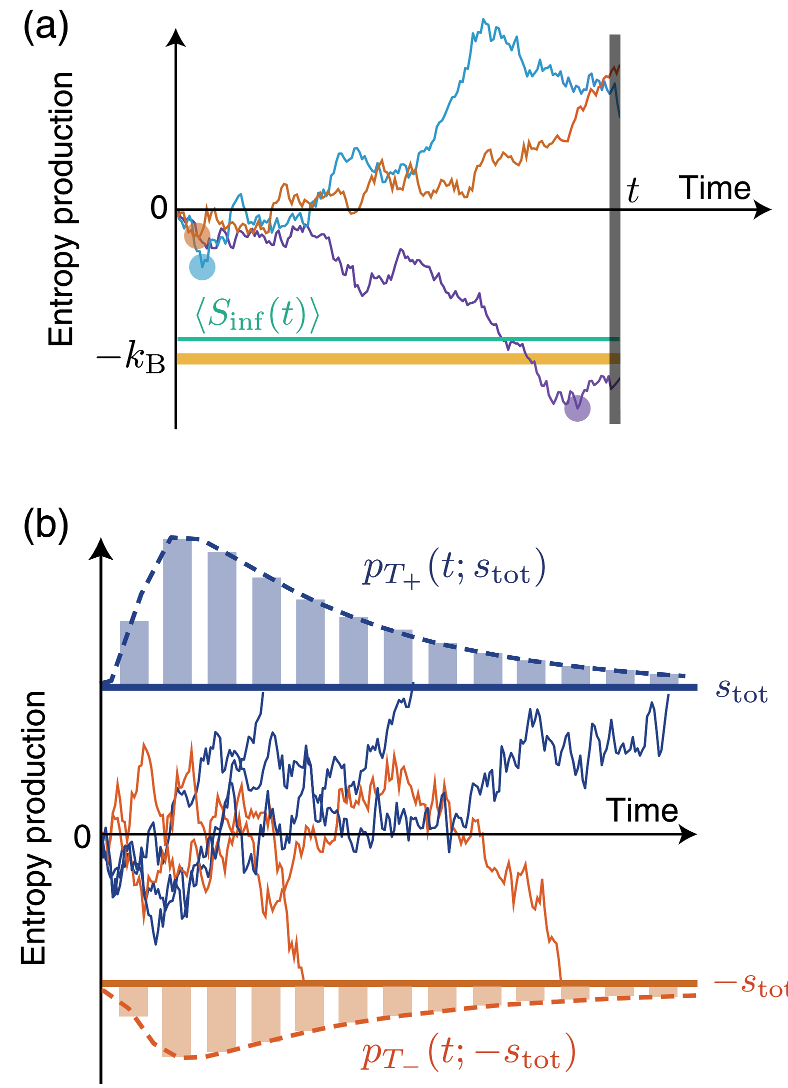


Two Refreshing Views of Fluctuation Theorems Through Kinematics Elements and Exponential Martingale

Raphaël Chetrite · Shamik Gupta

Statistics of Infima and Stopping Times of Entropy Production and Applications to Active Molecular Processes

Izaak Neri,[1,2] Édgar Roldán,[1,3] and Frank Jülicher[1]



Martingale

Thanks!

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