American Options and Stochastic Interest Rates

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Abstract

We study American equity options in a stochastic interest rate framework of Vasicek type (Vasicek (1977)). We allow for a non-zero correlation between the innovations driving the equity price and the interest rate. We also allow for the interest rate to assume negative values, which is the case for some investment grade government bonds in Europe in recent years. We develop a bivariate discretization of the equity price and interest rate processes by matching their original moments. The discretized processes converge in distribution to their joint limiting distribution as the time step shrinks. The discretization, described by a recombining quadrinomial tree, has a quadratic computational complexity in the number of steps. We exploit our quadrinomial tree to evaluate American put options on the risky equity asset, characterizing also the optimal exercise policy. We analyze the two-dimensional free boundary, i.e. the underlying asset and the interest rate values that trigger the optimal exercise of the option. We document in the stochastic interest rate environment non-standard exercise policies associated with the double continuation region first described by Battauz et al. (2015) in constant interest rate framework.

JEL Classification: G13.

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1 Introduction

In an arbitrage-free financial market the role of the short-term interest rate is twofold: on one hand it represents the rate at which the equity price appreciates; on the other hand it drives the locally risk-free asset and the related discount rate. Therefore, neglecting the variability of short-term interest rates may induce significant mispricings on both interest rates and equity derivatives. This issue is particulary relevant when derivatives are path-dependent and therefore sensitive to the entire path of the short-term interest rate, and not just its expected value at maturity. American equity call and put options, due to the optionality of their exercise policy, fall within this category. In fact, the holder of an American option has to timely chose when to cash in by exercising the option, balancing the effects from the discount rate and from the rate of return of the underlying asset. When both of these effects depend on a stochastic process, the valuation of the option becomes tricky. Our paper offers an intuitive and effective lattice method to compute both the price and the optimal exercise policies of American options on a risky asset with constant volatility in a stochastic interest rate framework of Vasicek type (see Vasicek (1977)). We employ the Vasicek mean-reverting model for the interest rate, because it allows the interest rate to assume mildly negative values, as the ones documented in recent years in the Euro-zone[1]. The feasibility of negative interest rates within the Vasicek model, once a source of major criticism, has very recently become the reason of renewed interest in the model itself because of the aforementioned market circumstances. We also allow for a non-zero constant correlation between the Brownian innovations of the interest rate and the equity

[1]It is widely accepted to proxy the risk-free rate in Europe by the recently negative rates of German bonds.

price processes. A positive (resp. negative) correlation between the interest rate and the equity price corresponds to a negative (resp. positive) correlation between the bond price and the equity price. After 2000 the market observed persistent negative stock-bond correlation as shown by Connolly et al. (2005). Perego and Vermeulen (2016) find that the correlation between equities and bonds is now consistently negative also in the Eurozone but for Southern Europe. Thus, in line with the recent empirical evidence, in our numerical examples we consider a positive correlation between the interest rate and the equity price. The literature on American equity options has so far focused on alternative stochastic interest rates models, such as the CIR one, based on the seminal work of Cox et al. (1885) (See[2] Medvedev and Scaillet (2010) and Boyarchenko and Levendorskiĭ (2013)). Our paper is, to our knowledge, the first that addresses the evaluation of American equity options in a stochastic interest rate framework of Vasicek type, allowing for the possibility of negative interest rates (see Detemple (2014) for an exhaustive review of the state of the art, Fabozzi et al. (2016) for a new recent quasi-analytic method to price and hedge American options on a lognormal asset with constant interest rate and Jin et al. (2013) for a computationally effective pricing algorithm for American options in a multifactor setting). First attempts to evaluate the impact of stochastic interest rates on American derivatives date back to Amin and Jr. (1995) and Ho et al. (1997). Nevertheless, both of them proxy American with Bermudan options with few exercise dates. Although this allows them to obtain closed form

[2] Medvedev and Scaillet (2010) introduce an analytical approach to price American options using a short-maturity asymptotic expansion. They perform a throughout numerical investigation for American call and put options with both stochastic CIR interest rates and stochastic underlying's volatility. Analogously, Boyarchenko and Levendorskiĭ (2013) consider a stochastic volatility equity and stochastic interest rates depending on two CIR factors, allowing for non-zero correlations between all the underlying processes. They provide a sophisticated iterative algorithm to price American derivatives that exploits a sequence of embedded perpetual options and their pricing results are in line with those of the Longstaff and Schwartz method and the asymptotics of Medvedev and Scaillet (2010).

solutions for both the price of the options and their optimal exercise policy, the approximation of a continuum of exercise dates by just a couple of possible exercise dates leads to a heavy mispricing of the options and provides no accurate insight on the free boundary.

In the spirit of Cox et al. (1979), we propose a lattice-based approach to compute an American option's price, its optimal exercise policy and the related free boundary. Building on Nelson and Ramaswamy (1990), who provide a tree approximation for an univariate process, we construct a discrete joint approximation for the both the equity price and the interest rate processes. Hahn and Dyer (2008) develop a similar discretization for a correlated two-dimensional mean reverting process representing the price of two correlated commodities and they use it to evaluate the value of an oil and gas switching option. Our paper is different, as the mean reverting stochastic interest rate process enters the risk-neutral drift of our equity price, that has constant volatility and correlates with the interest rate. In this framework, we provide a throughout investigation of American equity call and put options and their free boundaries. Our findings contribute to the literature on American options with stochastic interest rates, that usually restricts on non-negative interest rates. In particular, we unveil two novel significant features of the free boundary that appear when the stochastic interest rate may take mildly negative values.

First, we show that for American put (resp. call) options the early exercise region is not always downward (resp. upward) connected. The early exercise region is downward (resp. upward) connected if optimal exercise at t of the put (resp. call) option for some underlying equity price implies optimal exercise at t for all lower (resp. greater) values of the underlying equity price. In a stochastic interest rate framework Detemple (2014) retrieves the free boundary by a discretization of an integral equation for the early exercise premium decomposition. For American call options he argues that the exercise region is connected in the upward direction. Our results show that this property holds true if interest rates are always non-negative, but may fail if the interest rates' positivity assumption is not satisfied. In this case, we document the existence of a non standard double continuation region first described by Battauz et al. (2015) in a constant interest rate framework. In particular, a non-standard additional continuation region appears where the option is most deeply in the money and the underlying pays a negative dividend.[3] Under these circumstances a mildly negative interest rate may lead to optimal postponment of the deeply in the money option as the holder is confident the option will still be in the money later and prefers to delay the cash-in.

Second, we show that early exercise may be optimal for an American call option even if the underlying equity does not pay any dividend. This happens when a mildly negative initial interest rate causes the underlying equity's drift to be negative as well, pushing the underlying equity towards the out of the money region. In this case, immediate exercise turns out to be optimal as soon as the option is sufficiently in the money. The critical equity price that triggers optimal early exercise is increasing with respect to the interest rate value, as the higher the interest rate, the higher the underlying equity drift, the lower the risk of ending up in the out of the money region for the call option, and thus the higher has to be the immediate payoff to be optimally exercised before maturity.

The remaining of the paper is organized as follows: in Section 2 we introduce the financial framework, describing the stochastic processes we have and the related traded assets. We

[3] A negative dividend can be interpreted as a storage cost for commodities (e.g. gold or silver) or as the result of the interplay of domestic and foreign interest rates when evaluating options on foreign equities (see Battauz et al. (2018), (2015)).

develop here the lattice-based discretization of the market, that we call *quadrinomial tree*. In Section 3 we deal with American put and call options describing numerical pricing results, showing the differences arising from the standard constant interest rate case and providing a graphical characterization of the optimal exercise policy and of the free boundaries. Section 4 concludes.

2 The market and the Quadrinomial Tree

2.1 The Assets in the Market

Consider a stylized financial market in a continuous time framework with investment horizon $T > 0$. A risky security $S(t)$ is traded. Following the seminal work of Vasicek (1977), we assume a mean-reverting stochastic process for the prevailing short term interest rate on the market $r(t)$. We allow for a non zero correlation between the innovations of S and r. A market player can invest in the short-term interest rate, which is locally risk-free, through the money market account $B(t)$.

The dynamics of the risky security, of the short-term interest rate and of the money market account under the risk-neutral measure $\mathbb{Q}$ are:

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{S(t)} & = (r(t) - q)\mathrm{d}t + \sigma_S \mathrm{d}W_S^{\mathbb{Q}}(t) \\ \mathrm{d}r(t) & = \kappa\left(\theta - r(t)\right)\mathrm{d}t + \sigma_r \mathrm{d}W_r^{\mathbb{Q}}(t) \\ \mathrm{d}B(t) & = r(t)B(t)\mathrm{d}t \end{cases} \tag{1}$$

with $\langle \mathrm{d}W_S^{\mathbb{Q}}(t), \mathrm{d}W_r^{\mathbb{Q}}(t)\rangle = \rho \mathrm{d}t$ and given some initial conditions $S(0) = S_0$, $r(0) = r_0$ and $B(0) = 1$. Moreover: q is the deterministic constant annual dividend rate of the equity, $\sigma_S > 0$ the volatility of the equity price, κ the speed of mean-reversion of the short-term interest rate, θ its long-run mean, $\sigma_r > 0$ the volatility of the short-term interest rate and $\rho \in [-1, 1]$ the correlation between the Brownian shocks on S and r.

System (1) can be rewritten equivalently in the following vectorial specification:

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{S(t)} &= \mu_S \mathrm{d}t + \nu_S \cdot \mathrm{d}W^{\mathbb{Q}}(t) \\ \mathrm{d}r(t) &= \mu_r \mathrm{d}t + \nu_r \cdot \mathrm{d}W^{\mathbb{Q}}(t) \end{cases} \tag{2}$$

where $\mu_S = (r(t) - q)$, $\mu_r = \kappa(\theta - r(t))$, $\nu_S = [\sigma_S \quad 0]$, $\nu_r = [\sigma_r \rho \quad \sigma_r \sqrt{1-\rho^2}]$, $W^{\mathbb{Q}}(t) = \left[W_1^{\mathbb{Q}}(t) \quad W_2^{\mathbb{Q}}(t)\right]'$ is a standard two-dimensional Brownian motion and $\cdot$ is the matrix product.

The explicit solution to the system of SDEs in (1) is

$$\begin{cases} S(t) &= S_0 \exp\left[\int_0^t r(s)\mathrm{d}s - \left(q + \frac{\sigma_S^2}{2}\right)t + \sigma_S W_S(t)\right] \\ r(t) &= r_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}) + \sigma_r \int_0^t e^{-\kappa(t-s)}\mathrm{d}W_r(s) \\ B(t) &= \exp\left[\int_0^t r(s)\mathrm{d}s\right] \end{cases} \tag{3}$$

The zero-coupon bond with maturity T pays 1 at its holder at T and its price at $t \in (0, T)$ is labelled with $p(t, T)$. By no arbitrage valuation, we have

$$p(t, T) = \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B(T)}\bigg| \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[\exp\left[-\int_0^T r(s)\mathrm{d}s\right]\bigg| \mathcal{F}_t\right],$$

that admits a closed formula solution as derived in Section 3.2.1 of Brigo and Mercurio (2007):

$$p(t, T) = e^{A(t,T) - B(t,T)r(t)} \tag{4}$$

where:

$$B(t, T) = \frac{1}{\kappa}\left(1 - e^{-\kappa(T-t)}\right) \tag{5}$$

$$A(t, T) = \left(\theta - \frac{\sigma_r^2}{2\kappa^2}\right)(B(t, T) - (T - t)) - \frac{\sigma_r^2 B^2(t, T)}{4\kappa}. \tag{6}$$

In this fairly general pricing framework, the price of European put and call options on S can be derived in closed formulae. Indeed, exploiting a change of numèraire as described in Geman et al. (1995), it is possible to obtain the following:

Proposition 1 ***(Value of the European put/call equity option)*** *In the financial market specified in* (1)*, the price at* $t \in [0, T]$ *of an European put option on* S *with strike* K *is equal to*

$$\pi_E^{put}(t, S(t), r(t)) = Kp(t, T)N(-\tilde{d}_2) - S(t)e^{-q(T-t)}N(-\tilde{d}_1) \tag{7}$$

with[4]*:*

$$\begin{aligned}
\tilde{d}_1 &= \frac{1}{\sqrt{\Sigma_{t,T}^2}}\left(\ln \frac{S(t)}{Kp(t,T)} + \frac{1}{2}\Sigma_{t,T}^2 - q(T-t)\right), \\
\tilde{d}_2 &= \tilde{d}_1 - \sqrt{\Sigma_{t,T}^2}, \\
\Sigma_{t,T}^2 &= \sigma_S^2(T-t) + 2\sigma_S\sigma_r\rho\left(\frac{-1+e^{-\kappa(T-t)}+\kappa(T-t)}{k^2}\right) + \\
&\qquad -\sigma_r^2\left(\frac{3+e^{-2\kappa(T-t)}-4e^{-\kappa(T-t)}-2\kappa(T-t)}{2k^3}\right).
\end{aligned}$$

The price at $t \in [0, T]$ *of an European call option on* S *with strike* K *is equal to*

$$\pi_E^{call}(t, S(t), r(t)) = S(t)e^{-q(T-t)}N(\tilde{d}_1) - Kp(t, T)N(\tilde{d}_2). \tag{8}$$

Proof. See Appendix B. ■

Within this market model, Bernard et al. (2008) price European barrier options by properly approximating the hitting time of the equity price of the exogenously given barrier. Unfortunately the same approach does not work when the derivative is of American style, as the barrier has to be endogenously determined. For this reason we develop in the next session the lattice discretization of our market that will allow us to compute the American free boundaries.

[4]Notice that the current value of the interest rate $r(t)$ enters the current price of the zero-coupon bond $p(t, T)$ in $\tilde{d}_1$ and $\tilde{d}_2$.

2.2 The Quadrinomial Tree

In their seminal work, Cox et al. (1979) show how to discretize a lognormal risky security and how to easily exploit such a binomial discretization in order to evaluate derivatives written on the primary asset. Embedding this geometric Brownian motion case into a more general class of diffusion processes, Nelson and Ramaswamy (1990) propose a one-dimensional scheme to properly define a binomial process that approximates a one-dimensional diffusion process. They do so by matching the diffunsion's instantaneous drift and its variance and imposing a recombining structure to their discretized process.

The discretization via a tree/lattice structure of correlated processes, possibly of different kind, is more challenging. Gamba and Trigeorgis (2007) model two or more correlated geometric Brownian motion representing the price processes of different risky assets exploiting a log-transformation of the processes first and then an orthogonal decomposition of the shocks. In this way they are able to efficiently price derivatives on five correlated assets. Moving away from lognormality, Hahn and Dyer (2008) construct a quadrinomial lattice to approximate two mean-reverting processes in order to model two correlated one-factor commodity prices and evaluate derivatives on them.

We propose here a quadrinomial tree to jointly model a mean-reverting process for the short term interest rate as suggested first by Vasicek (1977) and the process for the risky equity's price with constant volatility and the drift that embeds the stochastic interest rate as in (1). Non constant short-term interest rates are surely more suitable from an option pricing perspective and an Ornstein–Uhlenbeck process enables us to investigate some interesting features of options when the discount rate becomes slightly negative, as documented in the Euro zone in recent years.

We first show how to build the discretization of the processes $(S(t), r(t))$ described by

(1) and then we address the convergence issue of the discretization itself.

We apply Itô's Lemma to $Y(t) := \ln(S(t))$ and we get

$$\begin{cases} \mathrm{d}Y(t) & = \mu_Y \mathrm{d}t + \sigma_S \mathrm{d}W_S(t) \\ \mathrm{d}r(t) & = \mu_r \mathrm{d}t + \sigma_r \mathrm{d}W_r(t) \end{cases} \tag{9}$$

where $\mu_Y := \left(r(t) - q - \frac{\sigma_S^2}{2}\right)$ and $\mu_r := \kappa(\theta - r(t))$. Again, the vectorial version of (2) is:

$$\mathrm{d}\begin{bmatrix} Y(t) \\ r(t) \end{bmatrix} = \begin{bmatrix} \left(r(t) - q - \frac{\sigma_S^2}{2}\right) \\ \kappa(\theta - r(t)) \end{bmatrix} \mathrm{d}t + \begin{bmatrix} \sigma_S & 0 \\ \sigma_r \rho & \sigma_r\sqrt{1-\rho^2} \end{bmatrix} \cdot \mathrm{d}W(t). \tag{10}$$

We refer here to the general technique of Section 11.3 of Stroock and Varadhan (1997) exploiting the very convenient notation introduced in Section 3.2.1 of Prigent (2003). For the ease of the reader we recall here their template. Consider the following bivariate SDE:

$$\mathrm{d}X(t) = \mu(x,t)\mathrm{d}t + \sigma(x,t) \cdot \mathrm{d}W(t) \tag{11}$$

where $X(t)_{t\geq 0} = (Y(t), r(t))_{t\geq 0}$, $W(t)$ is a standard two-dimensional Brownian motion, $\mu(x,t) : (\mathbb{R} \times \mathbb{R}^+) \times \mathbb{R}^+ \to \mathbb{R}^2$, $\sigma(x,t) : (\mathbb{R} \times \mathbb{R}^+) \times \mathbb{R}^+ \to \mathbb{R}^{2\times 2}$ and an initial condition $X(0) = (x_0, r_0)$ is given.

Consider a discrete uniform partition of the time interval $[0,T]$ like $\left\{i\frac{T}{n}, i = 1, \ldots, n\right\}$ and define $\Delta t := \frac{T}{n}$. For each n consider a bivariate stochastic process $\{X_n\}$ on $[0,T]$ which is constant between the nodes of the partition. At each node, both of the components of X_n jump up (or down) a certain distance with a certain probability. The sizes of the jumps and the probabilities are allowed to be time-dependent and state-contingent. Since after any jump each component of X_n can assume two new different values, there will be globally four possible outcomes after each jump. For sake of clarity, fix n and consider the generic i-th step of the bivariate discrete process $X_i = (Y_i, r_i)$. In the following step, the process can assume only one of the following four values:

$$(Y_{i+1}, r_{i+1}) = \begin{cases} (Y_i + \Delta Y^+, r_i + \Delta r^+) & \text{with probability } q_{uu} \\ (Y_i + \Delta Y^+, r_i + \Delta r^-) & \text{with probability } q_{ud} \\ (Y_i + \Delta Y^-, r_i + \Delta r^+) & \text{with probability } q_{du} \\ (Y_i + \Delta Y^-, r_i + \Delta r^-) & \text{with probability } q_{dd} \end{cases} \tag{12}$$

where $\Delta Y^{\pm}, \Delta r^{\pm}$ are the jumping increments and the four transition probabilities are both time-dependent and state-contingent. Figure (1) provides a graphical intuition for the bivariate binomial discretization over one step.

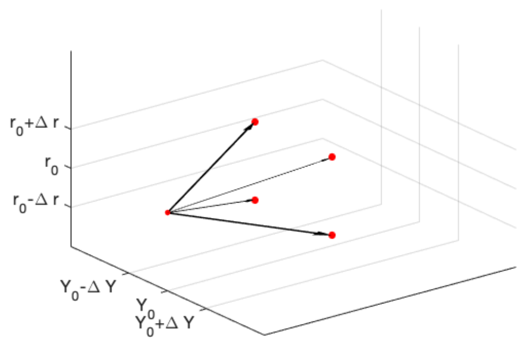

Figure 1: One step of the bivariate binomial discretization.

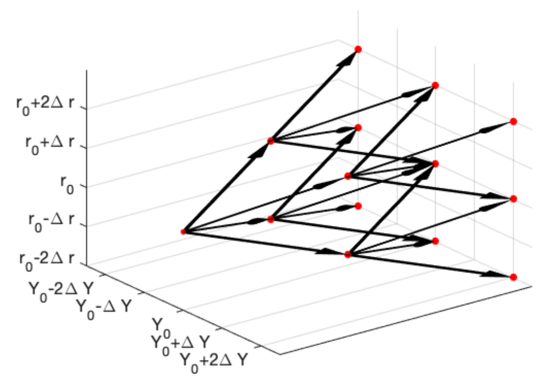

Figure 2: Two steps of the bivariate binomial discretization.

Globally, there are 8 parameters to pin down at each step: $\Delta Y^{\pm}$, $\Delta r^{\pm}$, q_{uu}, q_{ud}, q_{du} and q_{dd}.

In order to obtain a discretization that converges in distribution to the solution of (11), we need to match the first two moments of $Y(t)$ and $r(t)$ as well as their cross variation. In this way we impose five conditions on the eight parameters we need to determine. One more constraint has to be imposed on the four transition probabilities that need to sum up to one. Finally, we may want to impose a recombining structure to our quadrinomial tree in order to preserve tractability. Setting $\Delta Y^- = -\Delta Y^+ := \Delta Y$ and $\Delta r^- = -\Delta r^+ := \Delta r$

makes the number of different outcomes of our discretization grow quadratically (and non exponentially) in the number of steps. Figure (2) provides a graphical intuition of this trick: starting from (Y_0, r_0), after two steps the bivariate binomial process may assume nine possible values, namely all the possible ordered couples of $\{Y_0 - 2\Delta Y, Y_0, Y_0 + 2\Delta Y\}$ and of $\{r_0 - 2\Delta r, r_0, r_0 + 2\Delta r\}$. Thus, for a generic number of time steps n, the final possible outcomes of the discretization are $(n+1)^2$ rather than 2^{n+1}, the number of possible outcomes along a non recombining tree.

We now derive the explicit expressions of the increments and of the transition probabilities of our discretization for the bivariate SDEs (2).

Matching the first two moments of $Y(t)$ and $S(t)$ as well as their cross-variation, neglecting the Δt-second order terms, imposing the proper constraint on the probabilities and imposing a recombining tree as explained above lead to the following system of eight equations in eight unknowns:

$$\begin{cases} \mathbb{E}_t[\Delta Y] = & (q_{uu} + q_{ud})\Delta Y^+ + (q_{du} + q_{dd})\Delta Y^- & \overset{!}{=} \mu_Y \Delta t \\ \mathbb{E}_t[\Delta r] = & (q_{uu} + q_{du})\Delta r^+ + (q_{ud} + q_{dd})\Delta r^- & \overset{!}{=} \mu_r \Delta t \\ \mathbb{E}_t[\Delta Y^2] = & (q_{uu} + q_{ud})(\Delta Y^+)^2 + (q_{du} + q_{dd})(\Delta Y^-)^2 & \overset{!}{=} \sigma_S^2 \Delta t \\ \mathbb{E}_t[\Delta r^2] = & (q_{uu} + q_{du})(\Delta r^+)^2 + (q_{ud} + q_{dd})(\Delta r^-)^2 & \overset{!}{=} \sigma_r^2 \Delta t \\ \mathbb{E}_t[\Delta Y \Delta r] = & q_{uu}\Delta Y^+\Delta r^+ + q_{ud}\Delta Y^+\Delta r^- + & \\ & +q_{du}\Delta Y^-\Delta r^+ + q_{dd}\Delta Y^-\Delta r^- & \overset{!}{=} \rho\sigma_S\sigma_r\Delta t \\ & q_{uu} + q_{ud} + q_{du} + q_{dd} & \overset{!}{=} 1 \\ & \Delta Y^+ & \overset{!}{=} -\Delta Y^- \\ & \Delta r^+ & \overset{!}{=} -\Delta r^- \end{cases}$$

Imposing $\Delta Y^+ > \Delta Y^-$ and $\Delta r^+ > \Delta r^-$ we get:

$$\begin{aligned} \Delta Y^+ &= \sigma_S\sqrt{\Delta t} = -\Delta Y^- := \Delta Y \\ \Delta r^+ &= \sigma_r\sqrt{\Delta t} = -\Delta r^- := \Delta r \end{aligned} \tag{13}$$

$$
\begin{aligned}
q_{uu} &= \frac{\mu_Y \mu_r \Delta t + \mu_Y \Delta r + \mu_r \Delta Y + (1+\rho)\sigma_r \sigma_S}{4\sigma_r \sigma_S} \\
q_{ud} &= \frac{-\mu_Y \mu_r \Delta t + \mu_Y \Delta r - \mu_r \Delta Y + (1-\rho)\sigma_r \sigma_S}{4\sigma_r \sigma_S} \\
q_{du} &= \frac{-\mu_Y \mu_r \Delta t - \mu_Y \Delta r + \mu_r \Delta Y + (1-\rho)\sigma_r \sigma_S}{4\sigma_r \sigma_S} \\
q_{dd} &= \frac{\mu_Y \mu_r \Delta t - \mu_Y \Delta r - \mu_r \Delta Y + (1+\rho)\sigma_r \sigma_S}{4\sigma_r \sigma_S}.
\end{aligned} \tag{14}
$$

As noted in Nelson and Ramaswamy (1990), the four transition probabilities are not necessarily positive. In the limit, namely as $\Delta t \to 0$, we have $\Delta Y, \Delta r \to 0$ and, therefore, $q_{uu}, q_{dd} \to \frac{(1+\rho)}{4} > 0$ and $q_{ud}, q_{du} \to \frac{(1-\rho)}{4} > 0$. For $\Delta t > 0$, some of the four probabilities in (14) may become non-positive. In Appendix A we derive conditions leading to four non-negative probabilities is (14). Intuitively we get that q_{uu}, q_{du}, q_{ud} and q_{dd} are positive at any t as long as:

$$\underline{r} \leq r(t) \leq \overline{r}$$

where $\underline{r}$ and $\overline{r}$ depend on the parameters of the model but not on t. Namely, there exist two boundaries for the interest rate process outside which we need to "adjust" the transition probabilities in order to avoid the negative ones. We do so by setting equal to zero any transition probability that becomes negative and normalizing to 1 the other ones. This ensures that our bivariate discretization covers in the limit the entire support of the bivariate vector (Y, r).

Once we obtained a discretization of (2), we can map back the rate of return $Y(t)$ to the level $S(t)$ of the equity. In this way we have a lattice discretization of the solution of (2) and we name this discretization *quadrinomial tree.*

The following proposition shows that our quadrinomial tree converges in distribution to the solution of (2).

Proposition 2 ***(Convergence of the quadrinomial tree)*** *The bivariate discrete process $(X_i)_i$ defined in (12) with the parameters in (13) and (14) converges in distribution to*

$X(t) = (Y(t), r(t))$.

Proof. See Appendix B. ■

3 American Options

We now turn to American options. The holder of an American option can exercise it at any time before maturity T. This feature leads to an optimization problem as the holder tries to find the optimal exercise time that maximize her discounted profit. This flexibility does not come for free. Consequently, American options are usually more expansive than their European counterparts. This difference in the price is known as *early exercise premium*[5].

From now on, we will focus on American put (resp. call) option, whose final payoff is $\varphi(S) := (K - S)^+$ (resp. $\varphi(S) := (S - K)^+$). In a continuous time framework, as described in Section 21.5 of Björk (2009), the value at $t \leq T$ of an American put or call option on S with strike price K and maturity T is:

$$\begin{aligned} V(t) &= ess \sup_{t \leq \tau \leq T} \mathbb{E}^{\mathbb{Q}} \left[\frac{B(t)}{B(\tau)} \varphi(S(\tau)) \middle| \mathcal{F}_t \right] \\ &= ess \sup_{t \leq \tau \leq T} \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^\tau r(s) \mathrm{d}s} \varphi(S(\tau)) \middle| \mathcal{F}_t \right] \end{aligned} \tag{15}$$

where the *essential supremum* accounts for the fact that the sup is taken on an (uncountable) family of random variables defined up to zero-probability sets. In words, the value of the American option is determined by the optimal stopping time τ that maximizes the discounted payoff. It is well known that equation (15) admits no explicit formulation even in the standard Black ans Scholes market. In our case, since r is stochastic and not independent of S, we cannot split the conditional expected value in (15) into two simpler separate ones.

[5] Pressacco et al. (2008) perform a throughout comparison between lattice and analytical approximation of the early exercise premium in the Black-Scholes framework.

Nevertheless, the following Proposition shows that, as in the standard case of deterministic interest rate, the value of an American option $V(t)$ can be expressed as a deterministic function of time t (or, equivalently, of time to maturity $T-t$) and of the current value of both the underlying asset $S(t)$ and the short term interest rate $r(t)$. Moreover, this deterministic function inherits the same monotonicity properties with respect to t and S variables as in the constant interest rate enviroment.

Proposition 3 ***(Value of the American option as a deterministic function)*** *In the market described by* (1)*, the value of an American put option on* S (15) *is of the form:*

$$V(t) = F(t, S(t), r(t))$$

with $F : [0,T] \times \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{R}^+$ *given by:*

$$\begin{aligned} F(t,S,r) = \sup_{0 \le \eta \le T-t} \mathbb{E}^{\mathbb{Q}} \Bigg[& \exp\left(-\int_0^\eta r(s)\mathrm{d}s\right) \cdot \\ & \cdot \varphi\left(S \exp\left(\int_0^\eta r(s)\mathrm{d}s - \left(q + \frac{1}{2}\sigma_S^2\right)\eta + \sigma_S W_S(\eta)\right)\right)\Bigg]. \end{aligned} \tag{16}$$

The function F *is decreasing with respect to time* t*, convex with respect to* S *and increasing (resp. decreasing) in the call (resp. put) case. Moreover,* F *is increasing (resp. decreasing) in the call (resp. put) case with respect to* r*.*

Proof. See Appendix B. ■

At each point in time the holder of the option compares the immediate payoff she would get from an immediate exercise and the discounted future value of the option. As at any t both the present and the expected future values of the option depend on the current value of the underlying asset, S, and of the interest rate, r, it is convenient to divide the plane (S, r) into two complementary regions: the *early exercise region* and the *continuation region*. Provided that the holder of the option can distinguish the two, if for a given t she observes in the market a couple of values (S, r) that belongs to the early exercise region, the option will be exercised. If the observed couple belongs to the continuation one, the option will not be exercised at that moment.

We can now formally define the aforementioned regions. At each $t \in [0, T]$, the plane $(S, r) \in \mathbb{R}^+ \times \mathbb{R}$ can be divided into:

- the continuation region $CR(t) = \{(S, r) \in \mathbb{R}^+ \times \mathbb{R} : F(t, S, r) > \varphi(S)\}$, the set of couples (S, r) where it is optimal to continue the option at t; the r-section of the continuation region at r is $CR_r(t) = \{S \in \mathbb{R}^+ : F(t, S, r) > \varphi(S)\}$;
- the early exercise region $EER(t) = \{(S, r) \in \mathbb{R}^+ \times \mathbb{R} : F(t, S, r) = \varphi(S)\}$, the set of couples (S, r) where it is optimal to exercise the option at t; the r-section of the early exercise region at r is $EER_r(t) = \{S \in \mathbb{R}^+ : F(t, S, r) = \varphi(S)\}$.

When $r = r_0$ is deterministic and strictly positive, the two regions are separated by a *free boundary* $t \mapsto S^*(t)$, called *critical price*. It has no explicit expression but its asymptotic approximation as the time to maturity shrinks has been derived by Evans et al. (2002) and Lamberton and Villeneuve (2003). Chockalingam and Muthuraman (2015) approximate the free boundary with exponential functions and obtain a very accurate pricing algorithm in the standard Black-Scholes model.

For the American put option, the exercise region is downward connected, as shown in Figure (3): if it is optimal at t to exercise the option for some underlying values, it so for all the lower values of the underlying. When, on the contrary, $r = r_0 < 0$ and $r_0 - q - \frac{\sigma_S^2}{2} > 0$, Battauz et al. (2015) show that the early exercise region for the American put option lies in between two continuation regions. In this case there are two free boundaries: an upper one, $t \mapsto \bar{S}^*(t)$, and a lower one, $t \mapsto \underline{S}^*(t)$. Again, they have no explicit expressions but their asymptotic behaviours have been derived by Battauz et al. (2015). Therefore, the exercise region for the American put option is no more downward connected, as shown in Figure (4).

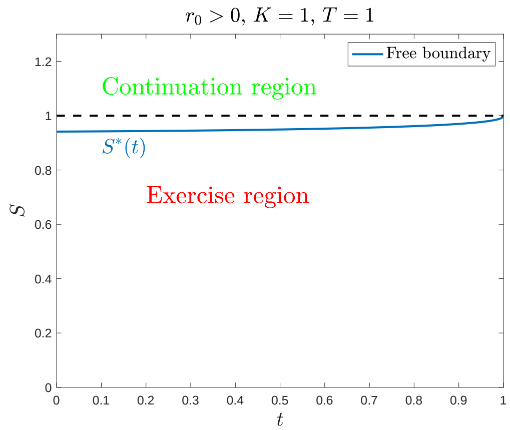

Figure 3: Free boundary of an American put option with $r_0 > 0$.

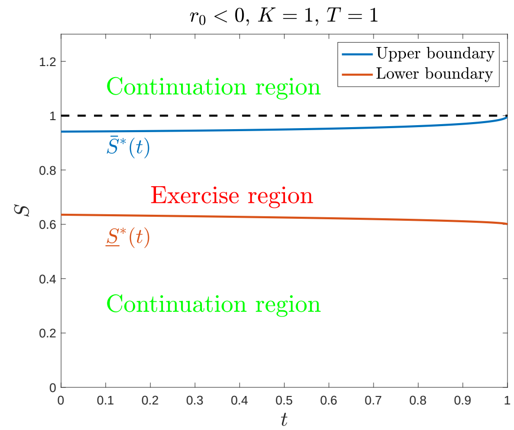

Figure 4: Free boundaries of an American put option with $\mu < 0$, $r_0 < 0$.

When r is constant, the intuition is the following. The value of the American put option, namely the counterpart of (16) is a deterministic function $G(t, S) : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ of the time and the current value of the underlying:

$$G(t, S) = \sup_{0 \leq \eta \leq T-t} \mathbb{E}^{\mathbb{Q}} \left[e^{-r\eta} \left(K - S \exp \left(\left(r - q - \frac{\sigma_S^2}{2} \right) \eta + \sigma_S W_S(\eta) \right) \right)^+ \right]. \quad (17)$$

Denote by $\mu := \left(r - q - \frac{\sigma_S^2}{2} \right)$ the drift of the equity price. The value of the risky asset is increasing in μ and, consequently, the value of the American put option is on average decreasing in μ. The value of the American option is also decreasing, on average, with respect to the discount rate r. Therefore, the impact on the value of the American option of r streams from two channels: the discount factor $e^{-r\eta}$ and the drift μ of the risky asset. When r decreases and becomes negative, the negative discount rate will make the holder of the American option eager to wait and cash in later on to gain from the negative interest rate. If also μ stays negative, the value of the underlying will decrease on average thus making the put option more valuable. Therefore, both of the effects will make the investor wait and exercise the option as late as possible, namely at maturity. In this case, the early exercise premium will disappear and the value of the American option will collapse into the value of its European counterpart. If, on the contrary, μ stays positive, the value of the underlying will increase on average thus making the put option less valuable. In this case the effects of the discount factor and of the the drift are not aligned and a trade-off arises: the investor would like to wait in order to cash in later on and gain from the negative interest rate but in this way she faces the risk of the payoff's depreciation, as the value underlying is expected to increase. In this case, $r < 0$ and $\mu = r - q - \frac{\sigma_S^2}{2} > 0$, the aforementioned tradeoff translates in a non-empty exercise region surrounded by a double continuation region. In particular, a non standard continuation region appears when the option is most deeply in the money. Here the risk of the payoff's depreciation is very mild and the preference for the postponement due to the negative interest rate prevails.

Notice that in the latter case, as $r < 0$ and $\sigma_S > 0$, $\mu > 0$ implies a continuous negative dividend yield $q < 0$, that can appear as a storage cost when dealing with options on commodities or as an "artificial drift" capturing the interplay of domestic and foreign interest rate when evaluating quanto options (see Battauz et al. (2018), (2015)). Similar arguments hold for the American call option.

When the interest rate is stochastic, both the discount factor $e^{-\int_0^\eta r(s)\mathrm{d}s}$ and the drift $\mu := \int_0^\eta r(s)\mathrm{d}s - \left(q + \frac{1}{2}\sigma_S^2\right)\eta$ in (16) depends on the whole path of r, from its current value up to the exercise time η. Therefore, at a given instant of time $t \in [0, T]$ and for a generic couple of state variables (S, r), the tradeoff between the discount factor and the expected drift of S, and consequently the presence of a singular or a double continuation region, depend on the all the possible future realizations of r, especially on its expected sign. The following Proposition formalizes this intuition and provides necessary conditions for the existence of optimal early exercise opportunities for American options when the current interest rate value determines the existence of a zero-coupon-bond price greater than 1. This is very likely to occur when the current interest rate value is non-positive.

Proposition 4 ***(Asymptotic necessary conditions for the existence of optimal early exercise opportunities)*** *In the market described by* (1)*, at any point in time t and given the current value of the interest rate $r(t) = r$, suppose that*

[NC0] $r\alpha - \theta(\alpha + (T - t)) > 0$ *with* $\alpha = \frac{e^{-\kappa(T-t)}-1}{\kappa} \leq 0$

Then the following are jointly necessary conditions for the existence of optimal exercise opportuinites at t, for $T - t \approx 0$:

[NC1] the dividend yield is non positive, $q \leq 0$;

[NC2] for some S, $\pi_E(t, S, r) = \varphi(S)$, where $\pi_E(t, S, r)$ is the value of the European put (resp. call) option defined in Proposition 1.

Proof. See Appendix B. ■

The condition *[NC0]* is very likely satisfied when $r < 0$, as the long-run mean of the interest rate θ is commonly assumed to be positive. *[NC1]* ensures that the discounted price of the risky security is not a supermartingale. If this was the case, we show in the proof that, under condition *[NC0]*, this would lead automatically to optimal exercise of the American put option at maturity only. A similar argument holds true for the American call option. For the American put option, if early exercise is optimal under condition *[NC0]*, then EER_r, the early exercise region section at r, is bounded by below by a strictly positive (non standard) lower boundary.

A change of numeraire allows to obtain the very same result for the American call option. We remark that our results cannot be obtained from standard symmetry results for American options (see Battauz et al. (2015) and the references therein) due to the stochasticity of our interest rates. In the standard Black-Scholes case, the American put-call symmetry swaps the constant interest rate with the constant dividend yield. Being our interest rate stochastic and our dividend yield constant, such symmetry result is not viable. Under *[NC0]*, *[NC2]* ensures that the price of the European option $\pi_E(t,S,r)$ does not dominate the immediate payoff value. If this was the case, the American option would dominate the immediate payoff value as well, thus preventing the existence of optimal early exercise opportunities. Although the formal proof of the necessary conditions in Proposition 4 requires the time to maturity to be small enough, we show in the following section that actually those conditions correctly identify nodes on the tree in which a double continuation region appears along the whole lifetime of the option (see Figure 7). In the following theorem we describe the main properties of the free boundary surface, under the assumption that the early exercise region is non-empty. We distinguish between the standard case of a non-negative interest rate and the case of a negative interest rate, where unusual optimal continuation policies may appear.

Theorem 5 ***(The free-boundary surface)***

1. *Suppose $r \geq 0$ and assume that $EER_r(\bar{t})$ is non-empty for some $\bar{t} \in (0,T)$. For the American put option*

$$\overline{S}^*(t,r) = \sup\{S \geq 0 : F(t,S,r) = \varphi(S)\} \tag{18}$$

defines the (standard upper) free boundary and early exercise is optimal at any $t \geq \bar{t}$ for $S(t), r(t) = r$ if $S(t) \leq \overline{S}^*(t,r)$. The free boundary $\overline{S}^*(t,r)$ is increasing with respect to $t \geq \bar{t}$ and $r \geq 0$.

For the American call option

$$\underline{S}^*(t,r) = \inf\{S \geq 0 : F(t,S,r) = \varphi(S)\} \tag{19}$$

defines the (standard lower) free boundary and early exercise is optimal at any $t \geq \bar{t}$ *for* $S(t), r(t) = r$ *if* $S(t) \geq \underline{S}^*(t,r)$*. The free boundary* $\underline{S}^*(t,r)$ *is decreasing with respect to* $t \geq \bar{t}$ *and increasing with respect to* $r \geq 0$*.*

2. *Suppose* $r < 0$ *and that the necessary conditions of Propositions 4 are satisfied with* $q < 0$ *and assume that* $EER_r(\bar{t})$ *is non-empty. Then the segment with extremes* $[\underline{S}^*(t,r), \overline{S}^*(t,r)]$ *(see equations (18), (19)) is non-empty for any* $t \in [\bar{t}, T]$*. The option is optimally exercised at any* $t \geq \bar{t}$ *for* $S(t), r(t) = r$ *whenever* $S(t) \in \left[\underline{S}^*(t,r), \overline{S}^*(t,r)\right]$*. The lower free boundary,* $\underline{S}^*(t,r)$*, is decreasing with respect to* t *and the upper free boundary* $\overline{S}^*(t,r)$ *is increasing with respect to* t *for any* $t \geq \bar{t}$*. For the American put*

$$\frac{rK}{q} \leq \underline{S}^*(t,r) < \overline{S}^*(t,r) \leq K.$$

Their limits at maturity are $\lim_{t\to T} \overline{S}^*(t,r) = K = \underline{S}^*(T,r)$ *and* $\overline{S}^*(T^-,r) = \lim_{t\to T} \underline{S}^*(t,r) = \frac{rK}{q} > \overline{S}^*(T,r) = 0$*. The lower free boundary,* $\underline{S}^*(t,r)$*, is decreasing with respect to* r *and the upper free boundary* $\overline{S}^*(t,r)$ *is increasing with respect to* r*.*

For the American call

$$K \leq \underline{S}^*(t,r) < \overline{S}^*(t,r) \leq \frac{rK}{q}.$$

The lower free boundary, $\underline{S}^*(t,r)$*, is decreasing with respect to* t *and the upper free boundary* $\overline{S}^*(t,r)$ *is increasing with respect to* t *for any* $t \geq \bar{t}$*. The lower free boundary,* $\underline{S}^*(t,r)$*, is increasing with respect to* r *and the upper free boundary* $\overline{S}^*(t,r)$ *is decreasing with respect to* r*. Their limits at maturity are* $\lim_{t\to T} \underline{S}^*(t,r) = K = \underline{S}^*(T,r)$ *and* $\overline{S}^*(T^-,r) = \lim_{t\to T} \overline{S}^*(t,r) = \frac{rK}{q} < \overline{S}^*(T,r) = +\infty$*.*

3. *Suppose* $r < 0$ *and* $q = 0$. *Then the early exercise region for the American put option at* t *is empty.*

   For the American call, suppose $EER_r(\bar{t})$ *is non-empty for some* $\bar{t} \in (0, T)$. *Then early exercise is optimal at any* $t \geq \bar{t}$ *for* $S(t), r(t) = r$ *if* $S(t) \geq \underline{S}^*(t, r)$ *(see equation (19)). The free boundary* $\underline{S}^*(t, r)$ *is decreasing with respect to* $t \geq \bar{t}$ *and increasing with respect to* $r \geq 0$

Proof. See Appendix B. ■

3.1 Numerical examples

We now present and describe three illustrative numerical examples that show the optimal exercise strategies and the possible characterizations of the continuation region for the American put and call options in the market described by (1), highlighting the free boundary's features derived in Theorem (5).

We exploit our quadrinomial tree to evaluate American options by backward induction. Once the whole quadrinomial tree, namely all the couples (S, r) and the related transition probabilities, have been generated, we start from the values of the state variables S and r at maturity T. At maturity, the American option is exercised in all the nodes in which it is in the money; the resulting payoff is the value of the American option at T. At any other generic instant $t \in \{0, \Delta t, 2\Delta t, \ldots, T - \Delta t\}$, and for any couple $(S(t), r(t))$, we compute the immediate payoff $\varphi(S)$ and we compare it to the continuation value of the option. The continuation value is obtained as the discounted (by the current realization of $r(t)$) expected value (according the transition probabilities computed at $(S(t), r(t))$) of the four values of the American option at $t+\Delta t$ connected on the tree to the current node. From the comparison between the immediate exercise and the continuation value, we get the value of the American option in the node $(S(t), r(t))$. Going backward, we finally get the price of the American option at $t = 0$.

Proposition 2 showed that the quadrinomial tree we proposed converges in distribution to the bivariate process that solves (1), as the time step shrinks. Mulinacci and Pratelli (1998) prove that the convergence in distribution of the lattice-based approximation of the underlying state variables implies that the price of the American option evaluated according to the backward procedure described above converges to its theoretical value given by (15).

In all of the three following examples the parameters are: $T = 1$, $n = 125$, $S_0 = K = 1$, $\sigma_S = 0.15$, $r_0 = 0$, $\theta = 0.02$, $\kappa = 1$, $\sigma_r = 0.01$ and $\rho = 0.05$. The dividend yield q of the equity is the only parameter that varies across the examples: in the first one we set $q = 0$, in the second $q = 0.02$ and $q = -0.02$ in the last one.

For each example we:

- compute the value at inception of the European counterpart π_E obtained both with the formula of Proposition 1 and along the quadrinomial tree (the values obtained in the two ways are indistinguishable);
- compute the value at inception of the American option π_A along the quadrinomial tree;
- compute the price of the American option, $\pi_A^{r_0}$, evaluated along the standard binomial tree of Cox et al. (1979) with a deterministic interest rate $r = r_0 = 0$[6]. Our aim is to quantify the error that an "unsophisticated" investor would make by evalu-

[6] we also evaluate the American option with a deterministic interest rate equal to the expected value of r over the investment period; namely, we also set $r = \mathbb{E}^{\mathbb{Q}}[r(T)] = 1.26\%$. This exercise delivers qualitatively similar results. With respect to the last column of Table 1, the relative errors in this case are, respectively, 4.58%, 4.64%, 4.52%.

ating American options within a flat term structure framework rather than within a fluctuating one;

- graphically show how the single, or double (if any), free boundaries look like in the space (t, S, r). These graphs characterize the optimal exercise policy: at any t, the investor should look at the current values of $(S(t), r(t))$;
- graphically highlight the nodes of the quadrinomial tree where the necessary conditions of Proposition 4 are satisfied.

We first show the numerical results for the American put option that are summed up in Table 1.

Figure	q	π_E	π_A	$\pi_A^{r_0}$	$\lvert\pi_A - \pi_A^{r_0}\rvert/\pi_A$
5	0%	5.620%	5.712%	5.979%	4.67%
6	2%	6.565%	6.570%	6.962%	5.96%
7	-2%	4.763%	5.030%	5.230%	3.97%

Table 1: Results from the three numerical examples for the American put option.

First example: $q = 0\%$. If the underlying pays no dividend and its volatility is reasonably small, the expected drift of S basically coincides with $r(t)$. This splits the domain of r in two complementary regions according to the sign of r, as can be seen in the right panel of Figure 5 (that displays the free boundary section at $t = \frac{T}{2}$). In the left region where r and μ are both negative, the investor is willing to wait and postpone the exercise as much as possible in order to gain from both the negative discount rate and the implied expected depreciation of S. In the right region, on the contrary, where r and μ are both positive, we have the standard tradeoff between a positive discount rate (that makes the investor willing to exercise the option as soon as possible) and a negative expected drift of S (that makes the investor willing to wait for a larger payoff). This generates the standard upper boundary shown in the left panel of Figure 5. We notice that the standard upper boundary is increasing with respect to r. Indeed, early exercise is more profitable when r increases and S is likely to appreciate.

The investor who believes that the term structure is flat and evaluates the American put option with a constant discount rate equal to our r_0 makes a relative error almost equal to 5%. This figure is economically significant as it is greater than the maximal error due to suboptimal exercise delay of the option as estimated[7] in Chockalingam and Feng (2015).

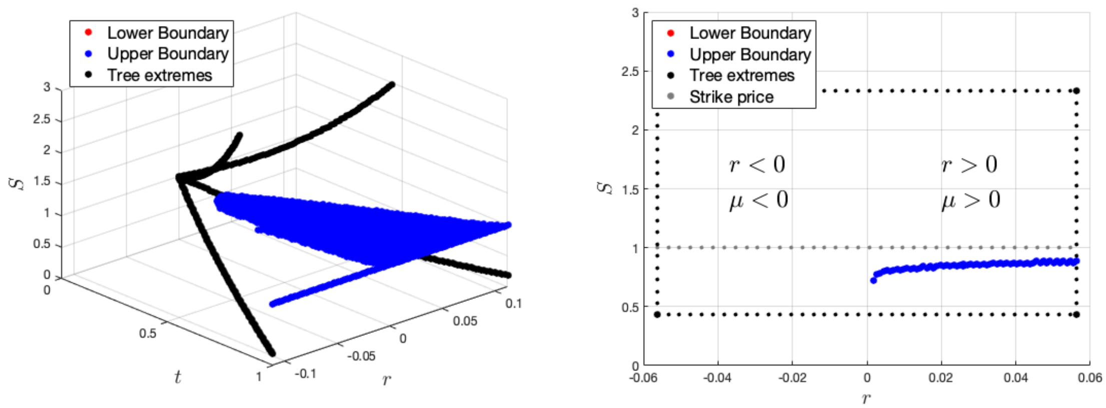

Figure 5: First example, American put: $q = 0\%$.

Second example: $q = 2\%$. If the underlying pays (positive) dividends, the instantaneous drift of S, $\mu(t)$, is equal to $r(t)$ plus a negative quantity ($-q - \frac{\sigma_S^2}{2} < 0$). This splits the domain of r into three complementary regions. The first one in which r and μ are both negative, the one in which r is positive but small so that μ is still negative, the last one

[7] Our relative error of 4.64% in the first line of Table 1 corresponds to an absolute pricing error of 27.8 bps. This figure is indeed significant compared to the maximal error obtained in Figure 3 by Chockalingam and Feng (2015). In particular, Figure 3, second row, right column, in Chockalingam and Feng (2015), displays a pricing error of 4 bps, after a rescaling to unit moneyness and with volatility equal to 20%.

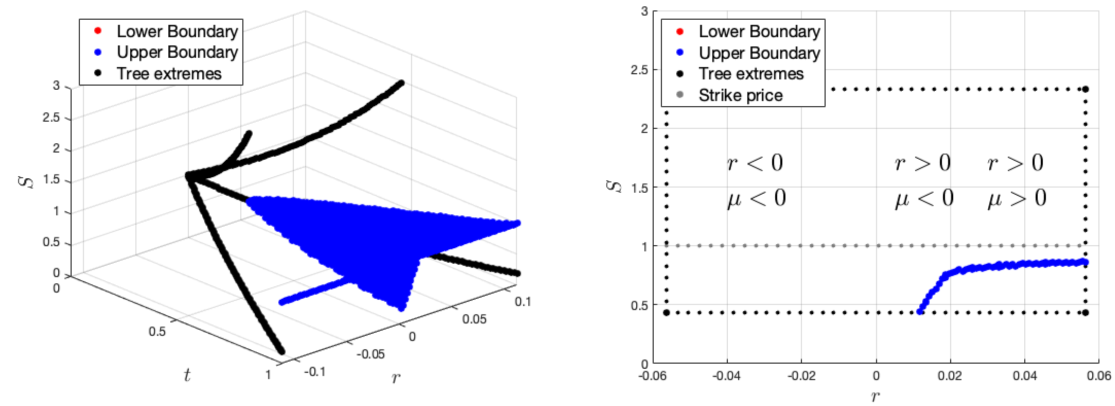

Figure 6: Second example, American put: $q = 2\%$.

in which r and μ are both positive. In the first one, the option is optimally exercised at maturity, as before. In the middle region there is a new tradeoff: the investor would like to cash in as soon as possible due to $r > 0$ but the value of S is expected to decrease as $\mu < 0$. This allows for a standard upper boundary. The critical price below which the investor will exercise, though, becomes smaller as r approaches 0: as r decreases the threat of the positive discount rate weakens and, therefore, the investor would postpone the exercise unless the underlying reaches a very low level. In other words, if the discount is not that strong, the investor prefers to gain the relative high dividend yield keeping the asset as long as possible. In the last region, we find the standard behaviour already outlined in the first example.

The investor who believes that the term structure is flat and evaluates the American option with a constant interest rate makes here an even higher relative error than before (5.78%).

Third example: $q = -2\%$. In the case of negative dividends[8], the instantaneous drift of S, $\mu(t)$, is equal to $r(t)$ plus a quantity which is now positive ($-q - \frac{\sigma_S^2}{2} > 0$). As a result, μ

[8] As previously discussed, negative dividends might model storage and insurance cost for commodities such as gold or domestic risk-neutral drifts of foreign equities in quanto options.

may be positive also when r is mildly negative. This splits again the domain of r into three complementary regions, as shown in the top-right panel of Figure 6: the one in which r and μ are both negative, the one in which r is negative but μ is positive and the last one in which r and μ are both positive. In the first region, the option is again optimally exercised at maturity as in the previous examples. In the middle section a double continuation region appears: this is the case in which the necessary conditions in Proposition 4 are satisfied as documented in the bottom panels of Figure 6. To the best of our knowledge, this is the first paper that documents the existence of a non standard double free boundary in a stochastic interest rates framework, generalizing the result obtained in the constant interest rates setting by Battauz et al. (2015). In the last region where both r and μ are positive, we find the standard behaviour already outlined in the first two examples.
Finally, this is the case in which the investor who believes that the term structure is flat and evaluates the American option with a constant discount rate makes the largest relative error (6.14%).
In Appendix C we provide additional plots of the free boundaries, that illustrate their time-dependence structure. In particular, we show that, for fixed values of $r = r(t)$, the upper critical price of the American put is increasing with respect to time t whereas the lower critical price (if any) is decreasing, as already documented in the constant interest rate framework by (Battauz et al., 2015).

We now turn to the American call options. Numerical pricing results for the American call option in the same scenarios analysed above for the American put option are summed up in Table 2. We notice that in all cases the investor who believes that the term structure is flat and evaluates the American call option with a constant discount rate equal to our r_0

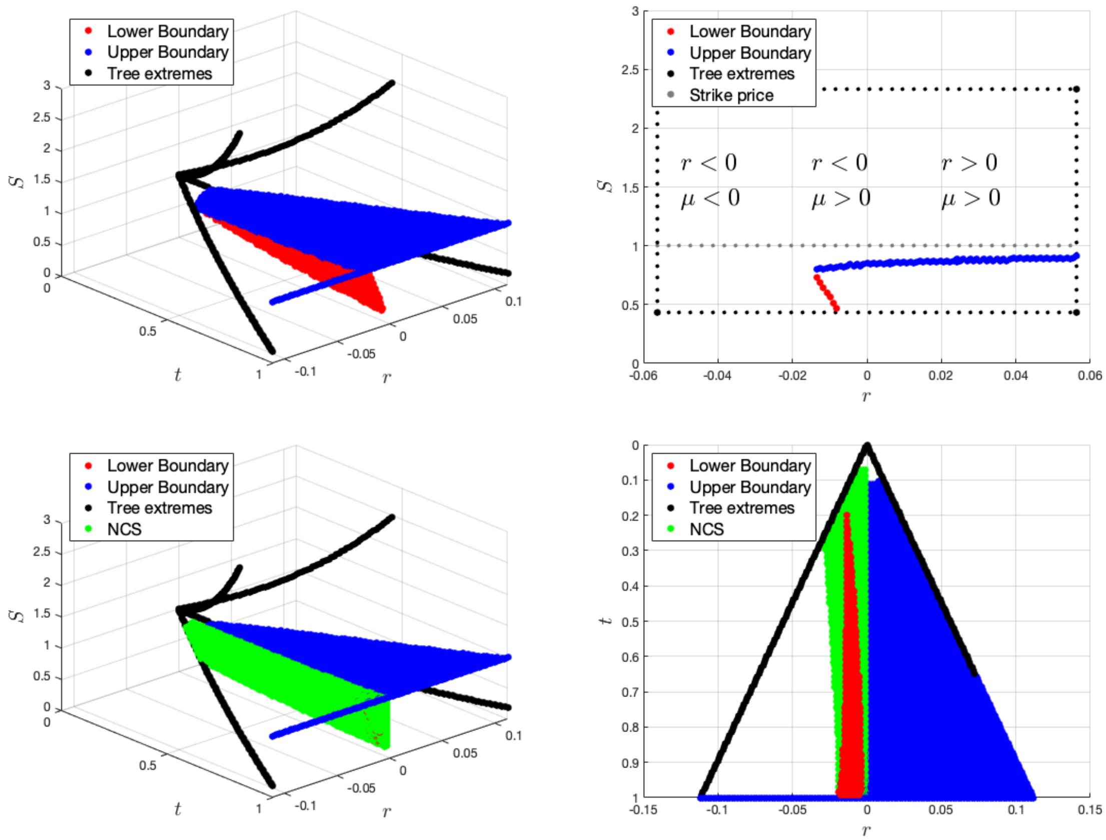

Figure 7: Third example, American put: $q = -2\%$. Green points in the bottom panels show the nodes of the quadrinomial tree in which necessary conditions *[NC0]*, *[NC1]* and *[NC2]* of Proposition 4 for a double continuation region hold simultaneously.

makes a relative error between 4% and 5%.

It is well known that American options on non-paying dividend assets do not display any early exercise premium. This is true under usual market circumstances, i.e. when interest rates are non negative. In fact, in this case, the zero-coupon bonds of any maturity have initial prices that are smaller than one, i.e. $p(0, \tau) < 1$ for any $\tau \in [0, T]$. This implies that the option is optimally exercised at maturity only, as Jensen's inequality implies that

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[(S(\tau)-K)^{+}\, e^{-\int_0^\tau r(s)\mathrm{d}s}\right] &\geq (S(0)-Kp(0,\tau))^{+} \\ &> (S(0)-K)^{+}\,.\end{aligned}$$

The same holds true if S pays a negative dividend yield as $\mathbb{E}^{\mathbb{Q}}\left[S(\tau)e^{-\int_0^\tau r(s)\mathrm{d}s}\right] = S(0)e^{-q\tau} > S(0)$.

Within our framework, interest rates are not always positive and zero-coupon bonds may have initial prices larger than one. Thus, early exercise may be optimal under some circumstances as one can indeed see in the following first example.

Figure	q	π_E	π_A	$\pi_A^{r_0}$	$\vert\pi_A - \pi_A^{r_0}\vert/\pi_A$
8	0%	6.339%	6.339%	5.979%	5.69%
9	2%	5.314%	5.396%	5.163%	4.32%
10	-2%	7.511%	7.511%	7.102%	5.44%

Table 2: Results from the three numerical examples for the American call option.

First example: $q = 0\%$. As explained above, early exercise may be optimal in this case only if zero-coupon bonds display initial prices larger than one for some maturity. This is the case portraited in Figure 8, where a (standard lower) free boundary for the American call option is documented for initial interest rates values smaller than 1%. To our knowledge, this is the first paper that shows the existence of optimal early exercise opportunities for an American call option when the dividend yield is zero. We notice that the critical price, and thus the continuation region, is increasing in r, as the increasing drift μ of S pushes the option towards the in the money region. The impact of these optimal early exercise opportunities on the price of the option, however, is negligible because the risk-neutral probability of the equity price entering the early exercise region is quite small, as one can see from the first row of Table 2.

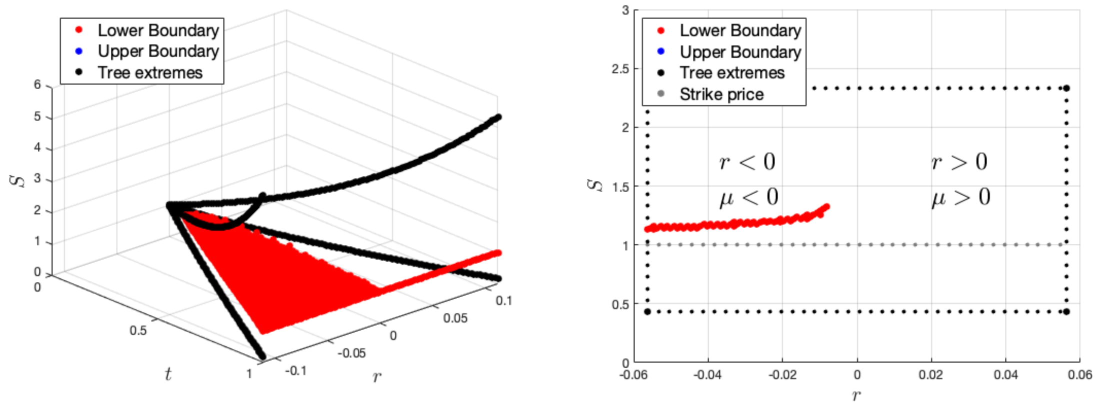

Figure 8: First example, American call: $q = 0\%$.

Second example: $q = 2\%$. When the dividend yield is positive, early exercises of the American call option become profitable. In Figure 9 we document the existence of a (lower standard) free boundary that is again increasing in r. Interestingly, the slope of the free boundary becomes steeper when μ, the drift of S, turns positive, and the continuation region increases substantially as S is expected to appreciate. Consequently, early exercise in this case is optimal only if S is very deeply in the money.

Third example: $q = -2\%$. As already discussed for the American put option example, when the dividend yield is negative, the instantaneous drift of S, μ, is always positive but for very negative values of r. As a result, early exercise for the American call option is never optimal unless r is very negative. In this case, for negative values of r, a non standard early exercise region appears surrounded by two continuation regions (see the top panels of Figure 10). However, as in the first example with $q = 0\%$, the early exercise premium does not significantly contribute to the price of the American call option because the equity price enters the early exercise region with a very small risk-neutral probability, as one can see from the third row of Table 2. The green dots in the bottom panels of Figure 10 mark

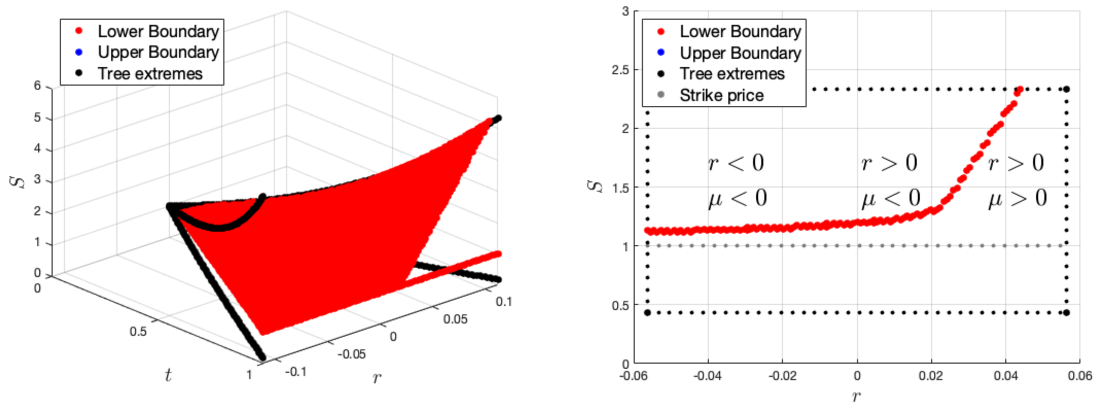

Figure 9: Second example, American call: $q = 2\%$.

the region where our necessary conditions for non standard early exercise of Proposition 4 are satisfied. We notice that this region overlaps very accurately with the area where early exercise is optimal as portrait in the top-left panel of Figure 10.

In Appendix C we analyze the time-dependence of the free boundaries of American call options with additional plots. In particular, we show that, for fixed values of $r = r(t)$, the upper critical price (if any) of the American call is decreasing with respect to time t whereas the lower critical price is increasing, thus confirming the results of (Battauz et al., 2015) in a constant interest rate enviroment.

4 Conclusions

In this paper we have studied American equity options in a correlated stochastic interest rate framework of Vasicek (1977) type. We have introduced a tractable lattice-based discretization of the equity price and interest rate processes by means of a quadrinomial tree. Our quadrinomial tree matches the joint discretized moments of the equity price and the

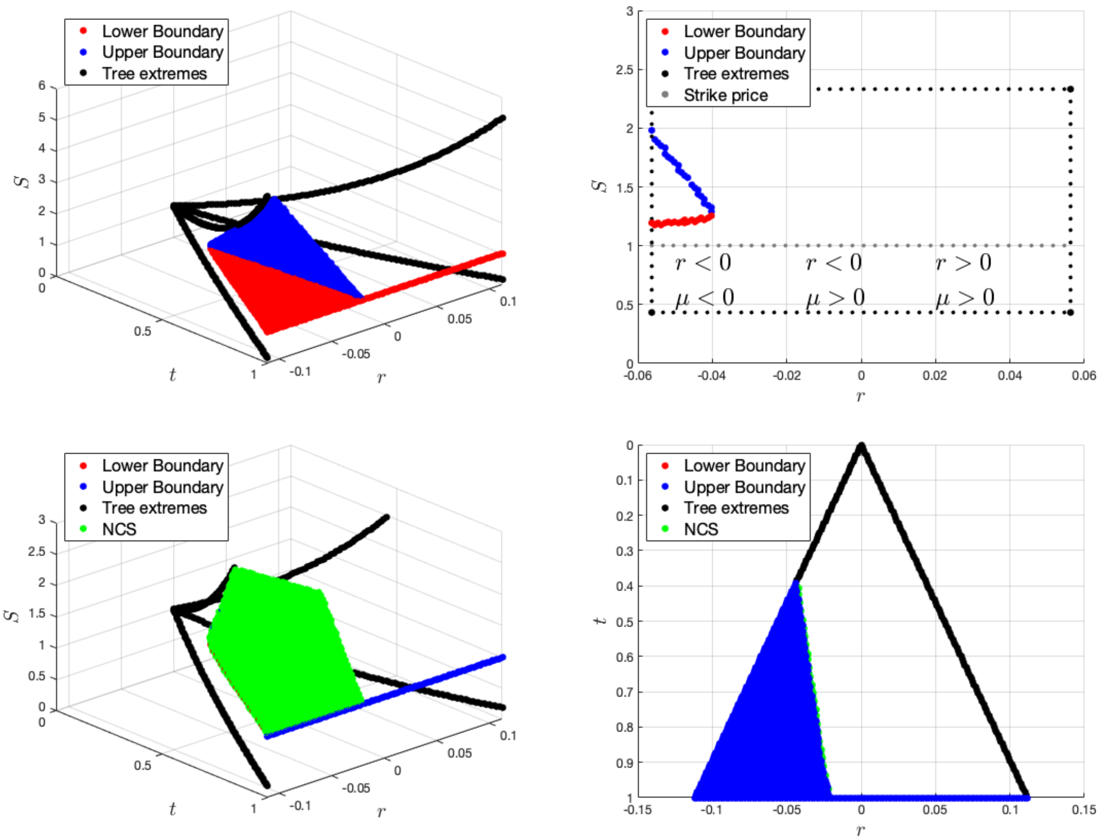

Figure 10: Third example, American call: $q = -2\%$. Green points in the bottom panels show the nodes of the quadrinomial tree in which necessary conditions *[NC0]*, *[NC1]* and *[NC2]* of Proposition 4 for a double continuation region hold simultaneously.

stochastic interest rate and converges in distribution to the continuous time original processes. This allowed us to employ our quadrinomial tree to characterize the two-dimensional free boundary for an American equity put and call option, that consists of the underlying asset and the interest rate values that trigger the optimal exercise of the option. Our results are in line with the existing literature when interest rates lie in the positive realm. In particular, for the American put options, the higher the dividend yield, the higher the benefits from deferring the option exercise. Moreover, in this case, the exercise region is downward connected with respect to the underlying asset value (see, for instance, Detemple (2014)). On the contrary, when interest rate are likely to assume even mildly negative values, optimal exercise policies change, depending on the tradeoff between the interest rate and the expected rate of return on the equity price. If such expected rate of return is negative, optimal exercise occurs at maturity only as the option goes (on average) deeper in the money as time goes by and the negative interest rates make the investor willing to cash in as late as possible. If the expected rate of return on the equity asset is positive, the option is expected to move towards the out of the money region. This effect is compensated by the preference to postponement due to negative interest rates. The tradeoff results in a non-standard double continuation region that violates the aforementioned downward connectedness of the exercise region for American put option. We quantified the pricing error that an investor would make assuming a constant interest rate and therefore neglecting the variability (and the related risk) of the term structure. Finally, we documented similar non standard optimal exercise policies also for American call options. In particular, we find that early exercise of the American call option might be optimal even when the equity does not pay any dividend.

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Appendix

A Bounds of the probabilities in the Quadrinomial Tree

Recall that at each t the four probabilities of an upward/downward movement of r/Y on the tree are:

$$
\begin{aligned}
q_{uu} &= \frac{\mu_Y \mu_r \Delta t + \mu_Y \Delta r^+ + \mu_r \Delta Y^+ + (1+\rho)\sigma_r \sigma_S}{4\sigma_r \sigma_S} \\
q_{ud} &= \frac{-\mu_Y \mu_r \Delta t + \mu_Y \Delta r^+ - \mu_r \Delta Y^+ + (1-\rho)\sigma_r \sigma_S}{4\sigma_r \sigma_S} \\
q_{du} &= \frac{-\mu_Y \mu_r \Delta t - \mu_Y \Delta r^+ + \mu_r \Delta Y^+ + (1-\rho)\sigma_r \sigma_S}{4\sigma_r \sigma_S} \\
q_{dd} &= \frac{\mu_Y \mu_r \Delta t - \mu_Y \Delta r^+ - \mu_r \Delta Y^+ + (1+\rho)\sigma_r \sigma_S}{4\sigma_r \sigma_S}
\end{aligned}
\tag{A1}
$$

with $\Delta r^+ = \sigma_r \sqrt{\Delta t}$, $\Delta Y^+ = \sigma_S \sqrt{\Delta t}$, $\mu_Y = r(t) - q - \frac{\sigma_S^2}{2}$ and $\mu_r = \kappa(\theta - r(t))$. From now on we light the notation writing r instead of $r(t)$. Nevertheless, it is crucial to remember that these probabilities are different for each node of the quadrinomial tree.

As already pointed out, the four probabilities sum up to one by construction. Unfortunately, they do not necessarily lie in (0,1). As a first control, we investigate what happens as the length of the time step goes to zero, namely, as $\Delta t \to 0$. We have

$$
\lim_{\Delta t \to 0} q_{uu} = \lim_{\Delta t \to 0} q_{dd} = \frac{1+\rho}{4},
$$

$$
\lim_{\Delta t \to 0} q_{ud} = \lim_{\Delta t \to 0} q_{du} = \frac{1-\rho}{4}
$$

which are all positive quantities (at least as $\rho \in (-1,1)$). Therefore, the problem of having possibly negative probabilities is only due to the discretization procedure.

For instance, with $n = 250$ steps and $T = 1$ (that corresponds to $\Delta t = 0.004$), we need to impose the positivity constraint on all the four numerators in (A1).

Imposing $q_{uu} \geq 0$ and solving with respect to r leads to:

$$
A_{uu} r^2 + B_{uu} r + C_{uu} \leq 0 \tag{A2}
$$

where:

$$
\begin{aligned}
A_{uu} = & \quad \kappa \\
B_{uu} = & \quad -\kappa\left(\theta + q + \frac{\sigma_S^2}{2} - \frac{\sigma_S}{\sqrt{\Delta t}}\right) - \frac{\sigma_r}{\sqrt{\Delta t}} \\
C_{uu} = & \quad -\kappa\theta\left(-q - \frac{\sigma_S^2}{2} + \frac{\sigma_S}{\sqrt{\Delta t}}\right) - \frac{\sigma_r}{\sqrt{\Delta t}}\left(-q - \frac{\sigma_S^2}{2}\right) - \frac{(1+\rho)\sigma_r\sigma_S}{\Delta t}.
\end{aligned}
$$

Provided that the discriminant of equation (A2) is positive, which surely holds true as $\Delta t \to 0$, the solution is $\underline{r}_{uu} \leq r \leq \overline{r}_{uu}$, where, of course,

$$
\underline{r}_{uu} = \frac{-B_{uu} - \sqrt{B_{uu}^2 - 4A_{uu}C_{uu}}}{2A_{uu}} \quad \text{and} \quad \overline{r}_{uu} = \frac{-B_{uu} + \sqrt{B_{uu}^2 - 4A_{uu}C_{uu}}}{2A_{uu}}.
$$

Similarly, we can work out all of the other probabilities.

Imposing $q_{ud} \geq 0$ leads to:

$$
A_{ud}r^2 + B_{ud}r + C_{ud} \geq 0
$$

where:

$$
\begin{aligned}
A_{ud} = & \quad \kappa \\
B_{ud} = & \quad -\kappa\left(\theta + q + \frac{\sigma_S^2}{2} - \frac{\sigma_S}{\sqrt{\Delta t}}\right) + \frac{\sigma_r}{\sqrt{\Delta t}} \\
C_{ud} = & \quad -\kappa\theta\left(-q - \frac{\sigma_S^2}{2} + \frac{\sigma_S}{\sqrt{\Delta t}}\right) - \frac{\sigma_r}{\sqrt{\Delta t}}\left(q + \frac{\sigma_S^2}{2}\right) + \frac{(1-\rho)\sigma_r\sigma_S}{\Delta t},
\end{aligned}
$$

that is solved by $r \leq \underline{r}_{ud} \cup r \geq \overline{r}_{ud}$.

Imposing $q_{du} \geq 0$ leads to:

$$
A_{du}r^2 + B_{du}r + C_{du} \geq 0
$$

where:

$$
\begin{aligned}
A_{du} = & \quad k \\
B_{du} = & \quad -\kappa\left(\theta + q + \frac{\sigma_S^2}{2} + \frac{\sigma_S}{\sqrt{\Delta t}}\right) - \frac{\sigma_r}{\sqrt{\Delta t}} \\
C_{du} = & \quad -\kappa\theta\left(-q - \frac{\sigma_S^2}{2} - \frac{\sigma_S}{\sqrt{\Delta t}}\right) + \frac{\sigma_r}{\sqrt{\Delta t}}\left(q + \frac{\sigma_S^2}{2}\right) + \frac{(1-\rho)\sigma_r\sigma_S}{\Delta t},
\end{aligned}
$$

that is solved by $r \leq \underline{r}_{du} \cup r \geq \overline{r}_{du}$.

Finally, imposing $q_{dd} \geq 0$ leads to:

$$A_{dd}r^2 + B_{dd}r + C_{dd} \leq 0$$

where:

$$\begin{aligned}
A_{dd} = &\quad \kappa \\
B_{dd} = &\quad -\kappa\left(\theta + q + \frac{\sigma_S^2}{2} + \frac{\sigma_S}{\sqrt{\Delta t}}\right) + \frac{\sigma_r}{\sqrt{\Delta t}} \\
C_{dd} = &\quad -\kappa\theta\left(-q - \frac{\sigma_S^2}{2} - \frac{\sigma_S}{\sqrt{\Delta t}}\right) + \frac{\sigma_r}{\sqrt{\Delta t}}\left(-q - \frac{\sigma_S^2}{2}\right) - \frac{(1+\rho)\sigma_r\sigma_S}{\Delta t}.
\end{aligned}$$

that is solved by $\underline{r}_{dd} \leq r \leq \overline{r}_{dd}$.

Summing up, probabilities in (A1) stay positive as long as r satisfies:

$$\begin{cases}
\underline{r}_{uu} \leq r \leq \overline{r}_{uu} \\
r \leq \underline{r}_{ud} \cup r \geq \overline{r}_{ud} \\
r \leq \underline{r}_{du} \cup r \geq \overline{r}_{du} \\
\underline{r}_{dd} \leq r \leq \overline{r}_{dd}
\end{cases}$$

The solution to the previous system of inequalities depends on the sign of the correlation ρ. Given the sign of ρ, the eight extremes values $\underline{r}_{uu}, \underline{r}_{ud}, \ldots, \overline{r}_{du}, \overline{r}_{dd}$ always satisfy the same chain of inequalities. Furthermore, notice that this eight values depend only on the parameters of the model and not on t.

When $\rho \in (0, 1]$, the only interval on which all of the inequalities hold true is $\overline{r}_{ud} \leq r \leq \underline{r}_{du}$ as it can be conveniently seen in Figure 11.

The intuition here is that when r and S move together and the discretization of r reaches values far away from its long run mean θ, a further movement of r away from θ and in the opposite direction of S is extremely unlikely and, eventually, happens "with a negative probability".

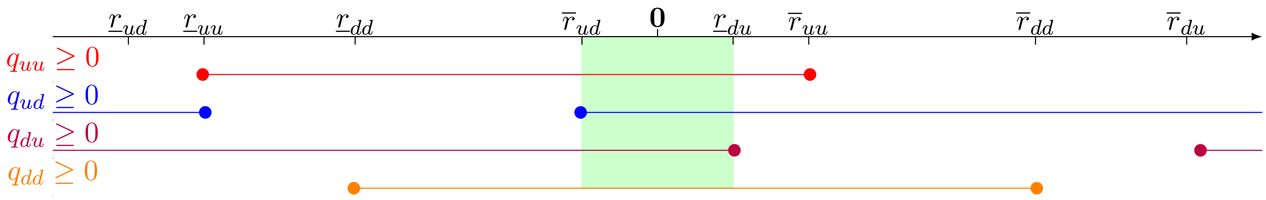

Figure 11: Graphical solution to the system of inequalities when $\rho \in (0, 1]$.

If, for example, $r(0) = 0$, $\theta = 0.02$, $\sigma_r = 0.01$, $\kappa = 0.7$, $S(0) = 1$, $\sigma_S = 0.15$, $q = 0$, $\rho = 0.5$, $T = 1$ and $n = 125$, after $m = 100$ steps, namely at $t = m \cdot \Delta t = m \cdot \frac{T}{n} = 0.8$, $r(t)$ spans the interval [-0.0885, 0.0885] and $S(t)$ the interval [-1.3282, 1.3282], both of them assuming $m = 101$ different values. Hence, at $t = 0.8$ there are $101^2 = 10201$ possible nodes on tree. As an instance, at the node $(S(t), r(t))_{t=0.8} = (0.5847, -0.0751)$ the four probabilities are:

$$q_{uu} = 0.4885$$

$$q_{ud} = -0.0143$$

$$q_{du} = 0.2780$$

$$q_{dd} = 0.2478.$$

Indeed, with the given parameters, probabilities are all positive as long as $\overline{r}_{ud} = -0.0660 \leq r(t) \leq 0.0861 = \underline{r}_{du}$, which is not our case. As $r(t)$ is extremely far away from its long-run mean and since $\rho > 0$ implies that r and S are likely to move together in the same direction, q_{ud}, namely the probability that r deviates even further from its long-run mean and also against S, becomes negative. Notice that $q_{ud} > q_{dd}$, meaning that the force that drives r towards its long-run mean prevails on the positive correlation between the two processes. When such a scenario happens, we adjust the probabilities by setting the negative one to 0 and normalizing to 1 the others. From the example above we would then get:

$$q_{uu} = 0.4816$$

$$q_{ud} = 0$$

$$q_{du} = 0.2741$$

$$q_{dd} = 0.2443.$$

A very similar situation happens when $\rho \in [-1, 0)$ and the four probabilities stay positive as long as $\underline{r}_{dd} \leq r \leq \overline{r}_{uu}$. Figure 12 shows the solution to the system of inequalities in this case. Now q_{uu} or q_{dd} might become negative. This is due to the negative correlation: as r and S are likely to move in the opposite direction, when r is far away from its long-run mean, moving even further in the same direction of S may result in a negative probability. Again, we correct for such a phenomenon with the normalization described above.

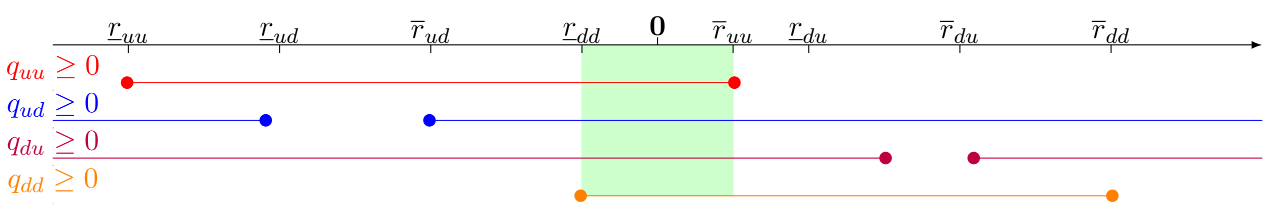

Figure 12: Graphical solution to the system of inequalities when $\rho \in [-1, 0)$.

For sake of completeness, we briefly discuss also the limit of zero correlation between r and S. When $\rho = 0$, $\underline{r}_{uu} = \underline{r}_{ud}$, $\overline{r}_{ud} = \underline{r}_{dd}$, $\overline{r}_{uu} = \underline{r}_{du}$ and $\overline{r}_{du} = \overline{r}_{dd}$. Hence, the two intervals we found for the two previous cases, $\overline{r}_{ud} \leq r \leq \underline{r}_{du}$ when $\rho \in (0, -1]$ and $\underline{r}_{dd} \leq r \leq \overline{r}_{uu}$ when $\rho \in [-1, 0)$, coincide. When $\rho = 0$, probabilities stay positive as long as r belong to that interval.

Since the support of the discretization of $r(t)$ is known at each t, we can retrieve the maximum t before which no normalization of the probabilities is needed.

Given the two thresholds $\underline{r}$ and $\overline{r}$ (where $\underline{r} = \overline{r}_{ud}$, $\overline{r} = \underline{r}_{du}$ if $\rho > 0$ and $\underline{r} = \underline{r}_{dd}$, $\overline{r} = \overline{r}_{uu}$ if $\rho < 0$) we can set $\underline{t}$ and $\overline{t}$ as:

$$\underline{t} := \min_{s \in \{0, \Delta t, 2\Delta t, \ldots, T\}} \{r(s) \geq \underline{r}\} \quad \text{and} \quad \overline{t} := \max_{s \in \{0, \Delta t, 2\Delta t, \ldots, T\}} \{r(s) \leq \overline{r}\}.$$

Given the binomial structure of the discretization, after m steps we have we have:

$$r(0) - m\Delta r^- = r(0) - m\sigma_r \Delta t \leq r(t) \leq r(0) + m\sigma_r \Delta t = r(0) + m\Delta r^+$$

and, therefore, from

$$\begin{aligned} r(0) - \underline{m}\sigma_r \Delta t &\geq \underline{r} \\ r(0) + \overline{m}\sigma_r \Delta t &\leq \overline{r} \end{aligned} \tag{A3}$$

we can explicitly compute:

$$\begin{aligned} \underline{t} = \underline{m}\Delta t &= \frac{r(0) - \underline{r}}{\sigma_r \sqrt{\Delta t}} \Delta t = \frac{r(0) - \underline{r}}{\sigma_r} \sqrt{\Delta t} \\ \overline{t} = \overline{m}\Delta t &= \frac{\overline{r} - r(0)}{\sigma_r \sqrt{\Delta t}} \Delta t = \frac{\overline{r} - r(0)}{\sigma_r} \sqrt{\Delta t}. \end{aligned}$$

Of course, neither $\underline{r}$, $\overline{r}$ nor $\underline{t}$, $\overline{t}$ are likely to correspond to any node of the discretized process $r(t)$ or to the discretized time line $\{0, \Delta t, 2\Delta t, \ldots, T\}$. In this case, we set $\underline{r}$, $\overline{r}$ and $\underline{t}$, $\overline{t}$ equal to the smallest values on the grid of $r(t)$ and t that satisfy the constraints in (A3). Going back to the numerical example above, we have that $\underline{t} = 0.5840$ and $\overline{t} = 0.7680$. A section of the quarinomial tree in this case is displayed in Figure 13.

B Proofs of the Claims

Proof of Proposition (1): value of the European put/call equity option. We first derive the price of the European put option at $t = 0$. As the payoff of the derivative depends only on the final value of S, the price at any t is obtained straightforwardly thanks to the Markovianity of (S, r) by replacing S_0 by $S(t)$, r_0 by $r(t)$ and T by $T - t$.

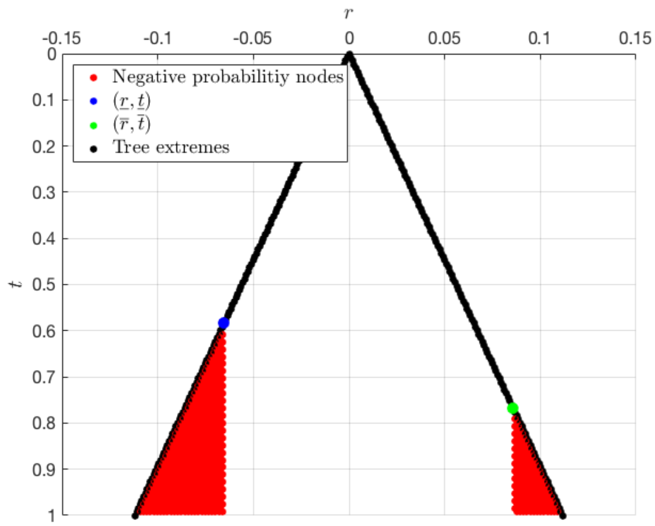

Figure 13: Section of the quadrinomial tree for $S = 0$. Red points indicate nodes at which one transition probability becomes negative. Parameters: $r(0) = 0$, $\theta = 0.02$, $\sigma_r = 0.01$, $\kappa = 0.7$, $S(0) = 1$, $\sigma_S = 0.15$, $q = 0$, $\rho = 0.5$, $T = 1$, $n = 125$.

In the market described by (1), the risk-neutral price at $t = 0$ of the European put option on S with strike price K and maturity T is given by:

$$\begin{aligned} \pi_E^{put}(0) &= \mathbb{E}^{\mathbb{Q}}[e^{-\int_0^T r(s)\mathrm{d}s}(K - S(T))^+] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\frac{(K - S(T))}{B(T)}\mathbf{1}_{\{K-S(T)>0\}}\right] \\ &= K\mathbb{E}^{\mathbb{Q}}\left[\frac{\mathbf{1}_{\{K-S(T)>0\}}}{B(T)}\right] - \mathbb{E}^{\mathbb{Q}}\left[\frac{S(T)\mathbf{1}_{\{K-S(T)>0\}}}{B(T)}\right]. \end{aligned} \tag{B1}$$

Since $B(T)$ depends on r which is correlated with S, in order to compute the two expected values we would need to know their joint distribution under $\mathbb{Q}$ and then evaluate a double integral. This turns out to be rather complicated. Nevertheless, we can greatly simplify the computation of the two expected values applying a change of numèraire.

We start from the first expectation in (B1). Consider the T-forward measure $\mathbb{Q}^T$, namely the martingale measure for the numèraire process $p(t, T)$. The Radon-Nikodym derivative of $\mathbb{Q}^T$ with respect to $\mathbb{Q}$ (whose numèraire process is the money market $B(t) = e^{\int_0^t r(s)\mathrm{d}s}$) is:

$$\frac{\mathrm{d}\mathbb{Q}^T}{\mathrm{d}\mathbb{Q}} = L^T(t) = \frac{p(t,T)}{B(t)p(0,T)} \text{ on } \mathcal{F}_t, 0 \leq t \leq T.$$

As $p(T,T) = 1$ and since $p(0,T)$ is a scalar, we get:

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\frac{\mathbf{1}_{\{K-S(T)>0\}}}{B(T)}\right] &= p(0,T)\mathbb{E}^{\mathbb{Q}}\left[\frac{p(T,T)}{B(T)p(0,T)}\mathbf{1}_{\{K-S(T)>0\}}\right] \\
&= p(0,T)\mathbb{E}^{\mathbb{Q}}[L^T(T)\mathbf{1}_{\{K-S(T)>0\}}] \\
&= p(0,T)\mathbb{E}^{\mathbb{Q}^T}[\mathbf{1}_{\{K-S(T)>0\}}] \\
&= p(0,T)\mathbb{Q}^T\left(S(T) < K\right) \\
&= p(0,T)\mathbb{Q}^T\left(\frac{S(T)}{p(T,T)} < K\right).
\end{aligned}$$

By definition, under the T-forward measure $\mathbb{Q}^T$ the discounted process of the risky asset, when accounting for the dividend, $Z_{0,T}(t) := \dfrac{S(t)e^{qt}}{p(t,T)}$ is a martingale. Applying the multidimensional Itō's Lemma to $Z_{0,T}(t)$ under $\mathbb{Q}$ we get:

$$\mathrm{d}Z_{0,T}(t) = (\dots)\,\mathrm{d}t + (\nu_S + B(t,T)\nu_r) \cdot \mathrm{d}W^{\mathbb{Q}}(t)$$

where, $\nu_S = [\sigma_S \quad 0]$, $\nu_r = [\sigma_r\rho \quad \sigma_r\sqrt{1-\rho^2}]$ and $W^{\mathbb{Q}}(t) = [W_1^{\mathbb{Q}}(t) \quad W_2^{\mathbb{Q}}(t)]'$ is standard two-dimensional Brownian motion under $\mathbb{Q}$. Since the volatility process $\sigma_{0,T}(t) := \nu_S + B(t,T)\nu_r$ is constant, we can apply a suitable change of measure from $\mathbb{Q}$ to $\mathbb{Q}^T$ to get rid of the deterministic drift of $Z_{0,T}(t)$. Under the T-forward measure we get:

$$\mathrm{d}Z_{0,T}(t) = +\sigma_{0,T}(t) \cdot \mathrm{d}W^{\mathbb{Q}^T}(t)$$

as we expected. The process $Z_{0,T}(t)$ is now a geometric Brownian motion driven by a bi-dimensional Wiener process. Hence, its solution is:

$$\begin{aligned}
Z_{0,T}(t) &= Z_{0,T}(0)\exp\left\{-\frac{1}{2}\int_0^t ||\sigma_{0,T}^2(s)||\mathrm{d}s + \int_0^t \sigma_{0,T}(s)\cdot \mathrm{d}W^{\mathbb{Q}^T}(s)\right\} \\
&= \frac{S(0)}{p(0,T)}\exp\left\{-\frac{1}{2}\int_0^t ||\sigma_{0,T}^2(s)||\mathrm{d}s + \int_0^t \sigma_{0,T}(s)\cdot \mathrm{d}W^{\mathbb{Q}^T}(s)\right\}.
\end{aligned}$$

Notice that, due to Itō's Isometry,

$$-\frac{1}{2}\int_0^t ||\sigma_{0,T}^2(s)||\mathrm{d}s + \int_0^t \sigma_{0,T}(s)\cdot \mathrm{d}W^{\mathbb{Q}^T}(s) \sim \mathcal{N}\left(-\frac{1}{2}\Sigma_{0,T}^2(t), \Sigma_{0,T}^2(t)\right)$$

where:

$$\begin{aligned}
\Sigma_{0,T}^2(t) :=& \int_0^t ||\sigma_{0,T}^2(s)||\mathrm{d}s \\
=& \int_0^t \sigma_S^2 + 2\sigma_S\sigma_r\rho B(s,T) + B(s,T)^2\sigma_r^2 \mathrm{d}s \\
=& \sigma_S^2 t + 2\sigma_S\sigma_r\rho\left(\frac{\kappa t - e^{-kT}\left(-1+e^{kt}\right)}{k^2}\right) + \sigma_r^2\left(\frac{e^{-2\kappa T}\left(-1+e^{2\kappa t}+4e^{\kappa T}-4^{\kappa(t+T)}+2e^{2\kappa T}\kappa t\right)}{2k^3}\right).
\end{aligned}$$

When $t = T$, the expression above simplifies to:

$$\Sigma_{0,T}^2(T) = \sigma_S^2 T + 2\sigma_S\sigma_r\rho\left(\frac{-1+e^{-\kappa T}+\kappa T}{k^2}\right) + \sigma_r^2\left(-\frac{3+e^{-2\kappa T}-4e^{-\kappa T}-2\kappa T}{2k^3}\right).$$

Finally, we can compute the T-forward probability that the put option closes in the money as:

$$\begin{aligned}
\mathbb{Q}^T\left(\frac{S(T)}{p(T,T)} < K\right) &= \mathbb{Q}^T\left(\frac{S(T)e^{qT}}{p(T,T)} < Ke^{qT}\right) \\
&= \mathbb{Q}^T\left(Z_{0,T}(T) < Ke^{qT}\right) \\
&= \mathbb{Q}^T\left(\frac{S(0)}{p(0,T)}\exp\left\{-\frac{1}{2}\int_0^T ||\sigma_{0,T}||^2(s)\mathrm{d}s + \int_0^T \sigma_{0,T}(s)\cdot \mathrm{d}W^{\mathbb{Q}^T}(s)\right\} < Ke^{qT}\right) \\
&= \mathbb{Q}^T\left(\mathcal{N}\left(-\frac{1}{2}\Sigma_{0,T}^2(T), \Sigma_{0,T}^2(T)\right) < \ln\frac{p(0,T)Ke^{qT}}{S(0)}\right) \\
&= N(-\tilde{d}_2)
\end{aligned}$$

with $\tilde{d}_2 = \frac{1}{\sqrt{\Sigma_{0,T}^2(T)}}\left(\ln\frac{S(0)}{p(0,T)K} - \frac{1}{2}\Sigma_{0,T}^2(T) - qT\right)$. Hence,

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{\mathbf{1}_{\{K-S(T)>0\}}}{B(T)}\right] = p(0,T)N(-\tilde{d}_2).$$

We now turn to the second expected value in (B1). Consider the martingale measure $\mathbb{Q}^S$ with numèraire process $S(t)e^{qt}$. The Radon-Nikodym derivative of $\mathbb{Q}^S$ with respect to $\mathbb{Q}$ is:

$$\frac{\mathrm{d}\mathbb{Q}^S}{\mathrm{d}\mathbb{Q}} = L^S(t) = \frac{S(t)e^{qt}}{S(0)B(t)} \text{ on } \mathcal{F}_t, 0 \le t \le T. \tag{B2}$$

As both $S(0)$ and e^{qT} are scalars, we have:

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\frac{S(T)\mathbf{1}_{\{K-S(T)>0\}}}{B(T)}\right] &= S(0)e^{-qT}\mathbb{E}^{\mathbb{Q}}\left[\frac{S(T)e^{qT}}{S(0)B(T)}\mathbf{1}_{\{K-S(T)>0\}}\right] \\
&= S(0)e^{-qT}\mathbb{E}^{\mathbb{Q}}\left[L^S(T)\mathbf{1}_{\{K-S(T)>0\}}\right] \\
&= S(0)e^{-qT}\mathbb{E}^{\mathbb{Q}^S}\left[\mathbf{1}_{\{K-S(T)>0\}}\right] \\
&= S(0)e^{-qT}\mathbb{Q}^S(S(T)<K).
\end{aligned}$$

Under $\mathbb{Q}^S$, the process $Y_{0,T}(t) := \dfrac{p(0,t)}{S(t)e^{qt}}$ is a martingale. Notice that $Y_{0,T}(t) = Z_{0,T}(t)^{-1}$. Then, Itō's Lemma tells us immediately that:

$$\mathrm{d}Y_{0,T}(t) = (\dots)\,\mathrm{d}t - (\nu_S + B(t,T)\nu_r)\cdot \mathrm{d}W^{\mathbb{Q}}(t)$$

and after a suitable change of measure we get that under $\mathbb{Q}^S$:

$$\mathrm{d}Y_{0,T}(t) = -\sigma_{0,T}(t)\cdot \mathrm{d}W^{\mathbb{Q}^S}(t).$$

As before, we get:

$$\begin{aligned}
Y_{0,T}(t) &= Y_{0,T}(0)\exp\left\{-\frac{1}{2}\int_0^t ||\sigma_{0,T}^2(s)||\mathrm{d}s - \int_0^t \sigma_{0,T}(s)\cdot \mathrm{d}W^{\mathbb{Q}^2}(s)\right\} \\
&= \frac{p(0,T)}{S(0)}\exp\left\{-\frac{1}{2}\int_0^t ||\sigma_{0,T}^2(s)||\mathrm{d}s - \int_0^t \sigma_{0,T}(s)\cdot \mathrm{d}W^{\mathbb{Q}^S}(s)\right\}.
\end{aligned}$$

and, since again $p(T,T) = 1$, the $\mathbb{Q}^S$ probability that the put option closes in the money is:

$$
\begin{aligned}
\mathbb{Q}^S\left(S(T) < K\right) &= \mathbb{Q}^S\left(\frac{1}{S(T)} > \frac{1}{K}\right) \\
&= \mathbb{Q}^S\left(\frac{p(T,T)}{S(T)e^{qT}} > \frac{1}{Ke^{qT}}\right) \\
&= \mathbb{Q}^S\left(Y_{0,T}(T) > \frac{1}{Ke^{qT}}\right) \\
&= \mathbb{Q}^T\left(\frac{p(0,T)}{S(0)}\exp\left\{-\frac{1}{2}\int_0^T ||\sigma_{0,T}^2(s)||\mathrm{d}s - \int_0^T \sigma_{0,T}(s)\cdot \mathrm{d}W^{\mathbb{Q}^T}(s)\right\} > \frac{1}{Ke^{qT}}\right) \\
&= \mathbb{Q}^T\left(\mathcal{N}\left(-\frac{1}{2}\Sigma_{0,T}^2(T), \Sigma_{0,T}^2(T)\right) > \ln\frac{S(0)}{p(0,T)Ke^{qT}}\right) \\
&= \mathbb{Q}^T\left(\mathcal{N}\left(0,1\right) > \frac{1}{\sqrt{\Sigma_{0,T}^2(T)}}\left(\ln\frac{S(0)}{p(0,T)K} + \frac{1}{2}\Sigma_{0,T}^2(T) - qT\right)\right) \\
&= N(-\tilde{d}_1)
\end{aligned}
$$

where $\tilde{d}_1 = \frac{1}{\sqrt{\Sigma_{0,T}^2(T)}}\left(\ln\frac{S(0)}{p(0,T)K} + \frac{1}{2}\Sigma_{0,T}^2(T) - qT\right) = \tilde{d}_2 + \sqrt{\Sigma_{0,T}^2(T)}$. Hence,

$$
\mathbb{E}^{\mathbb{Q}}\left[\frac{S(T)\mathbf{1}_{\{K-S(T)>0\}}}{B(T)}\right] = S(0)e^{-qT}N(-\tilde{d}_1).
$$

Finally, putting everything together we find the initial price of the put option:

$$
\begin{aligned}
\pi_E^{put}(0) &= K\mathbb{E}^{\mathbb{Q}}[p(0,T)\mathbf{1}_{\{K-S(T)>0\}}] - \mathbb{E}^{\mathbb{Q}}[p(0,T)S(T)\mathbf{1}_{\{K-S(T)>0\}}] \\
&= Kp(0,T)N(-\tilde{d}_2) - S(0)e^{-qT}N(-\tilde{d}_1).
\end{aligned}
$$

The price of the related call option can be derived by the put-call parity that at $t \in [0,T]$ reads

$$
\pi_E^{call}(t) = S(t)e^{-q(T-t)} - Kp(t,T) + \pi_E^{put}(t)
$$

as $e^{-q(T-t)}$ units of $S(t)$ at time t lead to one unit of S at time T by continuously investing the dividends in S itself.

Proof of Proposition (2): convergence of the quadrinomial tree. We now need to show that the bivariate discrete process $(X_i)_i$ defined in (12) with the parameters in (13) and (14) converges in distribution to $X(t) = (Y(t), r(t))$ that solves (9). With the notation of the general case in (11) and exploiting the result of Section 11.3 of Stroock and Varadhan (1997), the desired result holds true if the following four conditions are met:

(A1) the functions $\mu(x,t)$ and $\sigma(x,t)$ are continuous and $\sigma(x,t)$ is non negative;

(A2) with probability 1 a solution $(X_t)_t$ to the SDE:

$$X_t = X_0 + \int_0^t \mu(X_s, s)\mathrm{d}s + \int_0^t \sigma(X_s, s) \cdot \mathrm{d}W(s)$$

exists for $0 < t < +\infty$ and it is unique in law;

(A3) for all $\delta, T > 0$

$$\lim_{n\to+\infty} \sup_{||x||\leq\delta, 0\leq t\leq T} |\Delta Y^{\pm}| = 0$$

$$\lim_{n\to+\infty} \sup_{||x||\leq\delta, 0\leq t\leq T} |\Delta r^{\pm}| = 0;$$

(A4) let $X_{i,j}$ indicate the j-th entry of X_i and let $\mathcal{F}_i = \sigma(X_1, \ldots, X_i)$ be the filtration generated by the discrete bivariate process (X_i). Define:

$$\mu_i(x,t) := \begin{bmatrix} \mu_{i,1}(x,t) \\ \mu_{i,2}(x,t) \end{bmatrix} \text{ and } \sigma_i^2(x,t) := \begin{bmatrix} \sigma_{i,1}^2(x,t) \\ \sigma_{i,2}^2(x,t) \end{bmatrix}$$

where $\mu_{i,j}(x,t) = \dfrac{\mathbb{E}^{\mathbb{Q}}[X_{i+1,j} - X_{i,j}|\mathcal{F}_i]}{\frac{T}{n}}$ and $\sigma_{i,j}^2(x,t) = \dfrac{\mathbb{E}^{\mathbb{Q}}[(X_{i+1,j} - X_{i,j})^2|\mathcal{F}_i]}{\frac{T}{n}}$ for $j = 1, 2$. Let $\rho_i(x,t) = \dfrac{\mathbb{E}^{\mathbb{Q}}[(X_{i+1,1} - X_{i,1})(X_{i+1,2} - X_{i,2})|\mathcal{F}_i]}{\frac{T}{n}}$ and $\rho(x,t) = \sigma_1(x,t) \cdot \sigma_2(x,t)'$ where $\sigma_j(x,t)$ is the j-th row of $\sigma(x,t)$. Then, for all $\delta, T > 0$,

$$\lim_{n\to+\infty} \sup_{||x||\leq\delta, 0\leq t\leq T} ||\mu_i(x,t) - \mu(x,t)|| = 0$$

$$\lim_{n\to+\infty} \sup_{||x||\leq\delta, 0\leq t\leq T} ||\sigma_i^2(x,t) - \sigma^2(x,t) \cdot \mathbf{I}_2|| = 0$$

$$\lim_{n\to+\infty} \sup_{||x||\leq\delta, 0\leq t\leq T} |\rho_i(x,t) - \rho(x,t)| = 0$$

where $\mathbf{I}_n$ is the column vector with all of the n entries equal to one.

For our quadrinomial tree we have $X_t = [Y(t), \quad r(t)]'$,

$$\mu(X_t, t) = \begin{bmatrix} \left(r(t) - q - \frac{\sigma_S^2}{2}\right) \\ \kappa(\theta - r(t)) \end{bmatrix} \quad \text{and} \quad \sigma(X_t, t) = \begin{bmatrix} \sigma_S & 0 \\ \sigma_r\rho & \sigma_r\sqrt{1-\rho^2} \end{bmatrix}.$$

Assumption (A1) trivially holds true.

Assumption (A2) holds true if the standard conditions for the existence and the uniqueness of the solution to an SDE are met. According, e.g., to Proposition 5.1 in Björk (2009), it is sufficient to show that there exists a constant K such that the following are satisfied for all $x_i = [y_i, \quad r_i]'$, $i = 1, 2$ and t:

$$||\mu(x_1, t) - \mu(x_2, t)|| \leq K||x_1 - x_2||,$$

$$||\sigma(x_1, t) - \sigma(x_2, t)|| \leq K||x_1 - x_2||,$$

$$||\mu(x_1, t)|| + ||\sigma(x_1, t)|| \leq K\left(1 + ||x_1||\right).$$

Notice that the second and the third conditions involve the operator norm of a matrix $A \in \mathbb{R}^n$ defined as $||A|| := \sup_{||x||=1}\{||A \cdot x|| : x \in \mathbb{R}^n\}$.

As $||\mu(x_1, t) - \mu(x_2, t)|| = \sqrt{1+\kappa^2}|r_1 - r_2|$ and $(r_1 - r_2)^2 \leq ||x_1 - x_2||^2$, the first condition is surely satisfied for any $K \geq \sqrt{1+\kappa^2}$. As $\sigma(x_i, t)$ is actually constant and independent of x_i and t, $||\sigma(x_1, t) - \sigma(x_2, t)|| = 0$ and the second condition is surely satisfied for any $K \geq 0$. Finally, as

$$||\sigma(x_1), t)|| = \sigma_S^2 + \rho^2\frac{\sigma_r^2}{2} + |\rho|\frac{\sigma_r}{2}\sqrt{4\sigma_s^2 + \sigma_r^2}$$

is constant and as

$$||\mu(x_1, t)|| = \sqrt{\left(r_1 - q - \frac{\sigma_S^2}{2}\right)^2 + \kappa^2(\theta - r_1)^2}$$

can be bounded from above by $\sqrt{2(1+\kappa^2)r_1^2}$, we have

$$||\mu(x_1,t)|| + ||\sigma(x_1,t)|| \leq \sqrt{2(1+\kappa^2)}||x_1|| + ||\sigma(x_1,t)|| \leq K(1+||x_1||)$$

for any $K \geq \max\{\sqrt{2(1+\kappa^2)}, ||\sigma(x_1,t)||\}$. As the three conditions hold true simultaneously for any $K \geq \max\{\sqrt{2(1+\kappa^2)}, ||\sigma(x_1,t)||\}$, assumption (A2) is satisfied.

As the increments of the bivariate discrete process $\Delta Y^{\pm} = \pm\sigma_S\sqrt{\Delta t} = \pm\sigma_S\sqrt{\frac{T}{n}}$, $\Delta r^{\pm} = \pm\sigma_r\sqrt{\Delta t} = \pm\sigma_r\sqrt{\frac{T}{n}}$ are constant and do not depend neither on x_i, $i = 1, 2$, nor on t,

$$\sup_{||x||\leq\delta, 0\leq t\leq T} |\Delta Y^{\pm}| = |\Delta Y^{\pm}| = \sigma_S\sqrt{\frac{T}{n}},$$

$$\sup_{||x||\leq\delta, 0\leq t\leq T} |\Delta r^{\pm}| = |\Delta r^{\pm}| = \sigma_r\sqrt{\frac{T}{n}}.$$

As both of the sups are infinitesimal with respect to n, (A3) holds true as well.

As the parameters in (13) and (14) of the bivariate discretization $X_i = (Y_i, r_i)$ are chosen in order to match the first two moments and the cross-variation of $X(t) = (Y(t), r(t))$, we have $\mu_i(x,t) = \mu(x,t)$, $\sigma_i^2(x,t) = \sigma^2(x,t)\cdot\mathbf{I}_2$ and $\rho_i(x,t) = \rho(x,t)$. Hence, assumption (A4) is satisfied by construction.

Theorem 11.3.3 of Stroock and Varadhan (1997) allows us to conclude.

Proof of Proposition (3): value of the American option as a deterministic function. Let $\eta := \tau - t$. Then we can rewrite the value of the American option (15) as:

$$V(t) = ess \sup_{0\leq\eta\leq T-t} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{t+\eta} r(s)\mathrm{d}s}\ \varphi(S(t+\eta))|\,\mathcal{F}_t\right]. \tag{B3}$$

Recalling the dynamics of the equity price conditional on $S(t) = S$ we can further rewrite $\tilde{V}(t)$ as:

$$V(t) = ess \sup_{0\leq\eta\leq T-t} \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^{t+\eta} r(s)\mathrm{d}s\right)\cdot\right.$$

$$\cdot\varphi\left(S\exp\left(\int_t^{t+\eta} r(s)\mathrm{d}s - \left(q+\frac{1}{2}\sigma_S^2\right)\eta + \sigma_S(W_S(t+\eta)-W_S(t))\right)\right)\Bigg|\mathcal{F}_t\Bigg].$$

Therefore, V depends on the expectation of two random variables: the Brownian increment $W_S(t+\eta) - W_S(t)$ and the integral $\int_t^{t+\eta} r(s)\mathrm{d}s$, which appears both in the drift part of the underlying and in the discount factor. The first of the two random variables is $\mathcal{F}_t$-independent by definition:

$$W_S(t+\eta) - W_S(t) \perp\!\!\!\perp \mathcal{F}_t,$$

and, moreover,

$$W_S(t+\eta) - W_S(t) \overset{\mathbb{Q}}{\sim} W_S(\eta).$$

We now show that also $\int_t^{t+\eta} r(s)\mathrm{d}s$ is independent of $\mathcal{F}_t$ and that $\int_t^{t+\eta} r(s)\mathrm{d}s \overset{\mathbb{Q}}{\sim} \int_0^{\eta} r(s)\mathrm{d}s$ as well. Recalling the solution to the SDE driving the short term interest rate conditional on $r(t) = r$ we have:

$$\begin{aligned}\int_t^{t+\eta} r(s)\mathrm{d}s &= \int_t^{t+\eta}\left[re^{-\kappa(s-t)} + \theta(1-e^{-\kappa(s-t)}) + \sigma_r\int_t^s e^{-\kappa(s-y)}\mathrm{d}W_r(y)\right]\mathrm{d}s\\ &= -\frac{e^{-\kappa\eta}}{\kappa}(r-\theta) + \frac{r-\theta}{\kappa} + \theta\eta + \sigma_r\int_t^{t+\eta}\int_t^s e^{-\kappa(s-y)}\mathrm{d}W_r(y)\mathrm{d}s.\end{aligned}$$

The constant $\alpha := -\frac{e^{-\kappa\eta}}{\kappa}(r-\theta) + \frac{r-\theta}{\kappa} + \theta\eta$ does not depend on t. Exploiting the definition of stochastic integral with $\{t_i\}_{i=1,\dots,N}$ such that $t_0 = t$, $t_N = t+\eta$ and $||\{t_i\}|| \to 0$, we get:

$$\int_t^{t+\eta} r(s)\mathrm{d}s = \alpha + \sigma_r\int_t^{t+\eta}\sum_{i=0}^{N-1} e^{-\kappa(s-t_i)}(W_r(t_{i+1}) - W_r(t_i))\mathrm{d}s.$$

Since $t_{i+1} > t_i > t$ for any $i = 1,\dots,N-1$, $W_r(t_{i+1}) - W_r(t_i) \perp\!\!\!\perp \mathcal{F}_t$ by definition and for any value of s. Hence,

$$\sum_{i=0}^{N-1} e^{-\kappa(s-t_i)}(W_r(t_{i+1}) - W_r(t_i)) \perp\!\!\!\perp \mathcal{F}_t \quad \forall s.$$

Since the sum is independent of $\mathcal{F}_t$ for any s, the outer integral in ds preserves such independence. As a result,

$$\int_t^{t+\eta} r(s)\mathrm{d}s \perp\!\!\!\perp \mathcal{F}_t.$$

Furthermore, we need to show that the distribution of

$$\int_t^{t+\eta} r(s)\mathrm{d}s$$

does not depend on t. Recalling that:

$$\int_t^{t+\eta} r(s)\mathrm{d}s = \alpha + \sigma_r \int_t^{t+\eta} \int_t^s e^{-\kappa(s-y)}\mathrm{d}W_r(y)\mathrm{d}s$$

and setting $a := s - t$ in the outer integral in $\mathrm{d}s$, we get:

$$\int_t^{t+\eta} r(s)\mathrm{d}s = \alpha + \sigma_r \int_0^{\eta} \int_t^{a+t} e^{-\kappa(a+t-y)}\mathrm{d}W_r(y)\mathrm{d}a.$$

The argument of the inner stochastic integral is deterministic in y and, therefore:

$$\begin{aligned}\int_t^{a+t} e^{-\kappa(s-y)}\mathrm{d}W_r(y) &\overset{\mathbb{Q}}{\sim} \mathcal{N}\left(0, \int_t^{a+t} e^{-2\kappa(s-y)}\mathrm{d}y\right) \\ &\overset{\mathbb{Q}}{\sim} \mathcal{N}\left(0, \frac{1}{2\kappa}\left(1 - e^{-2\kappa a}\right)\right),\end{aligned}$$

which does not depend on t. Thanks to a little abuse of notation we see that:

$$\int_t^{t+\eta} r(s)\mathrm{d}s = \alpha + \sigma_r \int_0^{\eta} \mathcal{N}\left(0, \frac{1}{2\kappa}\left(1 - e^{-2\kappa a}\right)\right)\mathrm{d}a,$$

where the right-hand side of the equation does not depend on t. Hence:

$$\int_t^{t+\eta} r(s)\mathrm{d}s \overset{\mathbb{Q}}{\sim} \int_0^{\eta} r(s)\mathrm{d}s.$$

We now go back to the original expression (B3). Thanks to the independence of $\mathcal{F}_t$, the conditional expected value turns into the unconditional one:

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{t+\eta} r(s)\mathrm{d}s}\varphi(S(t+\eta))\middle| \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{t+\eta} r(s)\mathrm{d}s}\varphi(S(t+\eta))\right],$$

and setting $S(0) = S(t) = S$, $r(0) = r(t) = r$,

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{t+\eta} r(s)\mathrm{d}s}\varphi(S(t+\eta))\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{\eta} r(s)\mathrm{d}s}\varphi(S(\eta))\right].$$

Therefore, the value on an American option on S defined in (15) reduces to

$$V(t) = F(t, S(t), r(t))$$

with

$$F(t,S,r) = \sup_{0\le\eta\le T-t} \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^\eta r(s)\mathrm{d}s\right)\cdot\right.$$
$$\left.\cdot\varphi\left(S\exp\left(\int_0^\eta r(s)\mathrm{d}s - \left(q+\frac{1}{2}\sigma_S^2\right)\eta + \sigma_S W_S(\eta)\right)\right)\right].$$

where t enters only the upper bound of η, namely the time to maturity of the option. From this last expression it is immediate to see that F enjoys the same monotonicity properties of φ w.r.t.S, and that it is decreasing w.r.t. t, and convex w.r.t. S. For the put option we show now that F is decreasing in r. To this aim we rewrite

$$F(t,S,r) = \sup_{0\le\eta\le T-t} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^\eta r(s)\mathrm{d}s)}\left(K - Se^{\int_0^\eta r(s)\mathrm{d}s - \left(q+\frac{1}{2}\sigma_S^2\right)\eta + \sigma_S W_S(\eta)}\right)^+\right] \tag{B4}$$

where $r = r(0)$. If $r' > r$ then $F(t,S,r') \le F(t,S,r)$. In fact, $\int_0^\eta r(s)\mathrm{d}s$ started at $r(0) = r' > r$ is larger than $\int_0^\eta r(s)\mathrm{d}s$ started at $r(0) = r$. As the object of the expectation in (B4) is a decreasing function of $\int_0^\eta r(s)\mathrm{d}s$, we conclude that $F(t,S,r)$ is decreasing in r.

To show that the American call option is increasing with respect to r, we apply a change of numeraire to isolate the effect of the interest rate in the underlying drift only (as under the original risk neutral measure an increase in r has opposite effects in the discount factor and in the call's payoff).

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[(S(\tau)-K)^+ e^{-\int_0^\tau r(s)\mathrm{d}s}\right] &= \mathbb{E}^{\mathbb{Q}}\left[\frac{S(\tau)e^{q\tau}}{S(0)B(\tau)}\left(\frac{1}{K}-\frac{1}{S(\tau)}\right)^+ Ke^{-q\tau}S(0)\right]\\ &= \mathbb{E}^{\mathbb{Q}}\left[L^S(\tau)\left(\frac{1}{K}-\frac{1}{S(\tau)}\right)^+ Ke^{-q\tau}S(0)\right]\end{aligned}$$

where $L^S(\tau)$ is the Radon-Nikodym derivative of $\mathbb{Q}^S$ with respect to $\mathbb{Q}$ defined in (B2). Thus the call option is a put option under the new measure on K/S with strike $S(0)$ and interest rate q

$$\mathbb{E}^{\mathbb{Q}}\left[(S(\tau)-K)^{+}\,e^{-\int_0^\tau r(s)\mathrm{d}s}\right]=\mathbb{E}^{\mathbb{Q}^S}\left[\left(S(0)-\frac{K}{S(\tau)}\right)^{+}e^{-q\tau}\right]$$

Recalling the dynamics of the equity price and of the interest rate under $\mathbb{Q}$,

$$\begin{cases}\dfrac{\mathrm{d}S(t)}{S(t)} & =(r(t)-q)\mathrm{d}t+[\sigma_S \quad 0]\cdot\mathrm{d}W^{\mathbb{Q}}(t)\\ \mathrm{d}r(t) & =\kappa(\theta-r(t))\mathrm{d}t+[\sigma_r\rho \quad \sigma_r\sqrt{1-\rho^2}]\cdot\mathrm{d}W^{\mathbb{Q}}(t)\end{cases} \tag{B5}$$

Girsanov's theorem implies that $\mathrm{d}W^{\mathbb{Q}}(t)=\mathrm{d}W^{\mathbb{Q}^S}(t)+[\sigma_S \quad 0]'\mathrm{d}t$ and, therefore, (B5) becomes

$$\begin{cases}\dfrac{\mathrm{d}S(t)}{S(t)} & =(r(t)-q+\sigma_S^2)\mathrm{d}t+[\sigma_S \quad 0]\cdot\mathrm{d}W^{\mathbb{Q}^S}(t)\\ \mathrm{d}r(t) & =\kappa(\theta-r(t)+\frac{\rho\sigma_S\sigma_r}{\kappa})\mathrm{d}t+[\sigma_r\rho \quad \sigma_r\sqrt{1-\rho^2}]\cdot\mathrm{d}W^{\mathbb{Q}^S}(t)\end{cases} \tag{B6}$$

Ito's formula implies that

$$\mathrm{d}\left(\frac{1}{S(t)}\right)=\frac{1}{S(t)}\left((q-r(t))\mathrm{d}t-[\sigma_S \quad 0]\cdot\mathrm{d}W^{\mathbb{Q}^S}(t)\right)$$

and therfore the new underlying

$$\mathrm{d}\left(\frac{K}{S(t)}\right)=\frac{K}{S(t)}\left((q-r(t))\mathrm{d}t-[\sigma_S \quad 0]\cdot\mathrm{d}W^{\mathbb{Q}^S}(t)\right)$$

has drift $q-r(t)$. Thus the call option is a put option whose underlying under the new measure is

$$\frac{K}{S(t)}=\frac{K}{S(0)}e^{\int_0^\eta(q-r(s))\mathrm{d}s-\frac{1}{2}\sigma_S^2\eta-\sigma_S W_1^{\mathbb{Q}^S}(\eta)}$$

Thus, if the process r starts at $r(0)=r'>r$ the factor $\int_0^\eta(q-r(s))\mathrm{d}s$ is smaller than the one started at $r(0)=r$, and thus the put's payoff is larger, and the value of the American option larger as well. This shows that for the call option $r'>r$ implies $F(t,S,r')>F(t,S,r)$.

Proof of Proposition (4): asymptotic necessary conditions for the existence of a double continuation region. When the interest rate is constant, the value at t of the

American put option is $G(t, S(t))$, where G is a deterministic function defined in (17). As Battauz et al. (2015) show in Section 2, necessary conditions for the double continuation region to appear at a generic t are that the drift of S is positive and $G(t, 0) > K$.

According to Proposition (3), the value of the American put on S in the market described by (1) is of the form $V(t) = F(t, S(t), r(t))$ where F is a deterministic function implicitly defined by (16). Following the same logic of Battauz et al. (2015), we need to impose analogous conditions on $F(t, S(t), r(t))$.

The counterpart of $G(t, 0) > K$ in Battauz et al. (2015), is here $F(t, 0, r) > K$ that delivers *[NC0]*.

According to (16), we have:

$$F(t, 0, r) = K \cdot \sup_{0 \leq \eta \leq T-t} \mathbb{E}^{\mathbb{Q}} \left[\exp \left(- \int_0^\eta r(s) \mathrm{d}s \right) \right].$$

Therefore, we need to check under which condition the supremum on the right hand side of the last equation is strictly greater than 1. Namely, we have to derive conditions under which there exists a $\eta \in [0, T-t]$ such that the supremumis strictly greater than 1. Exploiting Jensen's inequality and the uniform integrability of $r(s)$, we get:

$$\mathbb{E}^{\mathbb{Q}} \left[\exp \left(- \int_0^\eta r(s) \mathrm{d}s \right) \right] \geq \exp \left(-\mathbb{E}^{\mathbb{Q}} \left[\int_0^\eta r(s) \mathrm{d}s \right] \right) = \exp \left(- \int_0^\eta \mathbb{E}^{\mathbb{Q}} \left[r(s) \right] \mathrm{d}s \right).$$

As before, thanks to (3), we have:

$$\mathbb{E}^{\mathbb{Q}} \left[\exp \left(- \int_0^\eta r(s) \mathrm{d}s \right) \right] \geq \exp \left(- \int_0^\eta r e^{-\kappa s} + \theta (1 - e^{-\kappa s}) \mathrm{d}s \right) = \exp \left(r\alpha - \theta(\alpha + \eta) \right)$$

where we set $\alpha := \dfrac{e^{-\kappa\eta} - 1}{\kappa}$. Notice that $\alpha \leq 0$ for any κ and $\eta \in [0, T-t]$.

If $r\alpha - \theta(\alpha + \eta) > 0$, then $F(t, 0, r) > K$.

For the American put option, under *[NC0]*, if *[NC1]* is not satisfied, i.e. $q > 0$, than the discounted risky security $\tilde{S}$ is driven by

$$\mathrm{d}\tilde{S}(t) = -q\mathrm{d}t + \sigma_S \mathrm{d}W_S^{\mathbb{Q}}(t),$$

and $\tilde{S}$ is a supermartingale. Thus, for any $t < \tau < T$,

$$\mathbb{E}^{\mathbb{Q}}\left[S(\tau)e^{-\int_t^\tau r(s)\mathrm{d}s} \middle| \mathcal{F}_t \right] \leq S(t)$$

and, by Jensen's inequality,

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[(K - S(\tau))^+ e^{-\int_t^\tau r(s)\mathrm{d}s} \middle| \mathcal{F}_t \right] &\geq \left(K\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^\tau r(s)\mathrm{d}s} \middle| \mathcal{F}_t \right] - S(t)e^{-q(\tau - t)} \right)^+ \\
&\geq (K - S(t))^+,
\end{aligned}$$

where the last inequalities holds under *[NC0]*. This shows that, for the American put option, under *[NC0]*, if *[NC1]* is violated, early exercise is never optimal at t.

We deal now with the American call option. For $0 < \tau < T$, we have by Jensen's inequality,

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[(S(\tau) - K)^+ e^{-\int_t^\tau r(s)\mathrm{d}s} \middle| \mathcal{F}_t \right] &\geq \left(S(t)e^{-q(\tau - t)} - K\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^\tau r(s)\mathrm{d}s} \middle| \mathcal{F}_t \right] \right)^+ \\
&= \left(S(t)e^{-q(\tau - t)} - Kp(t, \tau) \right)^+ \\
&\geq (S(t) - K)^+,
\end{aligned}$$

if $q \leq 0$ and $p(t, \tau) \leq 1$. Therefore, to ensure the existence of optimal early exercise opportunities for the American call option, we must assume that $q > 0$, or $q \leq 0$ and $p(t, \tau) > 1$ for some τ.

Under *[NC0]*, if *[NC2]* is not satisfied, then $\pi_A(t, S, r) \geq \pi_E(t, S, r) > (K - S)^+$, that means that early exercise is never optimal at t.

Proof of Theorem (5) The case $r \geq 0$ is standard (see Detemple, 2014), and therefore we focus on $r < 0$. The continuity, the monotonicity of the r-sections of the put option's free boundaries with respect to t and S and their limits as $t \to T^-$ follow by adapting the proof of Theorem 2.3 in (Battauz et al., 2015) where now the operator $\mathcal{L}$ becomes $\mathcal{L}F = \frac{\partial F}{\partial S}S(r-q)+\frac{\partial F}{\partial r}\kappa(\theta-r)+\frac{1}{2}\frac{\partial^2 F}{\partial S^2}\sigma_S^2 S^2+\frac{1}{2}\frac{\partial^2 F}{\partial r^2}\sigma_r^2+\frac{\partial^2 F}{\partial r \partial S}\rho\sigma_r\sigma_S$. The monotonicity properties of the free boundaries with respect to r follow from the monotonicity properties of F. In fact, let $r' > r$, and assume $S \in EER_r$. Then $(K-S)^+ \leq F(t,S,r') \leq F(t,S,r) = (K-S)^+$, where the first inequality follows from value dominance and the second one from the fact that F is decreasing in r. Thus if $S \in EER_r$, then $S \in EER'_r$, and $EER'_r \supseteq EER_r$. By passing to the infimum (resp. supremum) we conclude that the lower (resp. upper) free boundary is decreasing (resp. increasing) with respect to r.

For the call option, we start from the monotonicity properties of the free boundaries with respect to r. As the call option is increasing in r, we have that if $r' > r$ and $S \in EER'_r$ then $(S-K)^+ \leq F(t,S,r) \leq F(t,S,r') = (K-S)^+$, where the first inequality follows from value dominance and the second one from the fact that F is increasing in r. This means that $EER'_r \subseteq EER_r$. By passing to the infimum (resp. supremum) we conclude that the lower (resp. upper) free boundary is increasing (resp. decreasing) with respect to r.

For the other call option's properties, we cannot simply adapt the proof of Theorem 3.3 in (Battauz et al., 2015), as it relies on a simmetry result in a constant interest rate enviroment that fails to be applicable to our setting. The monotonicity properties of $\underline{S}^*$ and $\overline{S}^*$ with respect to t follow from the fact that F is decreasing with respect to t, similarly to the put case. We then prove the inequalities satisfied by the free boundaries. In the EER the function F satisfies

$$\frac{\partial F}{\partial t} + \mathcal{L}F \leq rF \tag{B7}$$

On the EER in the call case $F(t,S,r) = S-K$ and therefore equation (B7). simplifies to $1 \cdot S(r-q) \leq r(S-K)$. Thus $-Sq \leq -rK$ for all $S \in EER_r$, i.e. $S \leq \frac{r}{q}K$ for all $S \in EER_r$, as $q < 0$. By passing to the supremum we get $K \leq \underline{S}^*(t,r) < \overline{S}^*(t,r) \leq \frac{rK}{q}$.

At maturity $\underline{S}^*(T,r) = K$ and $\overline{S}^*(T,r) = +\infty$, as the option is exercised at T whenever $S(T) \geq K$.

We now show that $\underline{S}^*(T^-,r) = K$ and $\overline{S}^*(T^-,r) = \frac{rK}{q}$. By construction $\underline{S}^*(t,r) \geq K$ for all $t \in (\bar{t};T)$, and hence $\underline{S}^*(T^-,r) \geq K$. Suppose by contradiction that $\underline{S}^*(T^-,r) > K$. The set $(\bar{t};T) \times (K;\underline{S}^*(T^-,r)) \subset CR_r$ and therefore $(\mathcal{L}-r)F = -\frac{\partial}{\partial t}F \geq 0$, as F is decreasing w.r.t. t. As $t \uparrow T$ we have $(\mathcal{L}-r)F \to (\mathcal{L}-r)(S-K) = -qS + rK$ for $S \in (K;\underline{S}^*(T^-,r))$. This implies $-qS + rK \geq 0$ for $S \in (K;\underline{S}^*(T^-,r))$ and passing to the supremum over $S \in (K;\underline{S}^*(T^-,r))$ this delivers $\underline{S}^*(T^-,r) \geq \frac{rK}{q}$ which is a contradiction. We deal now with the upper free boundary limit. Suppose (by contradiction) that $\overline{S}^*(T^-,r) < \frac{rK}{q}$. But then the set $(\bar{t};T) \times \left(\overline{S}^*(T^-,r);\frac{rK}{q}\right) \subset CR_r$ and $(\mathcal{L}-r)F = -\frac{\partial}{\partial t}F \geq 0$ for $S \in \left(\overline{S}^*(T^-,r);\frac{rK}{q}\right)$. As $t \uparrow T$ we have $(\mathcal{L}-r)F \to (\mathcal{L}-r)(S-K) = -qS + rK$ for $S \in \left(\overline{S}^*(T^-,r);\frac{rK}{q}\right)$ (here the limits are in distribution). Then $-qS + rK \geq 0$ for all $S \in \left(\overline{S}^*(T^-,r);\frac{rK}{q}\right)$ and therefore also for the infimum $-q\overline{S}^*(T^-,r) + rK \geq 0$ that implies the contradiction $\overline{S}^*(T^-,r) \geq \frac{rK}{q}$.

C Additional numerical analysis of the free boundary

In this section we provide additional plots of the free boundaries. In particular, we show that, for fixed values of $r = r(t)$, the free boundaries sections as a function of time t display the same behavior found in the constant interest rate framework. In particular, for American put options the upper critical price is increasing with respect to time t whereas the lower critical price (if any) is decreasing (see Figure 14). For American call options the upper critical price (if any) is decreasing with respect to time t whereas the lower critical price is increasing (see Figure 15).

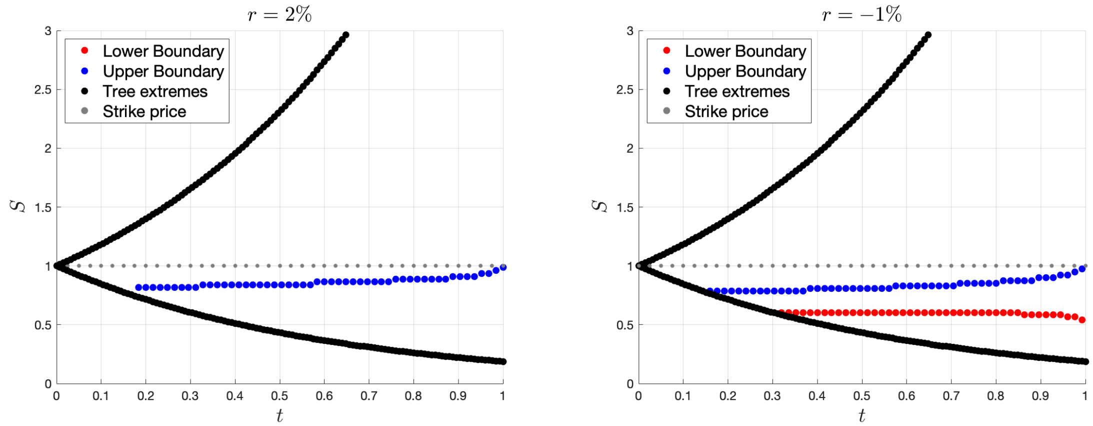

Figure 14: $r-$sections of free boundaries for the American put option. Left panel $r = 2\%$ and $q = 0\%$. Right panel $r = -1\%$ and $q = -2\%$.

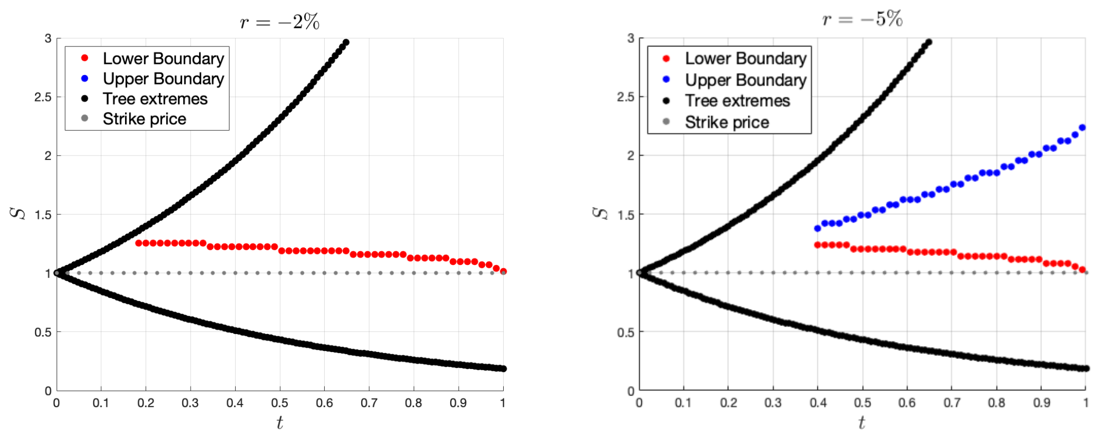

Figure 15: $r-$sections of free boundaries for the American call option. Left panel $r = -2\%$ and $q = 0\%$. Right panel $r = -5\%$ and $q = -2\%$.

The final table documents the impact of the correlation on the pricing of American equity options.

$\rho = -1, -0.5\%, -0.05\%, 0\%, 0.05\%, 0.5\%, 1$

ρ	q	π_E	π_A	$\pi_A^{r_0}$	$\lvert\pi_A - \pi_A^{r_0}\rvert/\pi_A$
American put option					
-50%	0%	5.540%	5.674%	5.979%	5.37%
	2%	6.486%	6.499%	6.962%	7.12%
	-2%	4.683%	5.002%	5.230%	4.55%
-5%	0%	5.606%	5.705%	5.979%	4.79%
	2%	6.551%	6.557%	6.962%	6.17%
	-2%	4.748%	5.025%	5.230%	4.07%
0%	0%	5.613%	5.709%	5.979%	4.72%
	2%	6.558%	6.563%	6.962%	6.08%
	-2%	4.755%	5.028%	5.230%	4.01%
5%	0%	5.620%	5.712%	5.979%	4.67%
	2%	6.565%	6.570%	6.962%	5.96%
	-2%	4.763%	5.030%	5.230%	3.97%
50%	0%	5.672%	5.745%	5.979%	4.06%
	2%	6.629%	6.630%	6.962%	5.00%
	-2%	4.827%	5.053%	5.230%	3.50%

Table 3: Results from the three numerical examples for the American put option.

ρ	q	π_E	π_A	$\pi_A^{r_0}$	$\lvert\pi_A - \pi_A^{r_0}\rvert/\pi_A$
American call option					
-50%	0%	6.269%	6.271%	5.979%	4.66%
	2%	5.235%	5.356%	5.163%	3.61%
	-2%	7.432%	7.432%	7.102%	4.44%
-5%	0%	6.335%	6.336%	5.979%	5.64%
	2%	5.230%	5.390%	5.163%	4.22%
	-2%	7.497%	7.497%	7.102%	5.26%
0%	0%	6.342%	6.343%	5.979%	5.75%
	2%	5.307%	5.393%	5.163%	4.27%
	-2%	7.505%	7.505%	7.102%	5.37%
5%	0%	6.339%	6.339%	5.979%	5.68%
	2%	5.314%	5.396%	5.163%	4.32%
	-2%	7.511%	7.511%	7.102%	5.45%
50%	0%	6.415%	6.415%	5.979%	6.80%
	2%	5.378%	5.431%	5.163%	4.94%
	-2%	7.576%	7.576%	7.102%	6.25%

Table 4: Results from the three numerical examples for the American call option.