Quantization of E&M field

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Wave - eq.

$\nabla B = 0$

No charge $\leftarrow$ $\nabla E = 0$

$\nabla \times E = -\frac{\partial B}{\partial t}$

$\nabla \times B = \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$

$\vec{B} = \nabla \times \vec{A}$ vector potential

$\vec{E} = -\nabla \phi - \frac{\partial A}{\partial t}$ Scalar potential

$\rightarrow$ Gauge Freedom

$$A \rightarrow A + \nabla \Lambda$$

$$\phi \rightarrow \phi - \frac{\partial \Lambda}{\partial t}$$

Coulomb Gauge (Radiation)

$$\begin{cases} \nabla . A = 0 \\ \phi = 0 \end{cases}$$

$$\nabla \times B = \mu_0 \varepsilon_0 \boxed{\frac{\partial E}{\partial t}} = \underbrace{\nabla \times (\nabla \times A)}$$

$$\underbrace{\nabla (\nabla . A)}_{0} - \nabla^2 A = \frac{\partial E}{\partial t}$$

$$\frac{\partial E}{\partial t} = -\frac{\partial A}{\partial t}$$

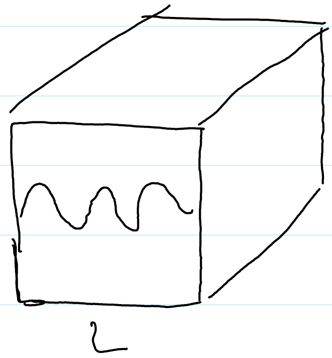
$$\Rightarrow \varepsilon_0 \mu_0 \frac{\partial^2 A}{\partial t^2} = \nabla^2 A \rightarrow \text{Wave eq.}$$

Wave speed $\leftarrow \frac{1}{c^2}$

$$A = A_0 e^{i(kx - \omega t)} + A^* e^{-i(kx - \omega t)}$$

$\longrightarrow$ This is for a specific $k \longrightarrow \omega$

Here I'm going to focus on a single mode. But the solution would involve all the different wave lengths (frequencies)



B.C.

$$A(x=0)=0 \Rightarrow A e^{i(kx-\omega t)} + A^* e^{-i(kx-\omega t)} = 0$$

$\forall t$

$$\Rightarrow A(x,t) = A \sin(kx) \cos(\omega t)$$

$$A(x=L)=0 \Rightarrow A(x,t) = A \sin(kL) \cos(\omega t) = 0$$

$$k_m = \frac{2\pi m}{L}$$

Energy of a single mode

$$H = \int_V \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) dV$$

$$E = \frac{-\partial A}{\partial t} = i\omega \left[A_0 e^{i(kx-\omega t)} - A_0^* e^{-i(kx-\omega t)} \right]$$

$$B = \nabla \times A = i\vec{k} \cdot \left[\vec{A}_0 e^{i(kx-\omega t)} - \vec{A}_0^* e^{-i(kx-\omega t)} \right]$$

$$\varepsilon_0 |E|^2 = \varepsilon_0 \omega^2 \left[2|A_0|^2 + \underbrace{A_0^2 e^{2i(kx-\omega t)}}_{\int dv \to 0} + \underbrace{A_0^{*2} e^{-2i(kx-\omega t)}}_{\int dv \to 0} \right]$$

$\int \quad \quad = V \omega^2 |A_0|^2$

$\int dV \to 0$ $\int dV \to 0$

$$\int \to \varepsilon_0 V \omega^2 |A_0|^2$$

$$\frac{1}{\mu_0} |B|^2 = \frac{1}{\mu_0} V k^2 |A_0|^2 \qquad k = \frac{\omega}{c} \to \omega^2 \varepsilon_0 V |A_0|^2$$

$$\Rightarrow H = 2 \varepsilon_0 V \omega^2 A_0 A_0^*$$

$$A = \frac{1}{\sqrt{4 \varepsilon_0 V \omega^2}} \left(\omega X - i P\right) \hat{e}$$

Quantization of light

$$E \rightsquigarrow \frac{1}{2} \int dV \left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right)$$

$$\to H = \frac{1}{2}\left(p^2 + \omega^2 q^2\right) \qquad \boxed{m=1}$$

$\to$ q & p form canonical conjugate variable.

$\{q, p\} = 1$ $\qquad [q, p] = i\hbar \mathbb{1}$

$$\frac{\hbar\omega}{2} \left(\qquad \right)$$

$$\hat{Q} = \sqrt{\frac{\omega}{\hbar}}\, q \qquad \hat{P} = \sqrt{\frac{1}{\hbar\omega}}\, p$$

$$\begin{cases} [\hat{Q}, \hat{P}] = i \\ H = \frac{\hbar\omega}{2}\left(\hat{Q}^2 + \hat{P}^2\right) \end{cases}$$

$$a = \frac{\hat{Q} + i\hat{P}}{\sqrt{2}}$$

$$a^\dagger = \frac{\hat{Q} - i\hat{P}}{} \qquad \to [a, a^\dagger] = -i[\hat{Q}, \hat{P}] \quad i$$

$$a^\dagger = \frac{\hat{Q} - i\hat{P}}{\sqrt{2}} \Rightarrow [a, a^\dagger] = -i[Q, P] + i[P, Q] = \boxed{1}$$

$[Q,P] = i$, $[P,Q] = -i$

$$H = \frac{\hbar\omega}{2}\left(\hat{Q}^2 + \hat{P}^2\right) = \frac{\hbar\omega}{2}\left[\underbrace{a^\dagger a + a a^\dagger}\right] = \left[2a^\dagger a + 1\right)\frac{\hbar\omega}{2}$$

This makes a huge difference

→ Look @ the classical calculation.

So

$$\rightarrow H = \frac{1}{2}\left[P^2 + \omega_u X^2\right] \rightarrow \text{H.O.}$$

$$\rightarrow \hat{H} = \frac{1}{2}\left(\hat{P}^2 + \omega_u \hat{X}^2\right)$$

$$\hat{P} \rightarrow \frac{P}{P_\cdot} = a - a^\dagger$$

$$\hat{X} \rightarrow X/X_\cdot = a + a^\dagger$$

$$H = \left(\underbrace{a^\dagger a}_{\hat{N}|n\rangle = n|n\rangle} + 1/2\right)\hbar\omega \rightarrow \text{Q.H.O.}$$

$\underbrace{\hat{N}|n\rangle = n|n\rangle}_{\text{Photons}}$ → Will come back to this.

Free field → no charge

$$H \rightarrow \underbrace{a^\dagger a} = \hat{N}$$ → It is normal so we know that:

$$\exists\ |n\rangle : \quad \hat{N}|n\rangle = n|n\rangle$$

$$[N, a] = -a \qquad\qquad \hat{N}\,\hat{a}|n\rangle = (n-1)\,a|n\rangle$$

$$\Rightarrow$$

$$[N, a^{\dagger}] = a^{\dagger} \qquad\qquad \hat{N}\,a^{\dagger}|n\rangle = (n+1)\,a^{\dagger}|n\rangle$$

$$\downarrow$$

They're ladder operators.

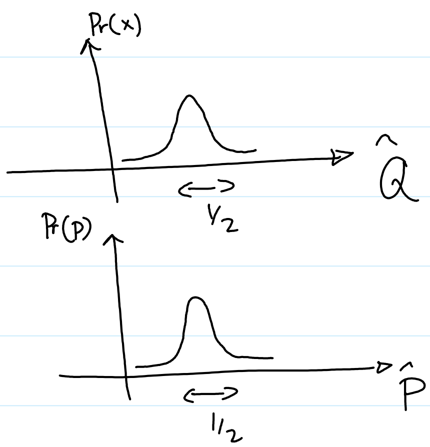
Energy is bounded from below so the ladder has to stop somewhere. or

$$a|n\rangle = C_n|n-1\rangle \Rightarrow \langle n|a^{\dagger}a|n\rangle = |C_n|^2 = n$$

$$\Rightarrow n \geq 0 \longrightarrow n \text{ has to be integer.}$$

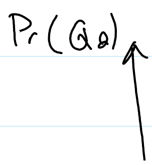
$$a|0\rangle = 0 \Rightarrow \psi_0(x) = \langle x|\underbrace{a}_{\frac{Q+iP}{\sqrt{2}}}|0\rangle = 0$$

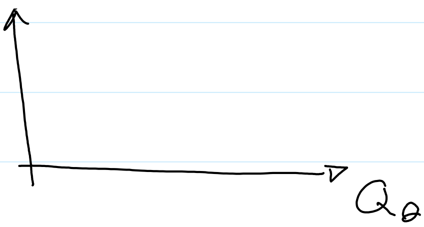
$$\rightarrow \left[x + i\left(-i\frac{\partial}{\partial Q}\right)\right]\psi_0(x) = 0 \Rightarrow \psi_0(x) = A\,e^{-Q^2}$$



$$\hat{Q}_\theta = e^{-i\theta H}\,\hat{Q}\,e^{i\theta H} = \hat{Q}\cos\theta + \hat{P}\sin\theta$$

$$\hat{P}_\theta = e^{-i\theta H}\,\hat{P}\,e^{i\theta H} = -\hat{Q}\sin\theta + \hat{P}\cos\theta$$





$W(Q, P)$:

$$Pr(Q_\theta) = \int W(Q_\theta, P_\theta)\, dP$$

...

$$W(q,p) = \frac{1}{2\pi}\int e^{ipx} \langle q-x|\rho|q+x\rangle\, dx$$

For pure states $\rho = |\psi\rangle\langle\psi|$

$$W(q,p) = \frac{1}{2\pi}\int e^{ipx}\, \psi(q-x)\, \psi^*(q+x)\, dx$$

$\longrightarrow$ We will come back to this.

How is $W_n(Q, P)$?

$|n\rangle \longrightarrow$ <u>Photons</u>

$\mathcal{E}_n = \underline{\hbar\omega}\,(n + 1/2)$ $\qquad \Delta\mathcal{E}_n = 0$

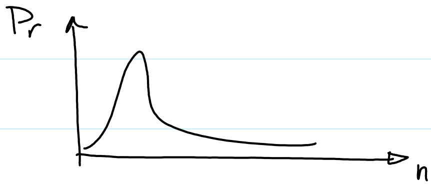
$$\begin{cases} \langle Q\rangle_n = 0 & \langle Q^2\rangle_n = \frac{\hbar\omega}{2}\left(n+1/2\right) \\ \langle P\rangle_n = 0 & \langle P^2\rangle = \frac{\hbar\omega}{2}\left(n+1/2\right) \end{cases}$$

$\longrightarrow$ Get back to this for wigner function

Coherent state

$$a|\alpha\rangle = \alpha|\alpha\rangle$$

$$|\alpha\rangle = \sum c_n |n\rangle \longrightarrow |\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum \frac{\alpha^n}{\sqrt{n!}} |n\rangle.$$



$$\mathcal{E}_\alpha = \langle\alpha|H|\alpha\rangle = \sum_{n=0}^{\infty} \mathcal{E}_n \Pr(n) = \cdots = |\alpha|^2$$

$$\Delta\mathcal{E}_\alpha = \sqrt{\langle\alpha|H^2|\alpha\rangle - \mathcal{E}_\alpha^2} = |\alpha| = \sqrt{n}$$

$$\langle \hat{E}_\alpha\rangle = \cdots = \mathrm{Re}(\alpha)$$

$$\langle E_\alpha^2\rangle = \cdots = \mathrm{Re}(\alpha) + \cdots$$

$$\left\} \begin{array}{l} \langle \hat{B}_\alpha\rangle \\ \langle B_\alpha^2\rangle \end{array}\right.$$

Displacement Operator

$$|\alpha\rangle = D(\alpha)|0\rangle \quad : \quad D(\alpha) = e^{(\eta a - \eta^* a^\dagger)}$$

$$D(\alpha) = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^\dagger} e^{-\alpha^* a}.$$

* $$D(\alpha)^\dagger \, a \, D(\alpha) = a + \alpha$$
$$D(\alpha)^\dagger \, a^\dagger \, D(\alpha) = a - \alpha$$

$$D(\alpha)^\dagger \, \hat{E} \, D(\alpha) = \dots$$

<u>Exercise</u>

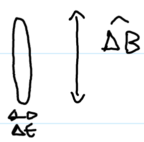
* $2\mathbb{1}\pi = \int d^2\alpha \, |\alpha\rangle\langle\alpha|$

* $\langle \alpha | \beta \rangle$

* Draw the W for $\left(\frac{|\alpha\rangle + |-\alpha\rangle}{2} \right)$.

<u>Squeezing</u>

$\longrightarrow$ What if we only want $\hat{E}$?



How to do this ?

$$S(\xi) = \exp\left(\tfrac{1}{2} \xi^* a^2 - \tfrac{1}{2} \xi a^{\dagger 2} \right) \quad , \xi = r e^{i\theta}$$

$$S^\dagger(\xi) \, a \, S(\xi) = a \cosh(r) - a^\dagger e^{i\theta} \sinh(r)$$

$$S^\dagger(\xi) \, a^\dagger \, S(\xi) = a^\dagger \cosh(r) - a \, e^{-i\theta} \sinh(r)$$

$$S^\dagger(\xi)\, a^\dagger\, S(\xi) = a^\dagger \cosh(r) - a\, e^{-i\theta} \sinh(r)$$

BCH:

$$e^{A}\, B\, e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \cdots$$

$$\frac{C_n}{n!}$$

$$A = \frac{1}{2}\left[\xi^* a^2 - \xi a^{\dagger 2}\right], \quad B = a$$

$$C_1 = [A,B] = \frac{1}{2}\xi a^\dagger \qquad C_2 = [A, a^\dagger]\left(\frac{1}{2}\xi\right) = \left(\frac{-1}{2!}\right)\underbrace{\xi\xi^*}_{r^2} a$$

$$\vdots \qquad\qquad \vdots$$

$$C_{2k+1} = \frac{r^{2k+1} e^{i\theta}}{(2k+1)!\, 2^{2k+1}}\, a^\dagger \qquad C_{2k} = \frac{r^{2k}}{(2k)!\, 2^{2k}}\, a$$

$$\sinh(r/2)\; a^\dagger \qquad\qquad \downarrow \; \cosh(r/2)\; a$$

$$\langle \xi | Q | \xi \rangle = \frac{1}{\sqrt{2}}\langle 0 | S^\dagger(\xi)\,(a + a^\dagger)\, S(\xi) | 0 \rangle$$

$$= 0$$

$$\langle \xi | P | \xi \rangle = 0$$

$$\langle \xi | Q^2 | \xi \rangle = \langle 0 | S^\dagger Q S^\dagger S Q S | 0 \rangle =$$

$$\cdots$$

$$= \frac{1}{2}\left[\cosh^2(r) + \sinh^2(r) \mp 2\cosh(r)\sinh(r)\cos\theta\right]$$

$\underline{\theta = 0} \longrightarrow$ Real ξ $\qquad (\cosh r \pm \sinh r)^2$

$$\langle \Delta Q \rangle = \underline{\underline{e^{-r}}} \qquad \langle \Delta P \rangle = \underline{\underline{e^{r}}}$$

$$\langle \Delta Q \rangle = \frac{e^{-r}}{\sqrt{2}} \qquad \langle \Delta P \rangle = \frac{e^{r}}{\sqrt{2}}$$

How to find the state.

$\underline{|\xi\rangle}$:

$$a|0\rangle = 0 \Rightarrow \underbrace{S^\dagger(\xi)\, a\, S^\dagger(\xi)}\, \underbrace{S(\xi)|0\rangle} = 0$$

$$\left[a \cosh(r) - a^\dagger e^{i\theta} \sinh(r)\right] |\xi\rangle = 0$$

$$|\xi\rangle = \frac{1}{\sqrt{\cosh r}} \sum_n (-1)^n \frac{\sqrt{(2n)!}}{2^n n!} e^{in\theta} \tanh^n r |2n\rangle.$$

For the plots of wigner functions, see https://github.com/sraeisi/QuantumOptics