Collision rate ansatz for quantum integrable systems

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based on arXiv: 2004.07113 with **H. Spohn**

Plan

- Ballistic and diffusive hydrodynamics
- Generalized hydrodynamics (GHD) and collision rate ansatz in integrable systems

Hydrodynamics for many-body systems

- Consider a one-dimensional interacting many-body systems with conservation laws $Q_j, j = 1,2,\ldots,N$.

- Hydrodynamics offers a universal language to characterise the long-wavelength dynamics of it.

- Continuity equations read

$$\partial_t \langle \mathfrak{q}_j \rangle + \partial_x \langle \mathfrak{j}_j \rangle = 0.$$

where $\langle \cdot \rangle$ is w.r.t. the initial ensemble. Hydro approximations then tell us to replace $\langle \mathfrak{j}_j \rangle$ with

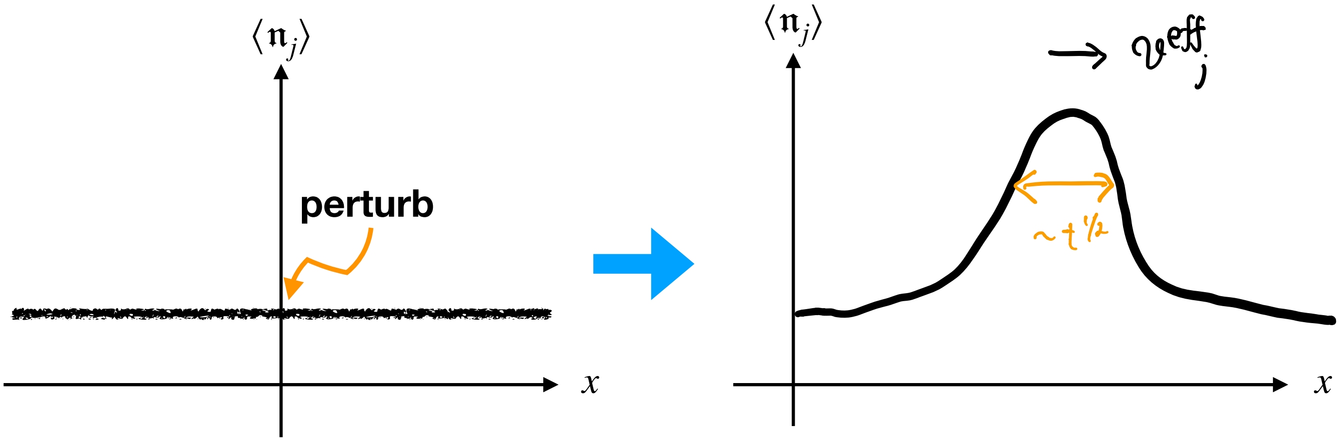
$$\langle \mathfrak{j}_j \rangle = A_j{}^k \langle \mathfrak{q}_k \rangle + \mathfrak{D}_j{}^k \partial_x \langle \mathfrak{q}_k \rangle +$$ ~~higher deriv. (shorter-wavelength)~~

Ballistic Diffusive

- $A_j{}^k = \dfrac{\partial \langle \mathfrak{j}_j \rangle}{\partial \langle \mathfrak{q}_k \rangle}$ governs the ballistic dynamics while $\mathfrak{D}_j{}^k$ controls the diffusive broadening of the ballistic trajectories.

- The eigenvalues of $A_j{}^k$, which we denote v_j^{eff}, plays an important role.

- Consider a weak perturbation to the background ensemble.



Collision rate ansatz in integrable systems

- In integrable systems, thermodynamic Bethe ansatz (TBA) allows us to write $\langle \mathfrak{q}_j \rangle$ and $\langle \mathfrak{j}_j \rangle$ as

$$\langle \mathfrak{q}_j \rangle = \int \mathrm{d}\theta \rho(\theta) h_j(\theta), \quad \langle \mathfrak{j}_j \rangle = \int \mathrm{d}\theta \rho(\theta) v^{\mathrm{eff}}(\theta) h_j(\theta) \, .$$

$$\Rightarrow \quad \boxed{\partial_t \rho(\theta) + \partial_x(\rho(\theta) v^{\mathrm{eff}}(\theta)) = 0}$$

[Bertini, Collura, De Nardis, Fagotti, 2016; Castro-Alvaredo, Doyon, Yoshimura, 2016]

- The root density $\rho(\theta)$ satisfies some integral equation, and can be computed by solving it. $h_j(\theta)$ is the one-particle eigenvalue of Q_j. $v^{\mathrm{eff}}(\theta)$ is the functional of $\rho(\theta)$, and its exact form is what we are concerned with.

- So far two different proofs of v^{eff} are available [Vu and Yoshimura, 2019; Borsi, Pozsgay, and Pristyák, 2020; Pozsgay, 2020].

- I will present a new proof that relies neither on the FF expansion nor deformations [Spohn, 2020; Yoshimura and Spohn, 2020].

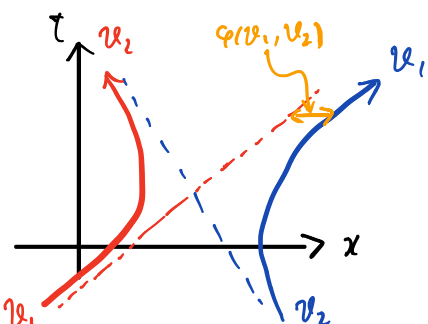
- $v^{\text{eff}}(\theta)$ admits an intuitive understanding from the scattering picture.

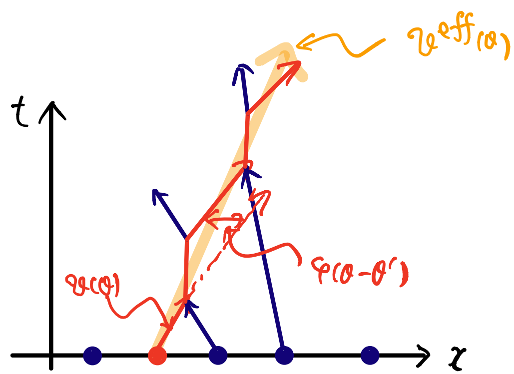
- In a fluid of quasi-particles, $v^{\text{eff}}(\theta)$ is the mean velocity of a tracer particle with the incoming velocity $v(\theta)$ when traveling over a large distance Δx for a long time Δt, i.e. $\Delta x = v^{\text{eff}}(\theta)\Delta t$. Δx can be decomposed as follows

$$\Delta x = v(\theta)\Delta t - \sum_{\substack{\text{quasi-particle is}\\ \text{located right to the tracer}}} \varphi(\theta-\theta_n) + \sum_{\substack{\text{quasi-particle is}\\ \text{located left to the tracer}}} \varphi(\theta-\theta_n)$$

jump distance

$$v^{\text{eff}}(\theta) = v(\theta) + \int \mathrm{d}\theta' \frac{\varphi(\theta-\theta')\rho(\theta')}{p'(\theta)}\left(v^{\text{eff}}(\theta') - v^{\text{eff}}(\theta)\right)$$





The proof

- Two inputs are needed. First, the charge-charge and charge-current susceptibility matrices are defined by

$$\begin{aligned} C_{ij} &= \int_{\mathbb{R}} \mathrm{d}x \langle \mathfrak{q}_i(x)\mathfrak{q}_j(0)\rangle^c_{\mathrm{GGE}} = -\frac{\partial\langle \mathfrak{q}_i\rangle}{\partial\mu^j} \\ B_{ij} &= \int_{\mathbb{R}} \mathrm{d}x \langle \mathfrak{j}_i(x)\mathfrak{q}_j(0)\rangle^c_{\mathrm{GGE}} = -\frac{\partial\langle \mathfrak{j}_i\rangle}{\partial\mu^j} \end{aligned}, \quad \rho_{\mathrm{GGE}} \sim e^{-\sum_j \mu^j Q_j}$$

- C_{ij} is obviously symmetric. Less obvious is that B_{ij} is actually also symmetric. The symmetry of B_{ij} is the first input to the proof.

- Second one is the existence of a self-conserved current. For instance, in Galilean systems, the particle current J_0 ($= \int \mathrm{d}x \mathfrak{j}_0(x)$) equals the momentum Q_1 (mass is set to 1). In the XXZ spin 1/2 chain, the energy current J_1 coincides with the charge Q_2.

- The starting point of the proof is to write $\langle \mathfrak{j}_j \rangle$, for some function $\bar{v}(\theta)$, as

$$\langle \mathfrak{j}_j \rangle = \int \mathrm{d}\theta \rho(\theta) \bar{v}(\theta) h_j(\theta),$$

which is always possible. We then show that $\bar{v}(\theta) = v^{\mathrm{eff}}(\theta)$.

- Suppose $J_a = Q_b$ for a pair a, b. Then $C_{bj} = B_{aj}$ holds. Using the symmetry of C and B, this implies $C_{jb} = B_{ja}$. Therefore 1) the symmetry of B and 2) the existence of a bridging pair $J_a = Q_b$ admit the identity

$$\frac{\partial \langle \mathfrak{q}_j \rangle}{\partial \mu^b} = \frac{\partial \langle \mathfrak{j}_j \rangle}{\partial \mu^a} .$$

- This (extremely) simple relation is the core identity in the proof.

- In integrable systems, this identity gives rise to

$$\partial_{\mu^a}(\rho\bar{v}) = \partial_{\mu^b}\rho\,.$$

- Elementary manipulations yield $\partial_{\mu^b}\rho = \partial_{\mu^a}(\rho v^{\mathrm{eff}})$ for systems known to possess a self-conserved current. $\rho(\bar{v} - v^{\mathrm{eff}})$ is therefore constant in μ^a.

- For instance, in the Lieb-Liniger model (Galilean invariant integrable field theory), $\rho(\theta) \to 0$ when $\mu^a \to \infty$. Hence the free constant must be zero, yielding $\bar{v}(\theta) = v^{\mathrm{eff}}(\theta)\,.$

- A similar reasoning is also possible in the XXZ spin-1/2 chain, and more broadly, in the XYZ spin-1/2 chain.

- The symmetry of B is always guaranteed so long as the clustering of correlation functions holds. Do we always have a self-conserved current in integrable systems?

→ **Quite often yes, but not always**
(one exception is the Fermi-Hubbard model).

Conclusion and outlook

- The effective velocity v^{eff} governs the ballistic transport.
- The symmetry of B matrix and the existence of a self-conserved current fixes the functional form of v^{eff} completely in integrable systems.
- Probably it is possible to extend our approach to the Fermi-Hubbard model upon appropriate generalisations.
- Our approach can also be applied to prove the form of generalised currents, which are currents generated by other conserved charges.
- Boost operators are also used to perform a long-range deformation to integrable spin chains [Pozsgay, 2020], from which the collision rate ansatz was also established. It is highly desired to understand the role of boost operators in GHD.

The boost operator

- It turns out that the boost operator gives a self-conserved current.
- Consider the XYZ spin-1/2 chain

$$H = -\frac{1}{2}\sum_{j\in\mathbb{Z}}(J_x S_j^x S_{j+1}^x + J_y S_j^y S_{j+1}^y + J_z S_j^z S_{j+1}^z)\,.$$

- The boost operator is defined for any local observable $\mathcal{O} = \sum_{j\in\mathbb{Z}} \mathfrak{o}(j)$ as $K[\mathcal{O}] = \sum_{j\in\mathbb{Z}} j\mathfrak{o}(j)$. In particular the boost operator associated to $H = Q_1$ generates conserved charges in a recursive fashion

$$[K[H], Q_n] = \mathrm{i}Q_{n+1}\,.$$

- The algebra generated by such commutation relations can be thought of as the Lattice Lorentz algebra.
- Such a boost operator also exists in other models as well, such as the quantum and classical Toda lattice.

- Recall the lattice continuity equation $\mathrm{i}[H, \mathfrak{q}_n(j)] = \mathfrak{j}_n(j) - \mathfrak{j}_n(j+1)$. This implies then

$$[K[Q_n], H] = \mathrm{i}J_n\,.$$

- Combining with the boost commutation relation, we find a self-conserved current $J_1 = Q_2$.

- In the XYZ spin-1/2 chain, we can take the scaling limit, under which the model becomes the massive sine-Gordon model, which is relativistic. In this case the first few boost commutation relations reduce to the usual Poincaré algebra

$$[H, P] = 0, \quad [K[H], H] = \mathrm{i}P, \quad [K[H], P] = \mathrm{i}H,$$

which obviously implies $J_H = P$.

- Similarly, Galilean invariant systems that conserve the particle number possess a bridging pair $J_0 = P$.

- A natural question is if there is any integrable system that fails to have a self-conserved current.

- A notable example of such integrable system is the Fermi-Hubbard model (FHM), in which even the energy current is not conserved unlike the XXZ spin-1/2 chain [Karrasch, Kennes, Heidrich-Meisner, 2016]. But to my knowledge the fact that there is no self-conserved current in FHM has not been proven yet.

- But the boost operator that satisfies the boost commutation relations does exist in FHM [Links, Zhou, Mckenzie, Gould; 2001]. The point is that the boost operator cannot be written as $K[H] = \sum_{j\in\mathbb{Z}} j\mathfrak{q}_1$.

- Finally, the collision rate ansatz for other flows that are generated by other conserved charges can also be proved in a similar manner.