QCD Physics for Colliders

Matteo Cacciari

LPTHE and Université de Paris

ICTP Summer School 2021

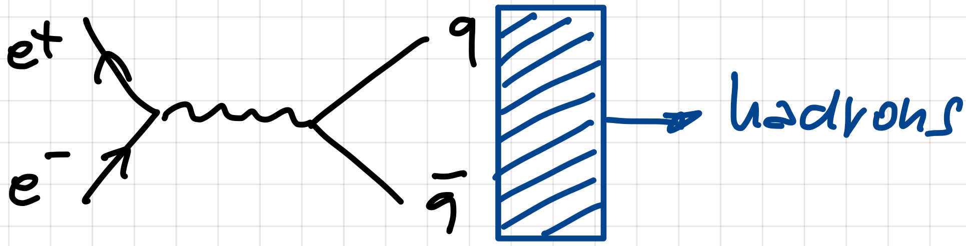
Lecture 4

$e^+e^- \to$ hadrons

IRC safety

$e^+e^- \to$ hadrons

Simplest "QCD" process



We actually study $e^+e^- \to q\bar{q}$ because

$$\sigma(e^+e^- \to \text{hadrons}) = \sigma(e^+e^- \to q\bar{q}(+\dots)) + \underbrace{\mathcal{O}\left(\left(\tfrac{1}{\sqrt{s}}\right)^4\right)}$$

non-perturbative corrections

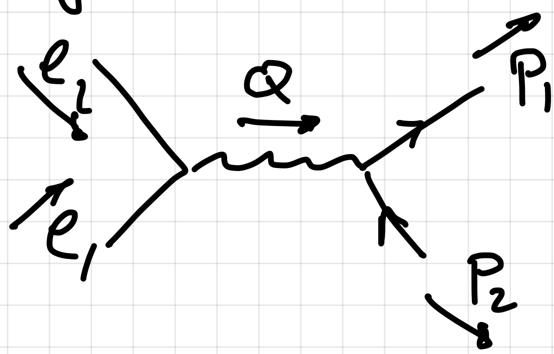
At leading order, actually pure e.m. process (but with quarks)

We can use this process to introduce/discuss a number of points

- characteristics of QCD emissions
  - factorisation of soft/coll radiation
- infrared structure. Need for real/virtual cancellations
- structure of higher order calculations
  - scale dependence
  - excursus: renormalons
- Infrared and collinear safety
  - IRC-safe observables. Event shapes, jets
- Parton-hadron duality, hadronisation corrections / power corrections

Start with leading order:

$e^+e^- \to q\bar{q}$



CM energy $= \sqrt{(e_1+e_2)^2} = \sqrt{(p_1+p_2)^2} = \sqrt{s}$

$$d\sigma = \frac{1}{2s}\,\overline{\sum |M|^2}\, d\phi_2$$

with $d\phi_2 = \dfrac{d^3p_1}{(2\pi)^3 2E_1}\,\dfrac{d^3p_2}{(2\pi)^3 2E_2}\,(2\pi)^4\,\delta^{(4)}(e_1+e_2-p_1-p_2)$

Total x-sect for massless quarks (for a single quark flavour)

$$\sigma = \frac{4\pi}{3}\,\frac{\alpha_{em}^2}{s}\,e_q^2\,N_c$$

number of colours. Experimental evidence that $N_c = 3$

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = N_c \sum e_q^2 = N_c\left[\overset{u}{\left(\tfrac{2}{3}\right)^2} + \overset{d}{\left(\tfrac{-1}{3}\right)^2} + \overset{s}{\left(\tfrac{-1}{3}\right)^2}\right]$$

$$= N_c\,\frac{6}{9} = N_c\,\frac{2}{3} = 2$$

σ and R in e^+e^- Collisions

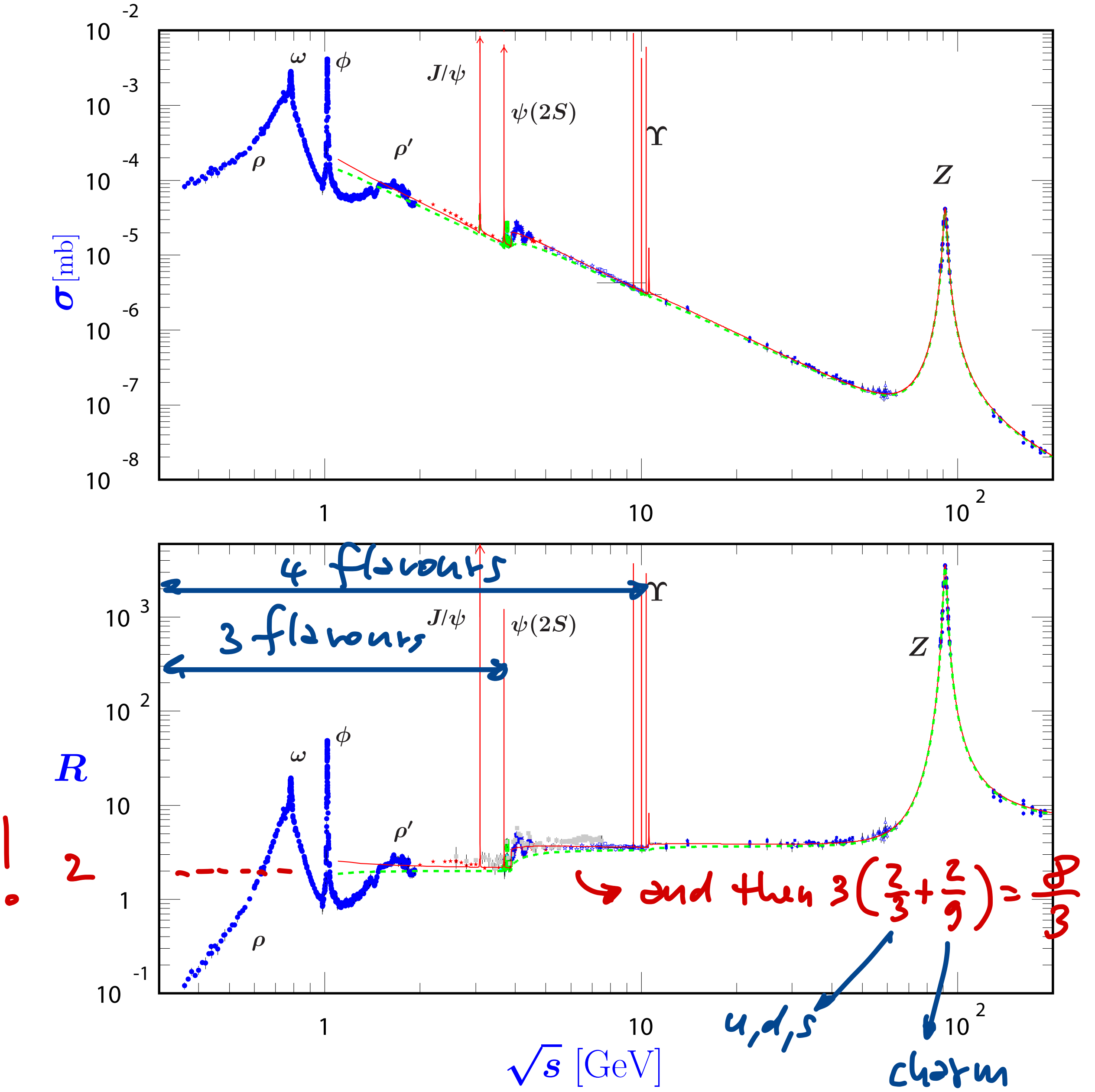
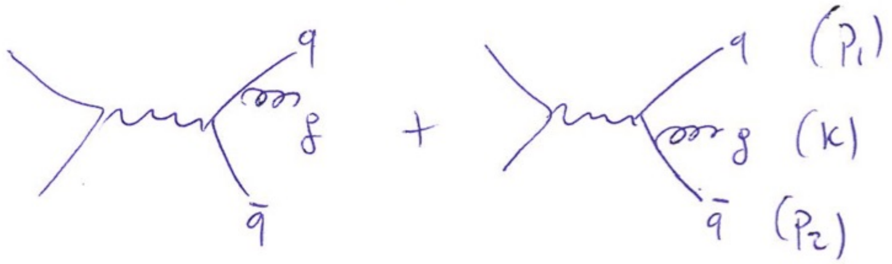


Figure 49.5: World data on the total cross section of $e^+e^- \to hadrons$ and the ratio $R(s) = \sigma(e^+e^- \to hadrons, s)/\sigma(e^+e^- \to \mu^+\mu^-, s)$. $\sigma(e^+e^- \to hadrons, s)$ is the experimental cross section corrected for initial state radiation and electron-positron vertex loops, $\sigma(e^+e^- \to \mu^+\mu^-, s) = 4\pi\alpha^2(s)/3s$. Data errors are total below 2 GeV and statistical above 2 GeV. The curves are an educative guide: the broken one (green) is a naive quark-parton model prediction, and the solid one (red) is 3-loop pQCD prediction (see "Quantum Chromodynamics" section of this *Review*, Eq. (9.7) or, for more details, K. G. Chetyrkin *et al.*, Nucl. Phys. **B586**, 56 (2000) (Erratum *ibid.* **B634**, 413 (2002)). Breit-Wigner parameterizations of J/ψ, $\psi(2S)$, and $\Upsilon(nS), n = 1, 2, 3, 4$ are also shown. The full list of references to the original data and the details of the R ratio extraction from them can be found in `[arXiv:hep-ph/0312114]`. Corresponding computer-readable data files are available at `http://pdg.lbl.gov/current/xsect/`. (Courtesy of the COMPAS (Protvino) and HEPDATA (Durham) Groups, May 2010.)

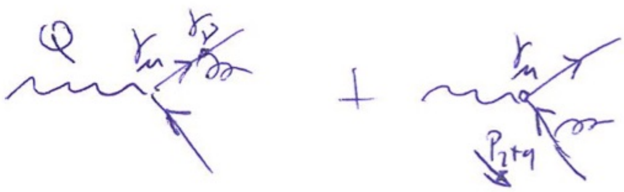
$$R = \frac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)} = N_c \sum e_q^2$$

Go now to first higher order: emission of a gluon

At first order in α_s



Consider just $\gamma^* \to q\bar{q}g$ (simpler)



$$M_\mu = \bar{u}(p_1)(i g_s \gamma_\nu t^A) \frac{i}{\not{p}_1+\not{k}} (+i e_q \gamma_\mu)\, v(p_2)\, \epsilon^{*\nu}(k)$$

$$+ \bar{u}(p_1)(+i e_q \gamma_\mu) \frac{(-i)}{\not{p}_2+\not{k}} (i g_s t^A \gamma_\nu)\, v(p_2)\, \epsilon^{*\nu}(k)$$

Make gluon <u>soft</u> $\Rightarrow$ $\boxed{k \ll p_{1,2}}$

$$\Rightarrow M_\mu \simeq \underbrace{\bar{u}(p_1)(i e_q \gamma_\mu)\, v(p_2)}_{\text{Born}}\, \underbrace{g_s t^A \left(\frac{p_1 \cdot \epsilon^*}{p_1 \cdot k} - \frac{p_2 \cdot \epsilon^*}{p_2 \cdot k} \right)}_{\text{eikonal current}}$$

Factorised form

Proof of eikonal factorisation

$$\bar{u}(p_1)\, i g_s \not\epsilon^* t^A \frac{i}{\not p_1 + \not k}\, i e_q \gamma_\mu v(p_2) =$$

$$= -\bar{u}(p_1)\, g_s t^A \not\epsilon^* \frac{\not p_1 + \not k}{(p_1 + k)^2} (i e_q \gamma_\mu)\, v(p_2)$$

(use $\not A \not B = 2A\cdot B - \not B \not A$)

$$= -\bar{u}(p_1)\, g_s t^A \left[2\epsilon^* \cdot (p_1 + k) - (\not p_1 + \not k)\not\epsilon^* \right] \frac{1}{(p_1 + k)^2} (i e_q \gamma_\mu)\, v(p_2)$$

(use $\bar{u}(p_1)\not p_1 = 0$ (Dirac) and $k \ll p_1$)

$$\simeq -g_s t^A\, \bar{u}(p_1) \left[2\epsilon^* \cdot p_1 \right] \frac{1}{(p_1 + k)^2} (i e_q \gamma_\mu)\, v(p_2)$$

$(p_1+k)^2 \simeq 2 p_1 \cdot k$

$$\simeq -g_s t^A \frac{p_1 \cdot \epsilon^*}{p_1 \cdot k} \underbrace{\bar{u}(p_1)(i e_q \gamma_\mu)\, v(p_2)}_{\text{usual QED vertex structure}}$$

$-$ or $+$?

and then similarly with the other term

$$\Rightarrow \sum_{\substack{col\\pol}} \left|M_{q\bar{q}g}\right|^2$$

$$= \sum_{\substack{col\\pol}} \left| \bar{u}(p_1)(ie_q\gamma_\mu)\, t^A\, v(p_2)\, g_s \left(\frac{p_1\cdot\epsilon^*}{p_1\cdot k} - \frac{p_2\cdot\epsilon^*}{p_2\cdot k}\right)\right|^2$$

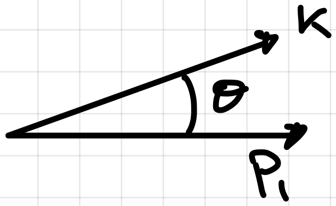
$$= \underbrace{\left|M_{q\bar{q}}\right|^2}_{\text{Born}}\ \underbrace{C_F\, g_s^2\, \frac{2\, p_1\cdot p_2}{(p_1\cdot k)(p_2\cdot k)}}_{\text{radiation}} \quad \leftarrow \text{factorisation}$$

Including the 3-particle phase space we have

$$\sum \left|M_{q\bar{q}g}\right|^2 d\phi_3 = \left|M_{q\bar{q}}\right|^2 d\phi_2\, \frac{d^3k}{(2\pi)^3(2E_g)}\, C_F\, g_s\, \frac{2p_1\cdot p_2}{(p_1\cdot k)(p_2\cdot k)}$$

$$= \left|M_{q\bar{q}}\right|^2 d\phi_2\ E_g\, dE_g\, d\cos\theta\, \frac{d\varphi}{2\pi}\, \frac{2\alpha_s C_F}{\pi}\, \frac{2p_1\cdot p_2}{(p_1\cdot k)(p_2\cdot k)}$$

with



In the soft limit ($k \ll p_1, p_2$)

$$s = Q^2 = (p_1 + p_2 + k)^2 \simeq (p_1 + p_2)^2 = 2p_1 \cdot p_2$$

and we can choose a frame where

$$p_1^0 = p_2^0 = \frac{\sqrt{s}}{2}$$

In this frame (and massless quarks)

$$\frac{2p_1 \cdot p_2}{(p_1 \cdot k)(p_2 \cdot k)} = \frac{s}{\left(p_1^0 k^0 - |\vec{p}_1||\vec{k}|\cos\theta_{p_1 k}\right)\left(p_2^0 k^0 - |\vec{p}_2||\vec{k}|\cos\theta_{p_2 k}\right)}$$

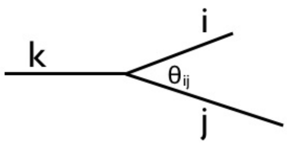
$$= \frac{s}{\frac{\sqrt{s}}{2} E_g (1 - \cos\theta) \frac{\sqrt{s}}{2} E_g (1 + \cos\theta)}$$

$$= \frac{4}{E_g^2 (1 - \cos^2\theta)}$$

$$\Rightarrow |M_{q\bar{q}g}|^2 d\phi_3 \simeq |M_{q\bar{q}}|^2 d\phi_2 \underbrace{\frac{2\alpha_s C_F}{\pi} \frac{dE_g}{E_g} \frac{d\theta}{\sin\theta} \frac{d\varphi}{2\pi}}_{dP}$$

dP ↑ emission of soft radiation

QCD emission probability



$$\frac{dP_{k\to ij}}{dE_i d\theta_{ij}} \sim \frac{\alpha_s}{\min(E_i, E_j)\theta_{ij}}$$

Singular in the soft ($E_{i,j}\to 0$) and
in the collinear ($\theta_{ij}\to 0$) limits.
Divergent upon integration.

The divergences can be cured by the addition of virtual corrections and/or **if** the definition of an observable is appropriate

Altarelli-Parisi kernel

Using the variables $E=(1-z)p$ and $k_t = E\theta$ we can rewrite

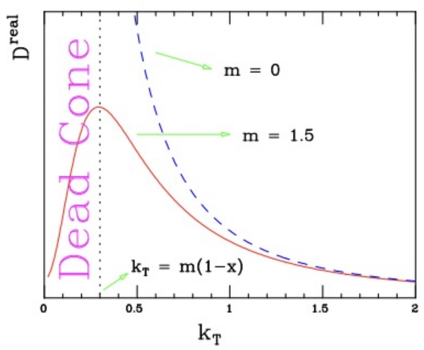
$$dS = \frac{2\alpha_s C_F}{\pi}\frac{dE}{E}\frac{d\theta}{\sin\theta}\frac{d\phi}{2\pi} \;\longrightarrow\; \frac{\alpha_s C_F}{\pi}\frac{1}{1-z}dz\frac{dk_t^2}{k_t^2}\frac{d\phi}{2\pi}$$

'almost' the Altarelli-Parisi splitting function P_{qq}

Massive quarks

If the quark is massive the collinear singularity is **screened**

$$\frac{\alpha_s C_F}{\pi}\frac{1}{1-z}dz\frac{dk_t^2}{k_t^2}\frac{d\phi}{2\pi} \to \frac{\alpha_s C_F}{\pi}\frac{1}{1-z}dz\frac{dk_t^2}{k_t^2+(1-z)^2m^2}\frac{d\phi}{2\pi}+\cdots$$



If one tries to calculate a total x-sect integrating over the gluon phase space, a double divergence appears:

$$\int_0 \frac{dE_g}{E_g}$$ is logarithmically

This is a SOFT divergence. It appears when the gluon is emitted with infinitesimally small energy

$$\int_0 \frac{d\theta}{\sin\theta} \sim \int_0 \frac{d\theta}{\theta}$$ is logarithmically div.

This is a COLLINEAR divergence. It appears when the gluon is emitted collinearly to a quark (NB this divergence does not appear if the quarks are massive)

Another way of seeing this:

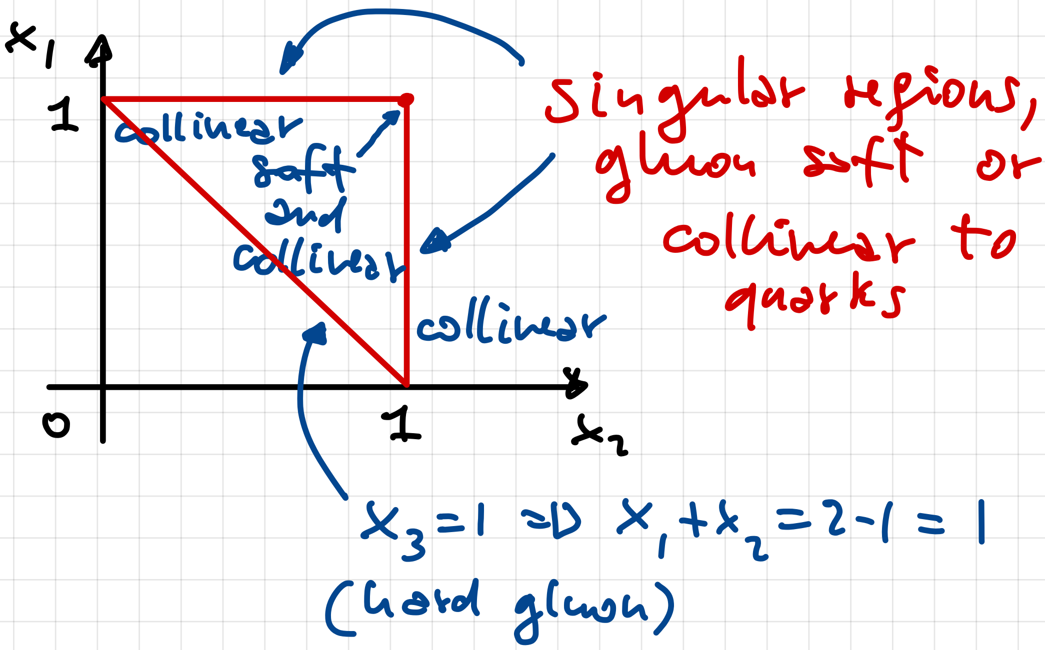
If we set $x_i \equiv \frac{2p_i \cdot Q}{Q^2}$ $\quad i=1,2,3 = q, \bar{q}, g$

we have that the full result is the remarkably simple

$$\frac{1}{\sigma_0}\frac{d\sigma}{dx_1 dx_2} = \frac{C_F \alpha_s}{2\pi} \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

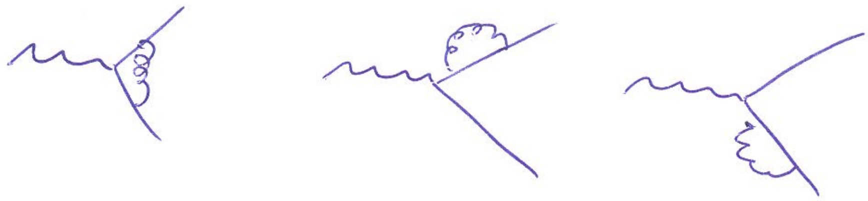
with $0 \le x_i \le 1$ and $x_1 + x_2 + x_3 = 2$

And one can check that in the soft limit this coincides with the previous result



Why is this divergent? Because if we want to calculate a <u>total</u> $e^+e^- \to$ hadrons cross section the emission of a gluon is not the full story a $O(\alpha_s)$.

We also have the virtual corrections:



This final state (no gluon) "looks like" (i.e. is degenerate with) the real one in the soft/collinear limit.

It must therefore be added to the real x-sect to yield a finite result. It's an example of the Kinoshita-Lee-Nauenberg theorem (inclusive observables are finite) (More later)

How to proceed?

1) Calculate both R and V x-sections with a regulator

2) Add them together, make the regulator disappear, get a finite limit.

We could use a small gluon mass, but the preferred regulator is dimensional regularization.

Work in $D = 4-2\epsilon$ dimensions. Infrared divergencies are regulated if $D>4$ ($\Rightarrow \epsilon < 0$)

One finds

$$\left|\ \ \right|^2 d\phi_3 = \alpha_{em}\, e_q^2\, \frac{\alpha_s}{2\pi}\, C_F\, N_c\, 2Q^2 \left(\frac{4\pi\mu^2}{Q^2}\right)^{2\epsilon} \frac{(1-\epsilon)}{\Gamma(2-2\epsilon)}$$

$$\frac{1}{\left[(1-x_1)(1-x_2)(x_1+x_2-1)\right]^{\epsilon}} \times$$

$$\times \Bigg[\underbrace{(1-\epsilon)\left(\frac{1-x_1}{1-x_2}+\frac{1-x_2}{1-x_1}\right)+\frac{2(x_1+x_2-1)}{(1-x_1)(1-x_2)}-2\epsilon}\Bigg]$$

$$\xrightarrow{\epsilon\to 0} \frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)}$$

$$\left(\text{and also} = \frac{x_1^2+x_2^2-\epsilon x_3^2}{(1-x_1)(1-x_2)}\right.$$

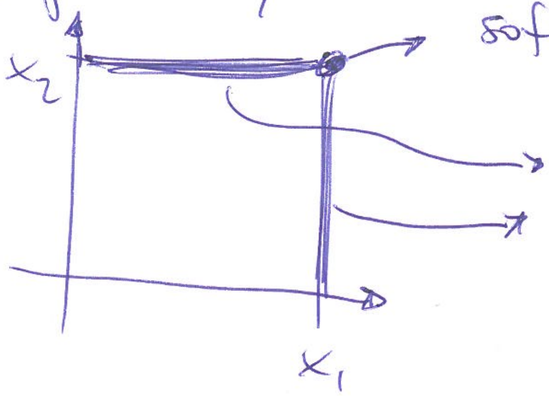
After integrating $\int_0^1 dx_1 \int_{1-x_1}^1 dx_2$ and dividing by the ~~[illegible]~~ flux and the propagator, $\frac{1}{2Q^4}$, and including the initial state e^+e^- one finds

$$\sigma_R^{(\epsilon)} = \sigma_0^{(\epsilon)} \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{Q^2}\right)^{\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-3\epsilon)} \left(\frac{2}{\epsilon^2}+\frac{3}{\epsilon}+\frac{19}{2}+\dots\right)$$

and

$$\sigma_0^{(\epsilon)} = \frac{4\pi\alpha_{em}^2}{Q^2} e_q^2 N_c \frac{(1-\epsilon)^2}{(3-2\epsilon)} \frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)} \left(\frac{4\pi\mu^2}{Q^2}\right)^{\epsilon}$$

$\frac{1}{\epsilon^2}$ and $\frac{1}{\epsilon}$ are the soft + collinear and the collinear singularity



the virtual contribution is instead given, after quite some work, by

$$\sigma_V^{(\epsilon)} = -\sigma_0^{(\epsilon)} \frac{\alpha_s}{2\pi} C_F \overbrace{\mathrm{Re}\left[(-1)^{\epsilon}\right]}^{\mathrm{Re}\left[e^{i\pi\epsilon}\right] = 1 - \frac{\pi^2}{2}\epsilon^2 + O(\epsilon^4)} \left(\frac{4\pi\mu^2}{Q^2}\right)^{\epsilon} \times$$

$$\times \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + 8 + \ldots\right)$$

The sum is therefore

$$\sigma = \left(\sigma_0^{(\epsilon)} + \sigma_R^{(\epsilon)} + \sigma_V^{(\epsilon)}\right)_{\epsilon \to 0} = \sigma_0 \left(1 + \frac{3}{4} C_F \frac{\alpha_s}{\pi} + O(\alpha_s^2)\right)$$

finite!

Simpler expressions in dimensional regularization

$D = 4-2\epsilon$, and $D>4$ ($\Rightarrow \epsilon<0$)

$$\sigma_R^{q\bar{q}g} = \sigma_0^{q\bar{q}} C_F \frac{\alpha_s}{2\pi} H(\epsilon)\left[\frac{2}{\epsilon^2}+\frac{3}{\epsilon}+\frac{19}{2}+\mathcal{O}(\epsilon)\right]$$

$$\sigma_V^{q\bar{q}} = \sigma_0^{q\bar{q}} C_F \frac{\alpha_s}{2\pi} H(\epsilon)\left[-\frac{2}{\epsilon^2}-\frac{3}{\epsilon}-8+\mathcal{O}(\epsilon)\right]$$

where $\sigma_0^{q\bar{q}} = N_c \sum e_q^2 \, \sigma_0(e^+e^- \to \mu^+\mu^-)$

and $H(\epsilon) = \dfrac{3(1-\epsilon)^2}{(3-2\epsilon)\Gamma(2-2\epsilon)} = 1+\mathcal{O}(\epsilon)$

We have therefore

$$\boxed{\frac{\lim_{\epsilon\to 0}\left(\sigma_0^{q\bar{q}}+\sigma_V^{q\bar{q}}+\sigma_R^{q\bar{q}g}\right)}{\sigma_0(e^+e^-\to\mu^+\mu^-)} = N_c \sum_{fl} e_q^2\left(1+\frac{\alpha_s}{\pi}+\mathcal{O}(\alpha_s^2)\right)}$$

While real and virtual corrections to the total x-section are separately divergent, their sum is <u>finite</u>

BN and KLN

This cancellation of divergencies in the total $e^+e^- \to q\bar{q}\,(+g)$ cross section is neither an accident nor a surprise.

Two theorems guarantee that this is the case to <u>all orders</u>

<u>Bloch-Nordsieck (1937):</u>
For abelian theories with massive fermions (i.e. like QED)
"Infrared singularities cancel when summing over all unobserved photons in the final state"

<u>Kinoshita, Lee + Nauenberg (1962 and 1964):</u>
"Infrared and collinear divergencies cancel when summing over all degenerate initial and final state"

Perturbative calculability in higher orders

Diverging real and virtual corrections show that calculating higher order corrections is delicate → we need the divergencies to cancel in perturbatively calculable quantities.

However, BN and KLN only apply to <u>fully inclusive</u> cross sections.

What do we do when we <u>want</u> <u>to calculate non inclusive quantities</u> (e.g. production of a Higgs with transverse momentum, recoiling against hard QCD radiation) or we <u>simply cannot sum over everything</u> (for instance, the initial state of a pp collision has specific hadrons, not "everything hadronic") ?

QCD calculations employ two strategies to ensure perturbative calculability (= finiteness)

① Factorisation

Absorb initial state collinear singularities into phenomenological (not calculable, but measurable) parton distribution functions in hadrons.

Factorisation 'theorems' ensure universality of PDFs, and hence their transportability from one ~~on~~ process to another

② Infrared and Collinear safety

KLN (full inclusivity) is a sufficient condition for finiteness, not a necessary one. One finds that less inclusive observables can still be finite in pQCD if properly defined.

Such conditions, that ensure calculability in pQCD, are called IRC safety

It is sufficient to be inclusive over the region where the cancellation takes place, provided the observable is properly behaved (next page)

IRC safety

Write the x-sect for an observable O calculated with up to n particles in the final state as

$$O = \frac{1}{\text{flux}} \sum_n \int \overline{|M_n|^2} \, d\phi_n \, S_n(\{p_i\})$$

If $S_n = 1 \Rightarrow O = \sigma_{tot}$ total x-sect

If $S_n = \delta(X - \chi_n(\{p_i\})) \Rightarrow O = \frac{d\sigma}{dX}$ diff. x-sect

When calculating higher order corrections to O, we need real corr. in the soft/collinear limit and virtual corrections to cancel, for finiteness of the calculation.

For the total x-sect, this happens (BN, KLN):

$$O \sim \int \overline{|M_{n+1}|^2} \, d\phi_{n+1} + \int \overline{|M_n|^2} \, d\phi_n \quad \text{finite}$$

This cancellation takes place in the soft/collinear region.

Whatever I do with the calculation, inserting now a $\chi_n(\{p_i\})$ function in place of the $S_n = 1$ for total x-sect, I must avoid spoiling it

$$\boxed{\text{I need } \chi_{n+1} \xrightarrow{\text{soft/coll}} \chi_n}$$

because in the total x-sect, case $S_{n+1} = S_n = 1$

In practice, this means that a soft emission or a collinear splitting must not change the observable:

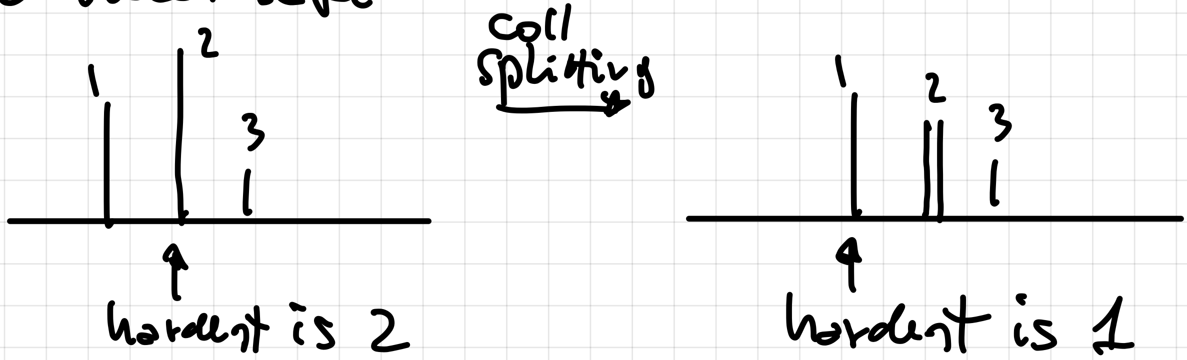
$$\begin{cases} \chi_{n+1}(p_1, p_2, \dots, \lambda p_n, (1-\lambda) p_{n+1}) = \chi_n(p_1, \dots p_n) \\ \chi_{n+1}(p_1, p_2, \dots p_n, 0) = \chi_n(p_1, \dots p_n) \end{cases}$$

Equivalently we can say that an IRC safe quantity must be invariant under the branching $\vec{p}_i \to \vec{p}_j + \vec{p}_k$ whenever $\vec{p}_j \parallel \vec{p}_k$ or either of them is soft

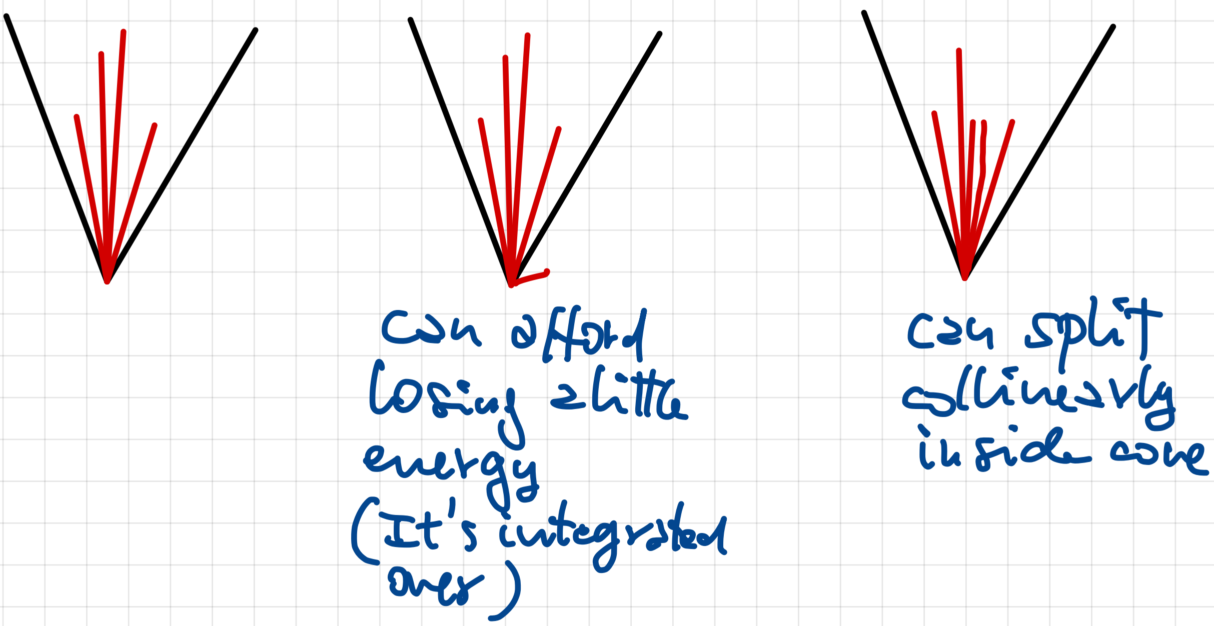
Examples

1) The multiplicity of gluons is not IRC safe (a soft emission or a collinear splitting change the number)

2) "Hardest particle" in an event is not collinear safe

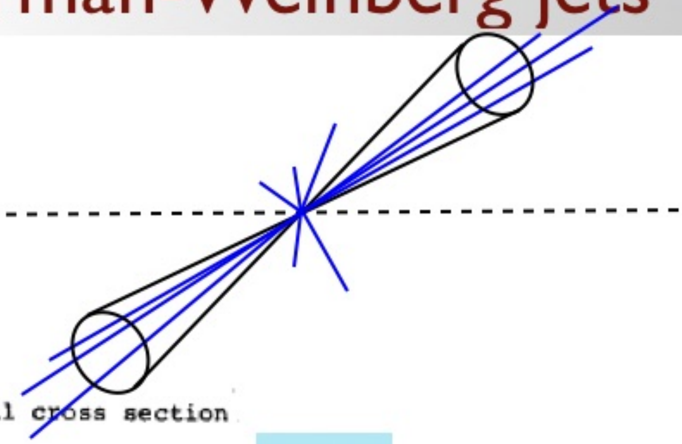


3) At least a certain amount of energy flowing into a cone with finite aperture is IRC safe



Sterman-Weinberg jets

The first rigorous definition of an **infrared and collinear safe** jet in QCD is due to Sterman and Weinberg, Phys. Rev. Lett. **39**, 1436 (1977):



To study jets, we consider the partial cross section $\sigma(E,\theta,\Omega,\epsilon,\delta)$ for e^+e^- hadron production events, in which all but a fraction $\epsilon \ll 1$ of the total e^+e^- energy E is emitted within some pair of oppositely directed cones of half-angle $\delta \ll 1$, lying within two fixed cones of solid angle Ω (with $\pi\delta^2 \ll \Omega \ll 1$) at an angle θ to the e^+e^- beam line. We expect this to be measur-

$$\sigma(E,\theta,\Omega,\epsilon,\delta) = (d\sigma/d\Omega)_0\,\Omega\left[1 - (g_E^2/3\pi^2)\left\{3\ln\delta + 4\ln\delta\,\ln 2\epsilon + \frac{\pi^3}{3} - \frac{5}{2}\right\}\right]$$

Calculable in pQCD (here is the result) but notice the soft and collinear large logs