Gravitational waves from the early Universe

Motivation: pre-CMB messengers!

Outline: I) GWs in vacuum – What is a GW?

II) The stochastic GW background

III) Cosmological sources / Probing BSM models

Literature:
- Maggiore, GWs, Vol I
- Maggiore, gr-qc/9909001
- Caprini & Figueroa, 1801.04268

1) Wave equation

Einstein's equations

$$G_{\mu\nu} \equiv \underbrace{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R}_{\text{curved spacetime } g_{\mu\nu}} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

curved spacetime $g_{\mu\nu}$ $\Longleftrightarrow$ energy momentum tensor of matter content

consider small departure from flat spacetime $\rightarrow G_{\mu\nu} = ?$

$$g_{\mu\nu} = \underset{(-+++)}{\eta_{\mu\nu}} + h_{\mu\nu}(x), \quad |h_{\mu\nu}| \ll 1$$

convention: raise/lower indices with $\eta_{\mu\nu}$

unblind!

Christoffel symbol

$$\Rightarrow \Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} [\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}]$$ definition

$$= \frac{1}{2} \eta^{\rho\sigma} [\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}] + \mathcal{O}(h^2)$$ linearization

Riemann curvature tensor

$$\Rightarrow R^{\mu}{}_{\nu\alpha\beta} = \partial_\alpha \Gamma^\mu_{\nu\beta} - \partial_\beta \Gamma^\mu_{\nu\alpha} + \underbrace{\Gamma^\mu_{\alpha\sigma}\Gamma^\sigma_{\nu\beta}}_{\mathcal{O}(h^2)} - \underbrace{\Gamma^\mu_{\beta\sigma}\Gamma^\sigma_{\nu\alpha}}_{\mathcal{O}(h^2)}$$

$$= \frac{1}{2}\left[\eta^{\mu\sigma}\left(\partial_\alpha\partial_\nu h_{\beta\sigma} - \underline{\partial_\alpha\partial_\beta h_{\nu\beta}} - \partial_\alpha\partial_\sigma h_{\nu\beta}\right) - (\alpha \leftrightarrow \beta)\right] + \mathcal{O}(h^2)$$

symmetric in $\alpha \leftrightarrow \beta$ $\rightarrow$ cancels

$\rightarrow$ 4 terms

$$\Rightarrow R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} \text{ Ricci tensor}, \quad R = g^{\mu\nu} R_{\mu\nu} \text{ Ricci scalar}$$

$$\ldots \Rightarrow \boxed{\begin{aligned} G_{\mu\nu} &= -\frac{1}{2}\left[\overset{\partial_\alpha\partial^\alpha}{\Box} \bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial^\rho\partial^\sigma \bar{h}_{\rho\sigma} - \partial^\rho\partial_\nu \bar{h}_{\mu\rho} - \partial^\rho\partial_\mu \bar{h}_{\nu\rho}\right] + \mathcal{O}(h^2) \quad (*) \\ &\text{with } \bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} \underset{h^\alpha{}_\alpha}{h} \quad \text{for compact notation} \end{aligned}}$$

Quiz symmetries:
GR: invariance under all coordinate trafos
linearized GR: invariance under infinitesimal coordinate trafos
+ finite global Poincaré trafos (translations + Lorentz trafos)

SVT decomposition and gauge fixing

symmetries:

general coordinate transformations $\xrightarrow{①}$ background $\to$ SO(3) global spatial rotations $\xrightarrow{②}$ Fourier trafo, direction $\vec{k}$ singled out $\to$ SO(2) helicity eigenstates

time-slicing to define $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

$X_{\vec{k}} \xrightarrow[\vec{k},\theta]{} e^{im\theta} X_{\vec{k}}$

$m=0 \to$ scalar
$m=\pm 1 \to$ vector
$m=\pm 2 \to$ tensor

e.g. for $g_{\mu\nu}$

$$①: \quad g_{\mu\nu} = \left(\begin{array}{c|c} g_{00} & g_{0i} \\ \hline g_{i0} & g_{ij} \end{array}\right)$$

Quiz: 10 dofs (symm. 4×4)

$$②: \quad g_{00} = -(1+2\Phi) \qquad \to 1 \text{ scalar } \Phi$$

Minkowski: $a=1$

$$g_{i0} = g_{0i} = 2a(\partial_i B - S_i), \quad \partial_i S^i = 0 \qquad \to 1 \text{ scalar } B,\ 1 \text{ vector } S_i$$

$$g_{ij} = a^2\left[(1-2\Psi)\delta_{ij} + 2\partial_{ij}F + (\partial_i T_j + \partial_j T_i) + t_{ij}\right], \quad \partial_i T^i = 0, \quad \begin{cases} \partial_i t_{ij} = 0 \\ t^i{}_i = 0 \end{cases}$$

$\hookrightarrow$ transverse traceless (TT)

$\to$ 2 scalars Ψ, F, 1 vector T_i, 1 tensor h_{ij}

$\Rightarrow$ 4 scalars (Φ, B, Ψ, F), 2 vectors (S_i, T_i), 1 tensor (t_{ij}).

decouple at linear order.

gauge invariance

$h_{\mu\nu} \to h_{\mu\nu} - \underbrace{(\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu)}_{\hat{=} h_{\mu\nu}}$ such that $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ still holds

consider infinitesimal coordinate transformation

$$x^\mu \to x^\mu + \varepsilon^\mu, \quad \varepsilon^\mu = (\varepsilon^0, \partial_i e + f_i), \quad \partial_i f^i = 0$$

$\to$ 2 scalars, 1 vector can be gauged away

$\to$ t_{ij} is gauge invariant.

Lorentz gauge $\partial_\mu \bar{h}^{\mu\nu} = 0$ (gauge fixing of scalars)

$$(*) \to \boxed{\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}}$$

linearized Einstein equation, GW equation

-4: 6 dofs

describes sourcing & propagation of GWs

In vacuum, $T_{\mu\nu} = 0$ $\rightarrow$ residual gauge freedom $\Box \varepsilon^\mu = 0$

$\Rightarrow$ transverse traceless gauge: $\bar{h} = 0$, $h_{0i} = 0$ $\rightarrow h_{\mu\nu} = \bar{h}_{\mu\nu}$

$$\Rightarrow \boxed{h^{TT}_{0\mu} = 0 , \quad h^{TT\,i}{}_{i} = 0 , \quad \partial^j h^{TT}_{ij} = 0} \quad (TT)$$

$\hookrightarrow h^{TT}_{ij} = t_{ij}$

$-4:$ 2 dofs

// Break //

Quiz: properties of GWs in vacuum: transverse, $v=c$, oscillates

2) plane wave solutions . $T_{\mu\nu} = 0$

ansatz: $h_{\mu\nu}(x) = A_{\mu\nu}(k) \sin(k^\alpha x_\alpha)$ dropped phase

$$\rightarrow \Box h_{\mu\nu} = \underbrace{k^\alpha k_\alpha}_{E^2 - p^2 = 0} A_{\mu\nu}(k) \sin(k^\alpha x_\alpha) \stackrel{!}{=} 0$$

$\rightarrow$ GWs travel at speed of light

TT gauge, $\hat{e}_z \parallel \vec{k}$:

$h_+ \equiv A_{xx} = -A_{yy}$ traceless

$h_\times = A_{xy} = A_{yx}$ symmetric

$\rightarrow$ 2 independent components

$$A_{\mu\nu}(k) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ wave, $v = c$, 2 dofs, transverse

effect of GW on test masses x^μ, $x^\mu + \xi^\mu$:

unblind

geodesic equation:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}(x) \frac{dx^\nu}{d\tau} \frac{dx^\nu}{d\tau} = 0 ,$$

same for $x^\mu + \xi^\mu$
τ = proper time along geodesic

$\rightarrow$ geodesic deviation equation to $\mathcal{O}(\xi)$:

$$\frac{d^2 \xi^\mu}{d\tau^2} + 2\Gamma^\mu_{\nu\rho}(x) \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} + \xi^\sigma \partial_\sigma \Gamma^\mu_{\nu\rho}(x) \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0$$

non-relativistic dectector ($\frac{dx_i}{d\tau} \ll \frac{dx_0}{d\tau}$), locally flat metric $\Gamma^\mu_{\nu\rho}(x^\mu) = 0$ at x^μ

$$\rightarrow \ddot{\xi}^i = -c^2 \underbrace{R^i{}_{0j0}}_{\hookrightarrow -\frac{1}{2c^2} \ddot{h}^{TT}_{ij}} \xi^j$$

(gauge invariant $\rightarrow$ compute in TT gauge)

$$\Rightarrow \boxed{\ddot{\xi}^i = \frac{1}{2} \ddot{h}^{TT}_{ij} \xi^j} \quad (**)$$

e.g. $\oplus$ polarization

$$h^{TT}_{ab} = h_+ \sin(\omega t) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad a,b = x,y$$

$$\xi_a(t) = (x_0 + \delta x(t),\; y_0 + \delta y(t))$$

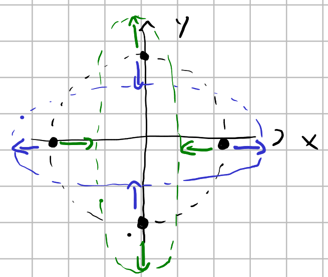
$$(**) \rightarrow \delta\ddot{x} = -\frac{h_+}{2}(x_0 + \delta x)\,\omega^2 \sin(\omega t)$$

$$\simeq -\frac{h_+}{2} x_0 \omega^2 \sin(\omega t)$$

$$\rightarrow \delta x(t) = \frac{h_+}{2} x_0 \sin(\omega t)$$

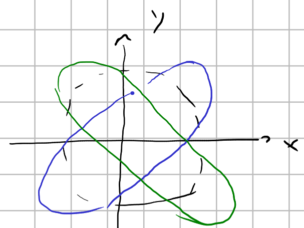
$$\delta\ddot{y} = \frac{h_+}{2} y_0 \omega^2 \sin(\omega t)$$

$$\rightarrow \delta y(t) = -\frac{h_+}{2} y_0 \sin(\omega t)$$

$\Rightarrow$ 

$\sin(\omega t) > 0$
$\sin(\omega t) < 0$

$\rightarrow$ "+" polarization.

analogous for $\otimes$ polarisation: 

$\Rightarrow$
- 2 dofs $\hat{=}$ 2 linear independent solutions, $\oplus$ and $\otimes$
- propagating transverse waves, deforming 2D plane
- spin 2 = tensor objects
- inherent prediction of GR

note:

- "area" (scalar) doesn't change
- chiral basis: $h_{L,R} = \frac{1}{\sqrt{2}}(h_+ \pm i h_\times)$