II The stochastic GW background (SGWB)

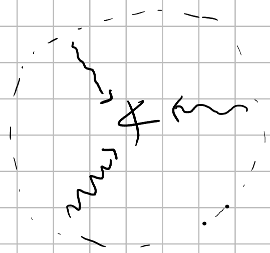
Recap: • linearized Einstein equation:

$$\Box h_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

→ 2 polarizations, spin 2, propagating at speed of light

TT-gauge: $h^{TT}_{0\mu} = 0$, $h^{TT\,i}_{\;i} = 0$, $\partial_i h^{TT}_{ij} = 0$

0) SGWB



- superposition of GWs with different $\vec{k}$ (frequency, direction), similar to CMB
- astrophysical and cosmological contributions
- typically: isotropic, unpolarized, gaussian
- spectral shape $\Omega_{gw} = \frac{1}{\rho_c} \frac{d\rho_{GW}}{d\ln k}$
- acts as additional "noise" in a GW detector

→ cosmological history book: a lot of information, tricky to decipher

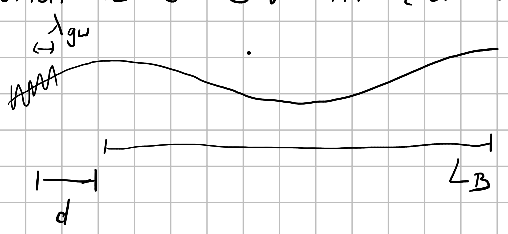
→ mention NanoGrav

Quiz: does the energy carried by GWs curve spacetime?
No / @ $\mathcal{O}(h)$ / @ $\mathcal{O}(h^2)$

1) Energy momentum tensor of GWs

what is a GW in curved space time?

space & time dependent background



seperation of scales:

$$\lambda_{gw} \ll d \ll L_B$$

(or $f_{gw} \gg f_B$ in time domain)

Expand $G_{\mu\nu}$ in powers of h

$$G_{\mu\nu} = \underbrace{G_{\mu\nu}^{(B)}}_{\mathcal{O}(h^0)} + \underbrace{G_{\mu\nu}^{(1)}}_{\mathcal{O}(h^1)} + \underbrace{G_{\mu\nu}^{(2)}}_{\mathcal{O}(h^2)} + \ldots$$

small k (L_B) — large k (λ_{gw}) — small & large k-modes: $|\vec{k}_1 + \vec{k}_2| \simeq 0$ for $\vec{k}_1 = -\vec{k}_2$ with $|k_i|$ large

focus on small-k part of Einstein equation

[high-k part: GW propagation on curved background]

$$G_{\mu\nu}^{B} = -\left[G_{\mu\nu}^{(2)}\right]^{\text{small-}\vec{k}} + \frac{8\pi G}{c^4}\left[T_{\mu\nu}\right]^{\text{small-}\vec{k}}$$

$$= \underbrace{-\langle G_{\mu\nu}^{(2)}\rangle_d}_{\equiv \frac{8\pi G}{c^4} t_{\mu\nu}} + \frac{8\pi G}{c^4}\langle T_{\mu\nu}\rangle_d$$

$\leftarrow$ GWs carry energy which curves background

$\rightarrow$ energy momentum tensor of GWs

explicit computation of $G_{\mu\nu}$ to 2nd order in TT gauge:

$$R_{\mu\nu}^{(2)} = \ldots = \frac{1}{4}\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta} + 12 \text{ Terms}$$

$\downarrow$ TT gauge, integration by parts [s. Maggiore]

$$\langle R_{\mu\nu}^{(2)}\rangle_d = -\frac{1}{4}\langle \partial_\mu h_{\alpha\beta}^{TT}\partial_\nu h^{TT\alpha\beta}\rangle, \quad \langle R^{(2)}\rangle = 0, \quad \langle R^{(1)}\rangle = 0$$

$$\Rightarrow \quad t_{\mu\nu} = \frac{c^4}{32\pi G}\langle \partial_\mu h_{\alpha\beta}^{TT}\partial_\nu h^{\alpha\beta,TT}\rangle$$

$$\Rightarrow \quad \boxed{\rho_{gw}(t,\vec{x}) = t_{00} = \frac{c^4}{32\pi G}\langle \dot{h}_{ij}^{TT}\dot{h}^{ij,TT}\rangle}$$

energy density of GWs

// Break

2) GWs in expanding FRW universe

$c=1$

$$ds^2 = -dt^2 + a^2(t)\, dx^i dx_i = a^2(\tau)\left(d\tau^2 - dx^2\right)$$

Quiz: how does amplitude of GWs redshift?

$$\rightarrow \Box \bar{h}_{\mu\nu}(\vec{x},\tau) - \underbrace{2\frac{a'}{a}\bar{h}'_{\mu\nu}(\vec{x},\tau)}_{=0 \text{ in static universe}} = -16\pi G\, T_{\mu\nu}$$

$$\xrightarrow{FT} \quad \tilde{h}''_\lambda(\vec{k},\tau) + \left(k^2 + \frac{a''}{a}\right)\tilde{h}_\lambda(\vec{k},\tau) = 16\pi G a\, \underbrace{T_\lambda(\vec{k},\tau)}_{\Lambda_{\lambda,\mu\nu}T^{\mu\nu}}$$

$\tilde{h} \equiv a h$

$\lambda = +, \times$

$\frac{\dot{a}}{a}$

- $k \gg aH$ (sub-horizon)

$\hookrightarrow \tilde{h}''_\lambda + k^2 h_\lambda = 0$

$\hookrightarrow h_\lambda = \frac{A_\lambda}{a}\cos(k\tau + \varphi)$ $\Rightarrow$ wave, decays as $1/a$

- $k \ll aH$ (super-horizon)

$2a'h'_\lambda + a h''_\lambda = 0$

$$\hookrightarrow h_\lambda = A_\lambda + \underbrace{B_\lambda \int_0^\tau \frac{d\tau'}{a^2(\tau')}}_{\text{decays in expanding universe}} \simeq \text{const}$$

$\Rightarrow$ GW are "frozen" outside Hubble horizon

$\rightarrow$ a useful parametrization:

$$h_{ij}^{TT}(\vec{x},\tau) = \sum_{\lambda=\times,+}\int d^3k\, \underbrace{h_\lambda(\vec{k})}_{\substack{\text{Fourier coefficient}\\ \text{at } \tau=\tau_*}}\, \underbrace{T_k(\tau)}_{\substack{\text{transfer function}\\ \simeq \frac{a(\tau_*)}{a(\tau)}}}\, \underbrace{e^\lambda_{ij}(\hat{k})}_{\text{polarization tensor}}\, e^{-i(k\tau - \vec{k}\vec{x})} + h.c.$$

τ_*: formation or horizon entry

- $\rho_{gw}(T) = \frac{1}{32\pi G} \langle \dot{h}_{ij}(x,\tau)\, \dot{h}^{ij}(x,\tau) \rangle$

- homogeneity & isotropy:

$$\langle h_\lambda(\vec{k})\, h_{\lambda'}(\vec{k}') \rangle = (2\pi)^3 \delta_{\lambda\lambda'}\, \underbrace{\delta(\vec{k}-\vec{k}')}_{\text{homogeneity}}\, P_\lambda(\underbrace{|\vec{k}|}_{\text{isotropy}})$$

$$\rightarrow \boxed{\rho_{gw}(\tau_0) = \frac{1}{32\pi G} \frac{1}{\pi^2 a^2(\tau_0)} \int k^2 dk\, \underbrace{\sum_\lambda P_\lambda(k)}_{\text{primordial power spectrum}}\, \underbrace{\frac{a^2(\tau_*)}{a^2(\tau_0)}}_{\text{cosmological history}}}$$

$$= \rho_c \int d\ln k\, \underbrace{\frac{1}{\rho_c} \frac{\partial \rho_{gw}}{\partial \ln k}}_{\equiv\, \Omega_{gw}(k,\tau_0)}$$

GW spectrum

↳ ~ GW 2-point function
→ (cross) correlation in GW detector

e.g. single field slow roll inflation:

$$P_\lambda(k) = \left(\frac{2}{M_P}\right)^2 \left(\frac{H_{inf}^2}{2k^3}\right), \quad \left(\Delta_t^2 = \frac{k^3}{2\pi^2} \sum_\lambda P_\lambda \simeq \text{const.}\right)$$

Quiz: what does $\Omega_{gw}(f)$ look like?

- flat
- broken power law
- peak at k_{CMB}
- peak at H_{inf}

$$\rightarrow \Omega_{gw}^0 = \frac{\Delta_t^2}{12} \frac{k^2}{a_0^2 H_0^2} \left(\frac{a(\tau_{*,k})}{a(\tau_0)}\right)^2 \leftarrow \quad a_* H_* = k \quad \text{horizon entry}$$

$$= a(\tau_*) H(\tau_*)$$

$$= \underbrace{\frac{\Delta t^2}{12}}_{\text{const.}} \underbrace{\left(\frac{k}{a_* H_*}\right)^2}_{=1} \underbrace{\frac{a_*^4 H_*^2}{a_0^4 H_0^2}}_{\text{evaluate in } \Lambda\text{CDM}}$$