ICTP SUMMER SCHOOL EXERCISE SESSION II ON HYPERKHLER MANIFOLDS

YAJNASENI DUTTA

Abstract. The following problem set was made to supplement lectures of Prof. Elham Izadi delivered at Trieste Algebraic Geometry Summer School (TAGSS) 2021 - *Hyperkähler and Prym varieties: classical and new results*. The exercises are based on a couple of fundamental results [Mat99] and [Mat05] in this area. Given a Lagrangian fibration of Hyperkähler manifold $f\colon X \to B$ the geometry and topology of B is heavily determined by X. In fact, Matsushita conjectured that $B \simeq \mathbb{P}^n$. It is known by works of Hwang [Hwa08] that if B is smooth then $B \simeq \mathbb{P}^n$. The conjecture is known to be true if $\dim B = 2$ by recent results of [BK18, HX20, Ou19]

3. Lagrangian fibrations

Definition 3.1. A compact Kähler manifold X is called irreducible symplectic, or hyperkähler if X is simply connected and $H^0(X, \Omega^2_X)$ is spanned by an everywhere non-degenerate 2-from σ.

These are higher-dimensional analogues of K3 surfaces. Let S be a K3 surface and $f\colon S \to C$ a proper surjective morphism on to a smooth irreducible curve with connected fibre [1].

Problem 3.2. *Show that $C \simeq \mathbb{P}^1$.* ***Hint.*** *Use that S is simply connected.*

Problem 3.3. *Show that the general fibres of f are elliptic curves.* ***Hint.*** *Use Adjunction.*

Problem 3.4. *Find an explicit fibration of the Fermat quartic $(x^4 + y^4 + z^4 + w^4 = 0) \subset \mathbb{P}^3$.* ***Hint.*** *rewrite as equality of two fractions.*

Let X be a hyperkähler manifold of dimension $2n$. The following exercises show how similar the situation is in the higher dimension. We will need to use the existence of the *Beauville–Bogomolov–Fujiki* quadratic form on $H^2(X, \mathbb{Z})$, which is a replacement for the intersection form on K3 surfaces. We briefly recall the definition and a few key properties that we may require for the exercises. The quadratic space $(H^2(X, \mathbb{R}), q_X)$ controls much of geometry of X and is a central gadget in the study of hyperkähler manifolds.

Theorem 3.5 (BBF form)**.** *There exists a nondegenerate quadratic form $q_X\colon H^2(X, \mathbb{R}) \to \mathbb{R}$ of signature $(3, b_2(X) - 3)$ with $b_2(X) := \dim H^2(X)$ satisfying for any $\alpha \in H^2(X, \mathbb{R})$,*

$$\int_X \alpha^{2n} = d_X \cdot q_X(\alpha)^n.$$

Here d_X is a constant that does not depend on α.

[1] we will call such a morphism *fibration* throughout the rest of the exercises.

As we will see during Prof. Izadi's talk q_X is a-priori dependant on the symplectic form $\sigma \in H^0(X, \Omega_X^2)$, however, up-to scaling it is independent of σ. Here are some key properties of q_X (we denote the associated bilinear form again by q_X).

- The scaled symplectic class σ satisfies $q_X(\sigma) = 0$ and $q_X(\sigma + \overline{\sigma}) = 1$.
- More generally, for $\alpha_i \in H^2(X)$ we have
$$\int_X \alpha_1 \cdots \alpha_{2n} = c_X \sum_{s \in S_n} q_X(\alpha_{s(1)}, \alpha_{s(2)}) \dots q_X(\alpha_{s(2n-1)}, \alpha_{s(2n-2)})$$
for some constant c_X depending only on X. As a consequence, we obtain $\int_X \sigma\overline{\sigma}\omega^{2n-2} = c' q_X(\omega)^{n-1}$.
- If a line bundle L is ample, $q_X(c_1(L)) > 0$. The Kähler cone is contained a connected component of $\{\alpha \in H^{1,1}(X, \mathbb{R}) | q_X(\alpha) > 0\}$. Partial converses to these statements exist. Although we won't need them for the exercises it might be of interest to some. If L is a line bundle with $q_X(L) > 0$ then X is projective [GHJ03, Prop. 26.13]. Furhtermore, if $q_X(\alpha) = 0$ and for every rational curves $C \subset X$ $\int_C \alpha > 0$, then α is a Kähler class [Bou01, Théorème 1.2].
- $H^{1,1}(X, \mathbb{C})$ is orthogonal to $H^{2,0}(X, \mathbb{C}) \oplus H^{0,2}(X, \mathbb{C})$ with respect to q_X.
- By [Bog96, Ver96] whenever there exists $0 \neq \beta \in H^2(X, \mathbb{C})$ that satisfies $q_X(\beta) = 0$, we have $\beta^n \neq 0$ and $\beta^{n+1} = 0$

We begin with a Hodge index-type theoerem.

Problem 3.6. *Given a divisor E on X, show that if E satisfies $E^{2n} = 0$ and $E \cdot A^{2n-1} = 0$ for some ample bundle A, then $E \sim 0$.* ***Hint.*** *Use $q_X(tE + A) = t^2 q(E) + 2tq(E, A) + q(A)$ and Theorem 3.5.*

Problem 3.7. *Given a divisor E on X, show that if E satisfies $E^{2n} = 0$ and $E \cdot A^{2n-1} > 0$ for some ample line bundle A, then $q_X(E, A) > 0$ and the following are true*
$$E^m \cdot A^{2n-m} = 0 \text{ ; for } m > n$$
$$E^m \cdot A^{2n-m} > 0 \text{ ; for } m \leq n$$
Hint. *Use the same polynomial as before.*

Problem 3.8. *Let $f : X \to B$ be a fibration of a hyperkähler manifold X[2]. Using the previous exercise show that $\dim B = n$.* ***Hint.*** *apply previous exercise on the pull-back of an ample class H on B.*

Problem 3.9. *Show that $\mathrm{Pic}(B)$ is of rank 1.* ***Hint.*** *Show for any divisor E in X with $E^{2n} = 0$ and $E^n \cdot (f^*H)^n = 0$ that $E \sim_{\mathbb{Q}} f^*H$.*

For the next exercise we need the definition of a Lagrangian (possibly singular) subvariety. Recall that

[2] you may assume both X and B are projective, although the results presented here work in a more general setting.

Definition 3.10. A subvariety $Y \subset X$ is said to be a Lagrangian subvariety if $\dim Y = \frac{1}{2}\dim X$ and there exists a resolution of singularities $\mu\colon Y' \to Y$ such that $\mu^*\sigma|_Y = 0$.

Problem 3.11. *Show that a general fibre of f is Lagrangian. By a classical theorem the general fibre of f is then complex tori. A more recent result of Voisin* [Cam06, Prop. 2.1] *or more generally* [Leh16, Theorem 1.1] *show that even if X is not projective, a Lagrangian subvariety of a hyperkähler manifold is always projective. Thus, a general fibre F is isomorphic to an abelian variety.* ***Hint.*** *Let A be an ample class on X. Argue that if $\sigma|_F \neq 0$, then $\int_F(\sigma\overline{\sigma}|_F)\cdot A|_F^{n-2} \neq 0$. Using the polynomial $q_X(tf^*H + A)^{n-1}$ and the second property listed above show that it is in fact 0.*

Problem 3.12. *Show that every fibre of f is Lagrangian and hence f is equi-dimensional.* ***Hint.*** *Use the map $H^2(X, \mathcal{O}_X) \to H^0(B, R^2f_*\mathcal{O}_X)$ induced by the Leray spectral sequence and that $R^2f_*\mathcal{O}_X$ is torsion free (a result of Kollár).*

Problem 3.13. *Show that B is $\mathbb{Q}$-factorial with at worst kawamata log terminal singularity.* ***Hint.*** *Use that f is equidimensional and* [KM08, Lemma 5.16] *which states that if the source of a finite surjective map between normal varieties is $\mathbb{Q}$-factorial and klt then so is the target.*

For the next exercise, recall and use the following

Definition 3.14 (Kodaira Dimension)**.** Let X be a $\mathbb{Q}$-factorial variety. Then

$$\kappa(X) = \sup_m \dim \overline{\phi_m(X)}$$

where $\phi_m : X \dashrightarrow \mathbb{P}^{P_m}$ is the rational map defined by the global sections of $\omega_X^{\otimes m}$ and $P_m = |\Gamma(X, \omega_X^{\otimes m})|$. Another way to interpret this is

$$\kappa(X) \coloneqq \operatorname{trdeg}_k(\bigoplus_m H^0(X, \omega_X^{\otimes m})) - 1$$

where the algebra structure on the right side is given by the multiplication map.

Iitaka's $C_{n,m}$ conjecture then states that

Conjecture 3.15. *Let $f : X \to B$ be a fibration of smooth projective varieties of dimension n and m,respectively, and let F be a general fibre of Then,*

$$\kappa(X) \geq \kappa(F) + \kappa(B)$$

The conjecture is known when the F is a minimal variety by a result of Kawamata [Kaw85, Theorem 1.1(2)].

Problem 3.16. *Assume B is smooth, show that B is Fano, i.e. the inverse of the canonical bundle of B is ample.* [3] ***Hint.*** *use that the picard rank of B is 1 and Kawamata's result above.*

[3] This is as well expected to be true when B is singular.

Problem 3.17. *Assume B is smooth. Let B^0 be the open set where f is smooth. Let $X^0 \coloneqq f^{-1}(B^0)$. Show that $R^i f^0_* \mathcal{O}_{X^0} = \Omega^i_{B^0}$.*

(Matsushita extends this equality over to the big open set U which includes the smooth points of the discriminant divisor D_f using Deligne canonical extension. Then using the reflexivity of $R^i f_* \mathcal{O}_X$ and the isomorphism $R^n f_* \mathcal{O}_X \simeq \omega_B$, he shows that $R^i f_* \mathcal{O}_X \simeq \Omega^i_B$.)

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Mathematik Zentrum, Universität Bonn, Endenicher Allee 60, Germany.
E-mail address: ydutta@uni-bonn.de
URL: http://www.math.uni-bonn.de/people/ydutta/