MINI-COURSE PRYM VARIETIES

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1. Exercises

(1) Let $A = V/\Lambda$ be a complex torus. Choose $\{e_i\}$ basis of V and $\lambda_1, \dots, \lambda_{2g}$ of Λ and Π the corresponding period matrix, i.e. w.r.t. these basis $A = \mathbb{C}^g/\Pi\mathbb{Z}^{2g}$. Show that A is an abelian variety if and only if there is a non degenerated alternating form $Z \in M_{2g}(\mathbb{Z})$ such that (i) $\Pi Z^{-1 t}\Pi = 0$ and (ii) $i\Pi Z^{-1 t}\overline{\Pi} > 0$. The conditions (i) and (ii) are called Riemann Relations. (If E is the alternating form on Λ defining the polarization, Z is the matrix of E on the basis $\{\lambda_i\}$). ([4, Thm. 4.2.1])

(2) If $0 \to X \to Y \to Z \to 0$ is an exact sequence of abelian varieties with Y principally polarized then $(\hat{Z}, X)$ is a pair of complementary abelian subvarieties of Y. ([4, Prop. 12.1.3])

(3) Show the universal property of the Jacobian. ([4, Thm. 11.4.1])

(4) Let $f : \widetilde{C} \to C$ be an étale double covering and σ the corresponding involution on $\widetilde{C}$. Show that $\operatorname{Ker}\operatorname{Nm}_f : J\widetilde{C} \to JC$ consists of 2 irreducible components that can be described as $P^0 := (1-\sigma)\operatorname{Pic}^0(\widetilde{C})$ and $P^1 := (1-\sigma)\operatorname{Pic}^1(\widetilde{C})$.

(5) Let $f : \widetilde{C} \to C$ a finite covering between smooth curves. Show that the pullback map $f^* : JC \to J\widetilde{C}$ is not injective if and only if f factorizes via a cyclic étale covering $\tilde{f}$ of degree ≥ 2. ([4, Prop. 11.4.3])

(6) Using Riemann Singularity theorem, show that $\dim \operatorname{Sing}\Theta = g - 3$ for the theta divisor $\Theta \in JC$ with C a non-hyperelliptic curve.

(7) Show that if $f : \widetilde{C} \to C$ is a double étale covering of hyperelliptic curves $\widetilde{C}, C$ then the Prym variety is isomorphic to a Jacobian of a hyperelliptic curve (find the curve). Show that any Jacobian of a hyperelliptic curve is a Prym variety for some covering. [15]

(8) Show that a genus 3 curve can be embedded in an abelian surface if and only if it is a covering of an elliptic curve. Can you find a quartic plane curve that is a 2021:1 covering of an elliptic curve?

(9) Find all cases of coverings for which the Prym variety is principally polarized. [4]

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