AUTUMN COLLEGE ON PLASMA PHYSICS
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One-Dimensional Spectral Studies in Single Fluid MHD for Stability of Fusion Plasmas

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These are preliminary lecture notes, intended only for distribution to participants.
One-dimensional Spectral Studies in Single Fluid MHD for Stability of Fusion Plasmas
Autumn College on Plasma Physics @ICTP (Oct. 26, 2001)
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0. Preliminary — finite dimensional matrix

1. MHD Equations

2. Displacement Vector and Its Norm

3. Reduced MHD Equations

4. Spectrum for Static Plasmas
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5. Spectral Studies for Shear Flow Plasmas
   (a) Linear Shear Flow Profile and Kelvin’s Method
   (b) Kelvin-Helmholtz Instability with Surface Wave Model
0. Preliminaries

Consider
\[ i\partial_t \psi = A\psi, \]  \hspace{1cm} (1)
where \( A \): \( N \times N \)-matrix, \( \psi(t) \): \( N \)-dimensional vector.

Scalar product
\[ (\phi | \psi) = \phi \cdot \bar{\psi} \]  \hspace{1cm} (2)

Eigenvalues and eigenvectors
\[ A\varphi_j = \lambda_j \varphi_j \]  \hspace{1cm} (3)

If \( A \) is Hermitian (selfadjoint), we can span whole vector space by orthogonal eigenvectors \( (N \text{ eigenvectors}) \)
\[ (\varphi_i | \varphi_j) = 0 \quad \text{for } i \neq j \quad (i, j = 1, \ldots, N) \]  \hspace{1cm} (4)
\[ (\varphi_j | \varphi_j) = 1 \quad (j = 1, \ldots, N) \]  \hspace{1cm} (5)

Projection
\[ \mathcal{P}_j = (\cdot | \varphi_j)\varphi_j \]  \hspace{1cm} (6)
\[ \psi = \sum_{j=1}^{N} (\psi | \varphi_j)\varphi_j \quad \text{for any } \psi \]  \hspace{1cm} (7)

Spectral resolution
\[ A = \sum_{j=1}^{N} \lambda_j \varphi_j(\cdot | \varphi_j) \]  \hspace{1cm} (8)
Evolution equation becomes

\[ i \partial_t \psi(t) = \sum_{j=1}^{N} \lambda_j \varphi_j (\psi(t) | \varphi_j) \]

\[ = \sum_{j=1}^{N} \lambda_j a_j(t) \varphi_j \]  \hspace{1cm} (9)

Taking a scalar product with \( \varphi_i \),

\[ i \partial_t a_i(t) = \lambda_i a_i(t) \]  \hspace{1cm} (10)

\[ \downarrow \]

\[ a_i(t) = a_i(0) \exp(-i\lambda_i t) \]  \hspace{1cm} (11)

Solution

\[ \psi(t) = \sum_{j=1}^{N} a_j(0) \exp(-i\lambda_j t) \varphi_j \]  \hspace{1cm} (12)
1. MHD Equations

\[ \partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (13) \]

\[ \rho(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) = \mathbf{j} \times \mathbf{B} - \nabla p \quad ( + \rho \mathbf{g}), \quad (14) \]

\[ \partial_t p + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = 0, \quad (15) \]

\[ \partial_t \mathbf{B} = -\nabla \times \mathbf{E}, \quad (16) \]

\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{j}, \quad (17) \]

\[ \mathbf{E} + \mathbf{v} \times \mathbf{B} = \mathbf{0}, \quad (18) \]

Comment: We can remove \( j, p, \) and \( \mathbf{E} \) from the system.

\( \Rightarrow \) seven waves
2. Linearized MHD Equation

Linearization

\[ \psi = \psi_0 + \psi_1, \quad (19) \]

where \(|\psi_1| \ll |\psi_0|\).

Equilibrium is written by \(\partial_t = 0\) in all equations.

Displacement vector \(\xi\)

\[ \partial_t \xi(x, t) = v_1(x, t), \quad \xi(x, 0) = 0. \quad (20) \]

Linearized MHD equation

\[ \partial_t^2 \xi = \mathcal{F}\xi \]

\[ = \frac{1}{\rho_0} \left[ \nabla (\gamma p_0 \nabla \cdot \xi + \xi \cdot \nabla p_0) \right. \]

\[ + \frac{1}{\mu_0} (\nabla \times B_0) \times [\nabla \times (\xi \times B_0)] \]

\[ + \frac{1}{\mu_0} [\nabla \times (\nabla \times (\xi \times B_0))] \times B_0 \right]. \quad (21) \]

Hermiticity of force operator \(\mathcal{F}\)

\[ \langle \eta | \mathcal{F} \xi \rangle = \langle \mathcal{F} \eta | \xi \rangle, \quad (22) \]

\[ \left( \langle \eta | \xi \rangle \equiv \frac{1}{2} \int_{\Omega} \rho_0 \eta \cdot \bar{\xi} \, dV. \right) \]

Stability theory

1. Energy principle

2. Spectral analysis
2. Linearized MHD Equation

All eigenvalues of force operator $\mathcal{F}$ are real.

$$\mathcal{F}\xi = \lambda\xi$$  \hspace{1cm} (23)

$$\langle\xi|\mathcal{F}\xi\rangle = \bar{\lambda}\langle\xi|\xi\rangle$$

$$\langle\mathcal{F}\xi|\xi\rangle = \lambda\langle\xi|\xi\rangle$$

$\Rightarrow \lambda = \bar{\lambda}$ : real

Now the eigenvalue is $\lambda = -\omega^2$ with $\exp(-i\omega t)$.

If we have any positive eigenvalue,

$$-\omega^2 = \lambda > 0 \Rightarrow \omega = \pm i\sqrt{\lambda}$$

$\Rightarrow \exp(\mp \sqrt{\lambda}t)$ : unstable

If we have all eigenvalues negative,

$$-\omega^2 = \lambda < 0 \Rightarrow \omega = \pm \sqrt{-\lambda}$$

$\Rightarrow \exp(\pm i\sqrt{-\lambda}t)$ : stable
3. Reduced MHD Equations

Low-$\beta$ tokamak ordering ($\epsilon = \text{minor r./major r.}$)

\[ B_z \sim 1 + \epsilon^2, \quad \nabla_\perp \sim 1, \quad B_\perp \sim \partial_z \sim \epsilon, \]
\[ v_\perp \sim \epsilon, \quad p \sim \epsilon^2, \quad \partial_t \sim v \cdot \nabla_\perp \sim \epsilon, \quad v_z \sim 0. \quad (24) \]

Static fluid $v_0 = 0$. \textit{Incompressible ($\nabla \cdot v = 0$)}

Stream and flux functions

\[ v_1 = \nabla \phi \times e_z, \quad B_1 = \nabla \psi \times e_z. \quad (25) \]

Two-fields RMHD equations (after linearization)

\[ \partial_t \Delta \phi = B_0 \cdot \nabla \Delta \psi + (\nabla j_0 \times e_z) \cdot \nabla \psi, \quad (26) \]
\[ \partial_t \psi = B_0 \cdot \nabla \phi. \quad (27) \]

Boundary conditions: $\phi = \psi = 0$ at the edge.

\(^1\text{H. R. Strauss, Phys. Fluids 19, 134 (1976).}\)
About norm

RMHD equation for homogeneous plasma

\[ \partial_t u = \mathcal{A} u, \]  

\[ \mathcal{A} = \begin{pmatrix} 0 & B \cdot \nabla \Delta \\ B \cdot \nabla \Delta^{-1} & 0 \end{pmatrix}. \]  

State vector \( u = [\Delta \phi, \psi]^T \).

Norm should be taken with the metric

\[ \mathcal{M} = \begin{pmatrix} -\Delta^{-1} & 0 \\ 0 & -\Delta \end{pmatrix}, \]  

as

\[ \langle u | u^\dagger \rangle \equiv \langle \Delta \phi | -\Delta^{-1} | \Delta \phi^\dagger \rangle + \langle \psi | -\Delta | \psi^\dagger \rangle. \]  

where \( \langle \phi | \phi^\dagger \rangle = \int \phi \bar{\phi}^\dagger \, dV \) denotes simple norm.

Physically, this norm corresponds to energy bilinear form

\[ \langle u | u \rangle = \int \Delta \bar{\phi}(-\Delta^{-1}) \Delta \phi + \bar{\psi}(-\Delta) \psi \, dV \]  

\[ = \int |\nabla \phi|^2 + |\nabla \psi|^2 \, dV, \]  

where \( \mathbf{v} = \nabla \phi \times \mathbf{e}_z, \, \mathbf{B} = \nabla \psi \times \mathbf{e}_z \).
About norm

Difficult for state vector $u$ in inhomogeneous case.

Combining two equations

$$
\partial_i^2 \Delta \phi = A_u \Delta \phi
$$

$$
= B \cdot \nabla \Delta B \cdot \nabla \Delta^{-1}(\Delta \phi)
$$

$$
+ (\nabla_j \times \mathbf{e}_z) \cdot \nabla B \cdot \nabla \Delta^{-1}(\Delta \phi) \quad (34)
$$

Define a scalar product as

$$
\langle \Delta \phi | \Delta \phi^\dagger \rangle = (\Delta \phi | - \Delta^{-1} | \Delta \phi^\dagger), \quad (35)
$$

then $A_u$ becomes Hermitian.
4. Spectrum for Static Plasmas \((\mathbf{v}_0 = 0)\)
(a) Continuous spectra in slab geometry

Equilibrium magnetic field

\[ \mathbf{B} = (0, B_y(x), B_z) \]  \hspace{1cm} (36)

Alfvén equation for eigenvalue \(\omega^2\)

\[
\frac{d}{dx} \left[ \left( \omega^2 - \omega_A^2 \right) \frac{d\phi}{dx} \right] - k_y^2 \left( \omega^2 - \omega_A^2 \right) \phi = 0, \tag{37}
\]

where \(\omega_A(x) = k \cdot \mathbf{B}(x) / \sqrt{\mu_0 \rho}\).

Regular singularity appears at \(x = x_s\) when \(\omega = \omega_A(x_s)\).

Solution is (due to Frobenius)

\[
\phi(x) = a_1 g_1(x) + a_2 \left[ g_1(x) \log |x - x_s| + g_2(x) \right]. \tag{38}
\]

where \(g_1(x)\) and \(g_2(x)\) are analytic functions around \(x_s\).

Note: this solution is non-square-integrable under previous norm

\[
\int (\log |x - x_s|) \Delta(\log |x - x_s|) \, dx \\
\sim - \int \frac{1}{(x - x_s)^2} \, dx \tag{39}
\]
There is no other solution in slab Alfvén equation.

\[ \omega^2_A = \inf_{x} \omega^2_A(x) : \text{ lower bound,} \]  \hspace{1cm} (40)

Dividing the singular factor

\[ \omega^2 - \omega^2_A = (\omega^2 - \omega^2_A) + (\omega^2_A - \omega^2), \]  \hspace{1cm} (41)

multiplying \( \tilde{\phi} \), and integrating with respect to \( x \)

\[
\begin{align*}
(\omega^2 - \omega^2_A) \int_{\Omega} \left( \left| \frac{d\phi}{dx} \right|^2 + k_y^2 |\phi|^2 \right) dx \\
= - \int_{\Omega} \omega^2_A \left( \left| \frac{d\phi}{dx} \right|^2 + k_y^2 |\phi|^2 \right) dx. \hspace{1cm} (42)
\end{align*}
\]

\[ \omega^2 \geq \omega^2_A \]  \hspace{1cm} (43)

Spectrum is

\[ \sigma = \sigma_c = \{ \omega^2 | \min_{x \in \Omega} \omega^2_A \leq \omega^2 \leq \max_{x \in \Omega} \omega^2_A \}. \]  \hspace{1cm} (44)
Solving by Laplace transform

\[ \tilde{\phi}(\omega) = \int_0^\infty \phi(t) e^{i\omega t} dt, \quad (45) \]

\[ \phi(t) = \frac{1}{2\pi i} \oint_C \tilde{\phi}(\omega) e^{-i\omega t} d\omega, \quad (46) \]

Branch cuts appear on \( \omega = \omega_A \)

Continuum damping

\[ \phi \propto \frac{1}{t} \exp[i\omega_A(x)t] + \frac{1}{t} \exp[-i\omega_A(x)t] \quad (47) \]

no stationary oscillation
(b) Instability in cylindrical geometry

Equilibrium magnetic field

$$ \mathbf{B} = (0, B_\theta(r), B_z) $$ (48)

Alfvén equation for eigenvalue $\omega^2$

$$ \frac{1}{r} \frac{d}{dr} \left[ r (\omega^2 - \omega_A^2) \frac{d\phi}{dr} \right] - \frac{m^2}{r^2} (\omega^2 - \omega_A^2) \phi + \frac{2 dF}{r dr} F \phi = 0, $$ (49)

Point spectra (kink modes) appear due to extra term.

Boundary conditions for $m \geq 1$ are

$$ \phi = 0 \quad \text{at} \quad r = 0, a $$ (50)
Simple ODEs

\[ u'' - (-1)u = 0, \quad v'' - (-4)v = 0 \quad (51) \]

\((-1 > -4)\)

have solutions

\[ u \sim \sin x, \quad v \sim \sin 2x, \quad (52) \]

\[ \text{slow} \quad \text{rapid} \]

\[ \Downarrow \text{generalize} \]

**Oscillation theorem (Sturm)**

For the ODE in the real domain

\[ \frac{d}{dr} \left[ K_1(r, \omega) \frac{du}{dr} \right] - G_1(r, \omega)u = 0 \quad (53) \]

\[ \frac{d}{dr} \left[ K_2(r, \omega) \frac{dv}{dr} \right] - G_2(r, \omega)v = 0 \quad (54) \]

If \( K_1 > K_2 > 0 \) and \( G_1 > G_2 \) for any \( r \), then \( v \) oscillates more rapidly than \( u \).
some more considerations...

Forget about the boundary condition at \( r = a \! \).

\[
\frac{d}{dr} \left[ r \frac{d}{dr} \left( r \frac{\left( \omega_2^2(x) - \omega^2 \right) \frac{d\phi}{dr} \right) \right] - \frac{m^2}{r} \frac{\left( \omega_2^2(x) - \omega^2 \right) \phi}{r} \frac{2dF}{dr} \frac{F \phi}{r} = 0, \quad K > 0 \tag{55}
\]

Suppose two neighboring solutions

\[
\phi_1 : \text{ solution for } \omega_1^2 \quad \text{with } \phi_1(0) = 0 \\
\phi_1 + \delta \phi : \text{ solution for } \omega_1^2 + \delta \omega^2 \quad \text{with } \phi_1 + \delta \phi(0) = 0
\]

If \( \delta \omega^2 > 0 \), then

\[
K(\omega_1^2) > K(\omega_1^2 + \delta \omega^2), \quad G(\omega_1^2) > G(\omega_1^2 + \delta \omega^2) \tag{56}
\]

\( \Downarrow \)

\( \phi_1 + \delta \phi \) oscillates more rapidly than \( \phi_1 \).

Spectra are

\[
\sigma = \sigma_c \oplus \sigma_p \tag{57}
\]

---

\(^1\)Goedbloed and Sakanaka, Phys. Fluids 17, 908 (1974).
(c) Interchange Instability in Stellarators\textsuperscript{2}

Stellarator ordering

\[ B \sim B_z + \varepsilon \sqrt[4]{\nabla \eta} + \varepsilon \]
\[ \rho \sim \varepsilon \]

helical field

RMHD equations for stellarators

\[ \partial_t \psi = \mathbf{B} \cdot \nabla \phi, \quad (58) \]
\[ \rho \frac{d \Delta \phi}{dt} = -\mathbf{B} \cdot \nabla j_z + \nabla \kappa \times \nabla p \cdot \mathbf{e}_z, \quad (59) \]
\[ \frac{dp}{dt} = 0, \quad (60) \]

where

\[ \mathbf{B} \cdot \nabla = B_0 \partial_z + \nabla \psi \times \mathbf{e}_z \cdot \nabla, \quad (61) \]
\[ \frac{d}{dt} = \partial_t + \nabla \phi \times \mathbf{e}_z \cdot \nabla, \quad (62) \]
\[ \kappa = \frac{2r \cos \theta}{R_0} + \frac{(\nabla \eta)^2}{B_0^2}, \quad (63) \]
\[ j_z = -\Delta A_z, \quad (64) \]
\[ A_z = \psi + \frac{1}{2B_0} \nabla \langle \eta \rangle \times \nabla \eta \cdot \mathbf{e}_z. \quad (65) \]

\textsuperscript{2}Tatsuno, et al., Nucl. Fusion 39, 1391 (1999).

Eigenvalue equation

\[
\frac{d^2 \phi}{dr^2} + \left[ \frac{1}{r} \frac{2m_i'(n - m_i)}{r^2 + (n - m_i)^2} \right] \frac{d\phi}{dr} - \left\{ \frac{m_i^2}{r^2} + \frac{1}{r^2 + (n - m_i)^2} \right\} \times \left[ \left( \frac{m_i'}{r} + m_i'' \right)(n - m_i) - \frac{D_s m_i^2}{r^2} \right] \phi = 0
\]

\(m(n)\): poloidal(toroidal) mode numbers,
\(\iota\): Rotational transform,
\(D_s = -\frac{1}{2} \beta_0 N p'(4r\iota + r^2 \iota')\): Instability drive,
\(\beta_0\): central toroidal beta,
\(N\): toroidal period number of the helical coils.
Numerical solution for the \((m, n) = (2, 1)\) mode.

Equilibrium profiles are
\[
\iota = 0.499 + 0.2r^2 \quad \text{(resonant),}
\]
\[
\iota = 0.501 + 0.2r^2 \quad \text{(non-resonant),}
\]
\[
p = p_0(1 - r^4) \quad \text{(for both)}.
\]

Beta dependence of growth rate

Eigenfunction near the beta limit
5. Spectral Studies for Shear Flow Plasmas
(a) Linear Shear Flow Profile and Kelvin’s Method

Consider linear shear flow profile in a slab plasma

\[ \mathbf{v}_0 = (0, \sigma x, 0) \quad \sigma > \text{const.} \]  \hspace{1cm} (66)

Non-Hermiticity only enters from \( \mathbf{v}_0 \cdot \nabla \) operator

\[ \partial_t u + \mathbf{v}_0 \cdot \nabla u = \mathcal{A}u \]  \hspace{1cm} (67)

\[ \overset{\text{coordinate transform}}{\longrightarrow} \overset{\text{spectral resolution}}{\longrightarrow} \]
\[ \mathcal{A}: \text{Hermitian (selfadjoint) operator} \]

Suppose we have a set of 'shearing modes' satisfying two conditions

- **Characteristic equation**

\[ \partial_t \tilde{\phi}(t; k, x) + \mathbf{v}_0 \cdot \nabla \tilde{\phi}(t; k, x) = 0. \]  \hspace{1cm} (68)

- **Eigenequation**

\[ \mathcal{A}\tilde{\phi}(t; k, x) = \lambda_k(t) \tilde{\phi}(t; k, x). \]  \hspace{1cm} (69)

---

\(^3\)Volponi, Yoshida, & Tatsuno, Phys. Plasmas 7, 2314 (2000).

We can decompose any function as

$$u = \int \hat{u}_k(t) \tilde{\varphi}(t; k, x) \, dk. \quad (70)$$

Plugging this expression into eq. (67),

$$\int [\partial_t \hat{u}_k(t)] \tilde{\varphi}(t; k, x) \, dk = \int \hat{u}_k(t) \lambda_k(t) \tilde{\varphi}(t; k, x) \, dk, \quad (71)$$

we can obtain the following ODE on time for each mode due to the orthogonality of the eigenvectors;

$$\text{d}_t \hat{u}_k(t) = \lambda_k(t) \hat{u}_k(t). \quad (72)$$

This is no longer a simple exponential evolution.

Kelvin’s mode
Transients and Secularities of Kelvin’s modes

\[ \psi \sim e^{i\omega t / t} \]

\[ \phi \sim t^{0.35} \]
(b) Kelvin-Helmholtz Instability with Surface Wave Model

Consider 2-D Euler fluid

\[ \mathbf{v}_0 = (0, v(x)) \]  \hspace{1cm} (73)

Generalized Rayleigh equation

\[
\begin{align*}
    \mathcal{K} f(x) &= -\Delta^{-1} \int_{-\infty}^{+\infty} \frac{e^{\frac{-k|x-\xi|}{2k}}}{2k} f(\xi) \, d\xi, \\
    \mathbf{i} \partial_t \Psi &= kv(x)\Psi + kw(x)\mathcal{K}\Psi.
\end{align*}
\]  \hspace{1cm} (74) \hspace{1cm} (75)

with \( \Psi = -\Delta \phi \): vorticity.

\( w(x) = v''(x) \) gives ordinary Rayleigh equation.
Norm again

In enstrophy norm

\[
\langle \Psi | kv\Psi^\dagger \rangle = \int kv\Psi\Psi^\dagger \, dx
\]

\[= \langle kv\Psi | \Psi^\dagger \rangle, \quad (76)\]

\[
\langle \Psi | kw\Delta^{-1}\Psi^\dagger \rangle = \langle \Delta^{-1}kw\Psi | \Psi^\dagger \rangle
\]

\[\neq \langle kw\Delta^{-1}\Psi | \Psi^\dagger \rangle. \quad (77)\]

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<thead>
<tr>
<th>operator ( kv(x) )</th>
<th>Hermitian</th>
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<tbody>
<tr>
<td>operator ( kw(x)K )</td>
<td>non-Hermitian</td>
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</table>

In energy norm

\[
\langle \Psi | kv\Psi^\dagger \rangle = - \int \Psi \Delta^{-1}(kv\Psi^\dagger) \, dV
\]

\[= - \int (kv\Delta^{-1}\Psi) \Delta\Delta^{-1}\Psi^\dagger \, dx \]

\[= - \int (\Delta kv\Delta^{-1}\Psi) \Delta^{-1}\Psi^\dagger \, dx \]

\[= \langle \Delta kv\Delta^{-1}\Psi | \Psi^\dagger \rangle. \quad (78)\]

\[
\langle \Psi | kw\Delta^{-1}\Psi^\dagger \rangle = - \int \Psi \Delta^{-1}(kw\Delta^{-1}\Psi^\dagger) \, dx
\]

\[= - \int (kw\Delta^{-1}\Psi) \Delta^{-1}\Psi^\dagger \, dx \]

\[= \langle kw\Delta^{-1}\Psi | \Psi^\dagger \rangle, \quad (79)\]

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</table>
Dividing vorticity field as

\[
\Psi = \alpha(t)\delta(x - a) + \tilde{\Psi}(x, t)
\]  

(80)

Evolution equation

\[
\frac{i}{\text{d}t} \alpha(t) = \frac{U}{2a} (2ka - 1)\alpha(t) - \frac{U}{2a} \int_{-\infty}^{\infty} e^{-k|a-\xi|} \tilde{\Psi}(\xi, t) \, d\xi,
\]

(81)

\[
i\partial_t \tilde{\Psi}(x, t) = kv(x) \tilde{\Psi}(x, t).
\]

(82)

Eigenvalue problem

\[
\lambda \varphi(x) = A \varphi(x)
\]

\[
= kv(x) \varphi(x) - \frac{U}{2a} \delta(x - a) \int_{-\infty}^{\infty} e^{-k|x-\xi|} \varphi(\xi) \, d\xi.
\]

(83)

Two kinds of eigenmodes exist

<table>
<thead>
<tr>
<th>point</th>
<th>eigenvalue</th>
<th>eigenfunction</th>
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<tbody>
<tr>
<td></td>
<td>( \lambda_1 = kU - U/2a )</td>
<td>( \varphi_1 = \delta(x - a) )</td>
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<th>eigenfunction</th>
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<tbody>
<tr>
<td></td>
<td>( \lambda_\mu = kU \mu/a )</td>
<td>( \varphi_\mu = \delta(x - \mu) + \frac{e^{-k(a-\mu)}}{2k(a-\mu) - 1} \delta(x - a) )</td>
</tr>
</tbody>
</table>

where \( \mu < a \land \mu \neq a - 1/2k \).
When \( \mu = a - 1/2k \), frequencies overlap \( (kU\mu/a = \lambda_1) \).

Eigenfunction in a wider sense: \( \varphi_2 = \delta(x - \mu_0) \)

\[
(\lambda_1 - \frac{A}{2a})\varphi_2 = \frac{U}{2a}e^{-k(a-\mu_0)}\varphi_1, \quad (84)
\]

\[
(\lambda_1 - \frac{A}{2})\varphi_2 = 0. \quad (85)
\]

Taking basis vector by \( \delta(x - a) \) and \( \delta(x - \mu) \),

\[
A = \begin{pmatrix}
\frac{kU - U}{2a} & \frac{U}{2a}e^{-k(a-\mu)} \\
0 & \frac{kU}{a}
\end{pmatrix}.
\quad (86)
\]

Including \( \mu = a - 1/2k \).

For an initial condition \( \Psi(0) = \varphi_2 \),

\[
\Psi(t) = i\frac{U}{2a}\sqrt{\frac{1}{t}}e^{i\lambda_1\mu_0 t/a} \quad (87)
\]

secularity
Linear Stability Theory
(from the viewpoint of spectral analysis)

to show instability — rather simple
you can show by finding one growing solution

to show stability — rather difficult
you must know all spectral properties
including exponential growth
secular \downarrow\non-Hermiticity