we obtain the final set of β-plane momentum equations for a $1\frac{1}{2}$ layer model

$$\begin{aligned} \frac{\partial u}{\partial t} - \beta y v &= -g' \frac{\partial h}{\partial x} - \varepsilon u + \frac{\tau^x}{\rho_0 H} \quad (1) \\ \frac{\partial v}{\partial t} + \beta y u &= -g' \frac{\partial h}{\partial y} - \varepsilon v + \frac{\tau^y}{\rho_0 H}. \quad (2) \end{aligned}$$

and with three unknowns (u, v, h) we need a third equation which is provided by the linearized mass conservation equation (which also includes on the rhs a rough linear parameterization of entrainment (mixing) at the base of the water layer above the thermocline)

$$\frac{\partial h}{\partial t} + H \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = -\varepsilon h. \quad (3)$$

1.2.3 Equatorial waves

The derivation here follows Gill [20]. Consider first the case of an equatorial Kelvin wave, which is a special solution of (1,2,3) for the case of zero meridional velocity ($v = 0$), no forcing and no dissipation. In this case, these equations reduce to

$$\begin{aligned} \frac{\partial u}{\partial t} &= -g' \frac{\partial h}{\partial x} \\ \beta y u &= -g' \frac{\partial h}{\partial y} \\ \frac{\partial h}{\partial t} + H \frac{\partial u}{\partial x} &= 0 \end{aligned}$$

Note the geostrophic balance in the y-momentum equation. Substituting $e^{i(kx-\omega t)}$ dependence for all three variables, we get from the first that $u = (kg'/\omega)h$, so that the third one gives the dispersion relation

$$\omega^2 = (g'H)k^2$$

which is the dispersion relation of a simple shallow water gravity wave. The second equation then gives $\beta y \frac{kg'}{\omega} h = -g' \frac{\partial h}{\partial y}$, or

$$\frac{\partial h}{\partial y} = -\frac{\beta k}{\omega} y h.$$

We are searching for equatorial-trapped solutions, and we note that the solution for the y-structure decays away from the equator only when $k > 0$. This implies that the wave solution we have found must be eastward propagating! Using the dispersion relation, with

$$c \equiv \sqrt{g'H} \approx (9.8 \times 10^2 * cm\, sec^{-2} \times 5 * 10^{-3} \times 100 \times 10^2 cm)^{1/2} \approx 2.2 m/sec$$

we finally have

$$h_{Kelvin}(x, y, t) \propto e^{-\frac{1}{2}(\beta/c)y^2} e^{i(kx-\omega t)}.$$

Note that the decay scale away from the equator is the equatorial Rossby radius of deformation defined as

$$L_{eq}^R \equiv \sqrt{c/2\beta} \approx (c/(2 \times 2.3 \times 10^{-11} m^{-1} sec^{-1}))^{1/2} \approx 220 km$$

Next is the derivation of the full set of equatorial waves, where we now do not assume that the meridional velocity v vanishes. Substitute $h(x,y,t) = h(y)e^{i(kx-\omega t)}$ dependence, and similarly for (u,v), and derive a single equation for h to find the parabolic cylinder equation (Gill, [20], section 11.6.1)

$$\frac{d^2v}{dy^2} + \left(\frac{\omega^2}{c^2} - k^2 - \frac{\beta k}{\omega} - \frac{\beta^2}{c^2}y^2\right) v = 0.$$

The solutions that vanish at $y \longrightarrow \pm\infty$ occur only for certain relations between the coefficients, and these relations serve as the dispersion relation

$$\frac{\omega^2}{c^2} - k^2 - \frac{\beta k}{\omega} = (2n+1)\frac{\beta}{c}. \tag{4}$$

Note that the Kelvin wave dispersion relation is formally a solution of this dispersion relation for $n = -1$ (simply check that $\omega = ck$ satisfies (4) for $n = -1$). The meridional structure of the waves in this case of equatorially trapped solutions is expressed in terms of the Hermit polynomials

$$v = 2^{-n/2} H_n((\beta/c)^{1/2} y)\exp(-\beta y^2/2c)\cos(kx - \omega t)$$

and is shown in Fig. 19, where

$$H_0 = 1; \qquad H_1 = 2x; \qquad H_2 = 4x^2 - 2; \qquad H_3 = 8x^3 - 12x; \qquad H_4 = 16x^4 - 48x^2 + 12$$

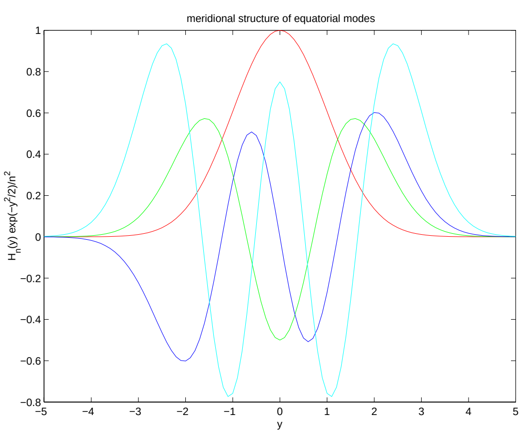

Figure 19: The latitudinal structure of the first few equatorial modes: $H_n(y)\exp(-y^2/2)/n^2$.

The dispersion relation is plotted in Fig. 20.

So, we have a complete set of waves, the Kelvin ($n = -1$), Yanai ($n = 0$), Rossby and Poincare ($n > 0$) waves. As seen in the plot, the dispersion relation includes two main sets of waves for $n > 0$. For high frequency, we can neglect the term $\frac{\beta k}{\omega}$, to find the Poincare gravity-inertial waves

$$\omega^2 \approx (2n+1)\beta c + k^2 c^2,$$

while for low frequency, we can neglect the term ω^2/c^2 in the dispersion relation to find the westward propagating Rossby wave dispersion relation

$$\omega = \frac{-\beta k}{k^2 + (2n+1)\beta/c}.$$

Typical speeds of long Rossby waves would therefore be

$$\omega/k = \frac{-c}{2n+1}$$

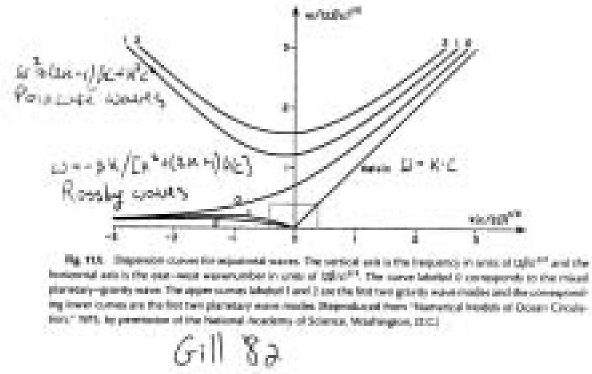

Figure 20: The Equatorial wave dispersion relation, (Gill [20], p 438, Fig. 11.1)/

so that the first Rossby mode ($n = 1$) travels at a 1/3 of the Kelvin wave speed, implying a roughly 2.5 months crossing time for Kelvin and 8 months for Rossby waves (based on 15,000 km basin width).

Note (Fig. 19) that the first Rossby mode has a zero at the equator and two maxima away from the equator, while the Kelvin wave has a maximum at the equator. This tells us something about how a random initial perturbation will project on the different modes. That is, a forcing pattern or an initial perturbation that is centered at the equator may be expected to excite Kelvin waves, while a forcing or initial perturbation that has components off the equator will tend to excite Rossby waves. The above discussion centers on the first baroclinic mode, but may be generalized to higher vertical baroclinic modes, although for our purposes this is not essential.

1.2.4 Ocean response to wind perturbation

Consider first the mean state of the thermocline. The steady state ($\partial u/\partial t = 0$) momentum equation (1) in a reduced gravity model, at $y = 0$ ($\beta y v = 0$) in the presence of easterly wind forcing and neglecting frictional effects ($-\varepsilon u = 0$) is

$$0 \approx -g' \frac{\partial h}{\partial x} + \frac{\tau^x}{\rho_0 H}$$

so that an easterly wind stress is balanced by a pressure gradient due to a thermocline tilt, with the thermocline closer to the surface in the East Pacific. This mean state of the thermocline results in the cold tongue there, as observed, via the mixing of cold sub-thermocline water with the surface water, as will be discussed more quantitatively below.

Regarding the interannual equatorial thermocline variability, at this stage we just note that a wind perturbation that corresponds to a weakening of the mean easterlies in the central Pacific affects the thermocline depth in the central Pacific. It creates downwelling Kelvin waves (that is, waves that propagate a downwelling signal, which means a thermocline deepening signal; these are waves that propagate a warm water surplus above the thermocline, and may therefore be called “warm” waves) and upwelling (i.e. cold) Rossby waves. The excitation of these waves by a wind anomaly will be examined more rigorously below.

1.2.5 Atmospheric response to SST anomalies

We now need to describe the atmospheric response to SST perturbations. Use Gill’s [19] model for this, whose equations are very much like the β plane ocean equations, except that the atmospheric time scales are much