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**CONFERENCE ON**
**"TYPICAL-CASE COMPLEXITY, RANDOMNESS AND ANALYSIS OF SEARCH ALGORITHMS"**

(5 - 7 September 2002)

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*" Smoothed analysis of algorithms: why the simplex algorithm usually takes polynomial time "*

presented by:

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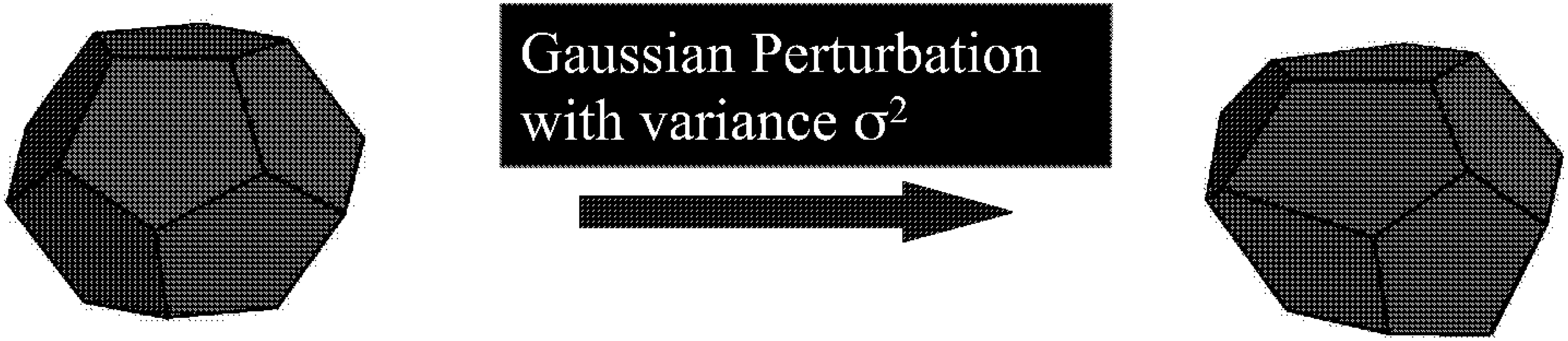



# Smoothed Analysis of Algorithms:

## Why The Simplex Method Usually Takes Polynomial Time

**Shang-Hua Teng**

**Boston University/Akamai**

**Joint work with Daniel Spielman (MIT)**

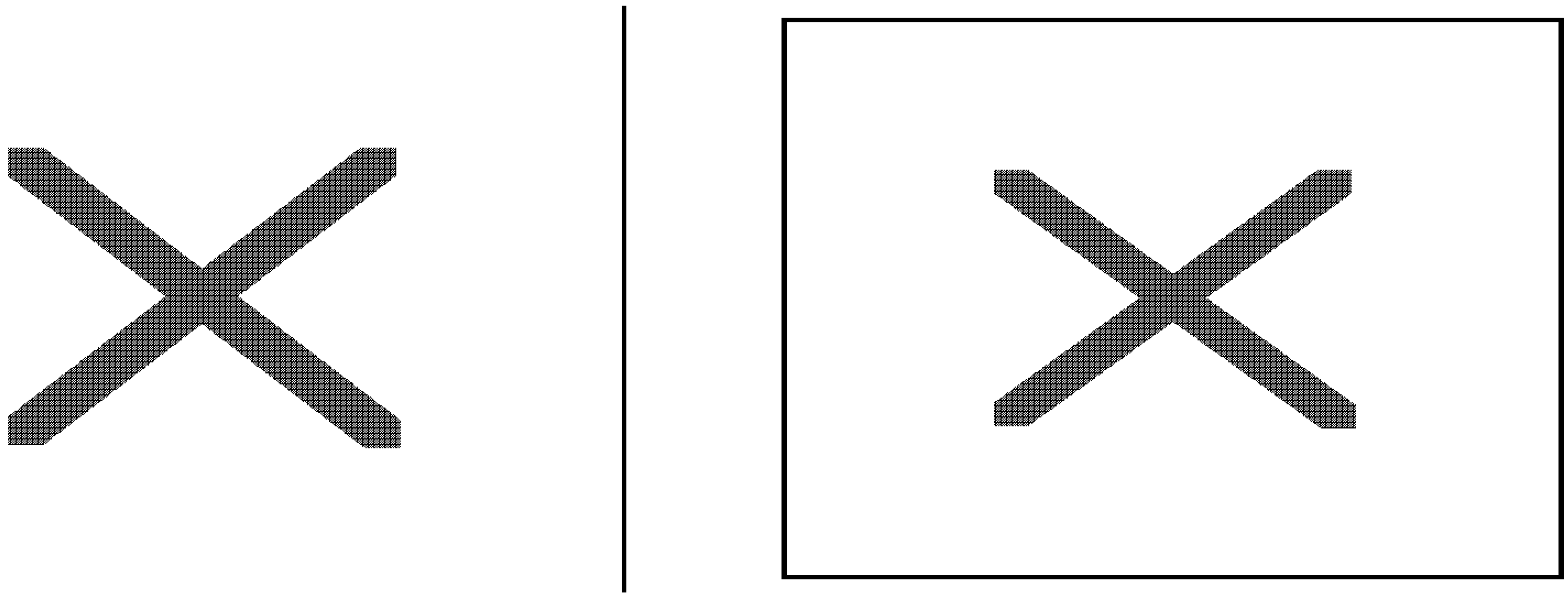
# Remarkable Algorithms and Heuristics

Work well in practice, but

Worst case: bad,
exponential,
contrived.

Average case: good,
polynomial,
meaningful?

# Random is not typical



# Smoothed Analysis of Algorithms:

worst case

$\max_x T(x)$

average case

$\text{avg}_r T(r)$

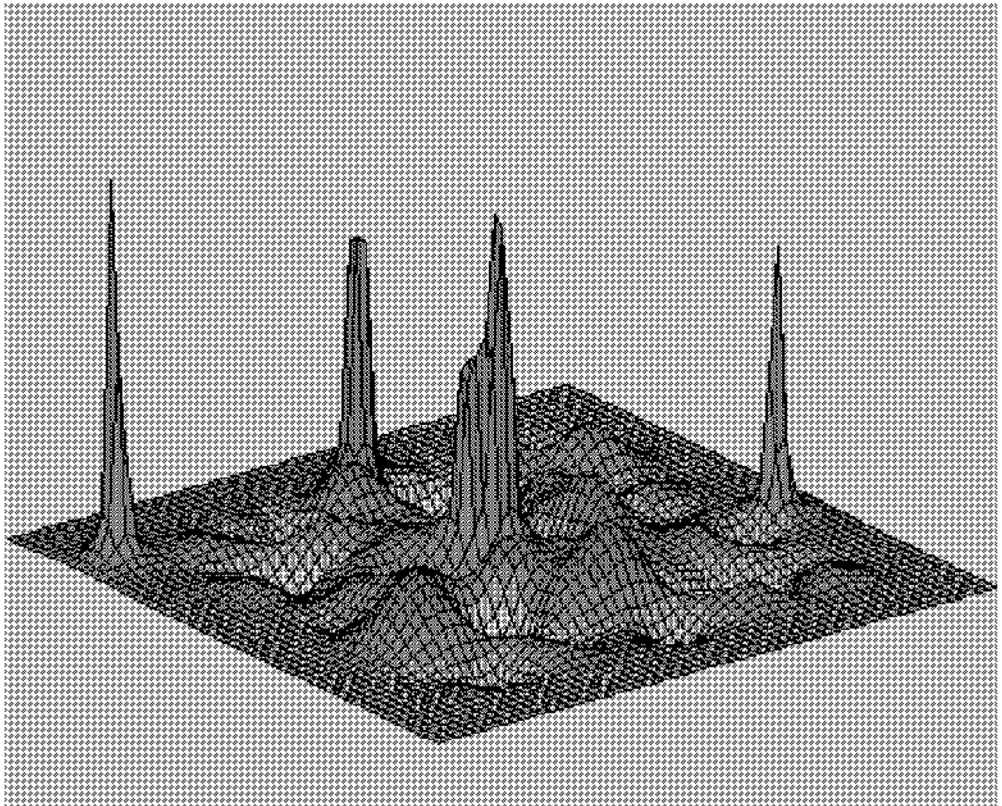
smoothed complexity

$\max_x \text{avg}_r T(x+\varepsilon r)$

# Smoothed Analysis of Algorithms

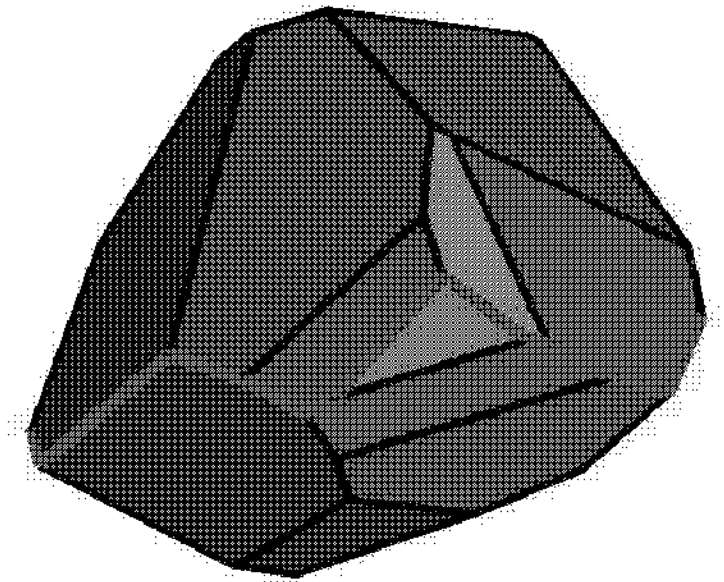
- **Interpolate between Worst case and Average Case.**
- **Consider neighborhood of *every* input instance**
- **If low, have to be unlucky to find bad input instance**

# Complexity Landscape



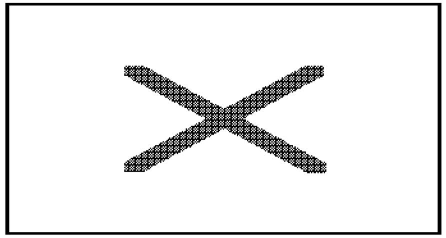
# Classical Example: Simplex Method for Linear Programming

$$\max \quad z^T x$$
$$\text{s.t.} \quad A x \le y$$



- Worst-Case: exponential
- Average-Case: polynomial
- Widely used in practice

# The Diet Problem



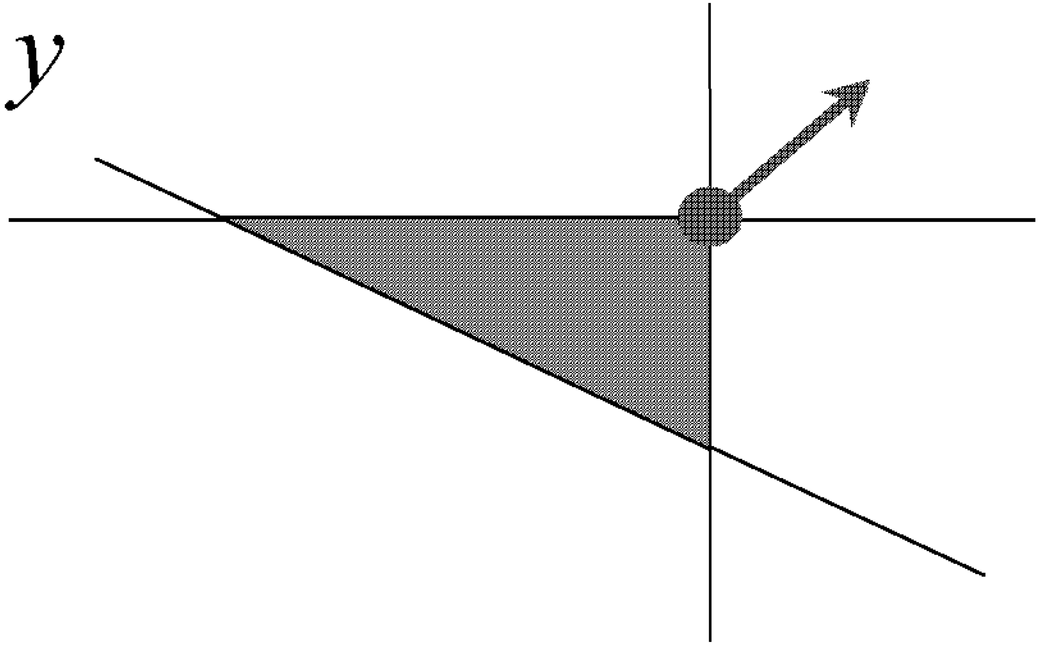
| | Carbs | Protein | Fat | Iron | Cost |
|---|---|---|---|---|---|
| **1 slice bread** | 30 | 5 | 1.5 | 10 | 30¢ |
| **1 cup yogurt** | 10 | 9 | 2.5 | 0 | 80¢ |
| **2tsp Peanut Butter** | 6 | 8 | 18 | 6 | 20¢ |
| **US RDA Minimum** | 300 | 50 | 70 | 100 | |

$$\text{Minimize } 30\,x_1 + 80\,x_2 + 20\,x_3$$

$$\begin{aligned} \text{s.t.} \quad 30x_1 + 10\,x_2 + 6\,x_3 &\geq 300 \\ 5x_1 + 9x_2 + 8x_3 &\geq 50 \\ 1.5x_1 + 2.5\,x_2 + 18\,x_3 &\geq 70 \\ 10x_1 + 6\,x_3 &\geq 100 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

# Linear Programming

$$\max \quad z^T x$$
$$\text{s.t.} \quad A\,x \le y$$



Max $\quad x_1 + x_2$

s.t $\quad x_1 \le 1$

$\quad x_2 \le 1$

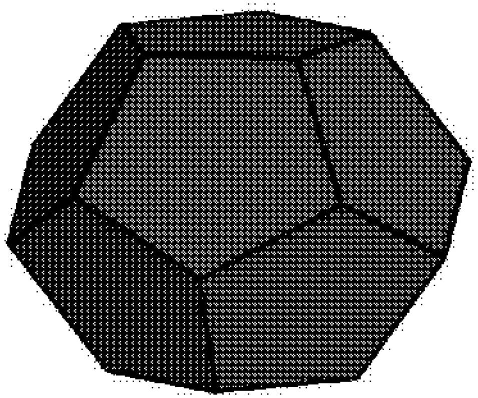
$\quad -x_1 - 2x_2 \le 1$

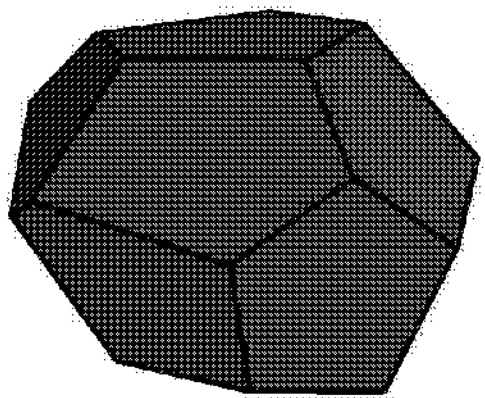
# Smoothed Analysis of Simplex Method

$$\begin{array}{ll} \max & z^T x \\ \text{s.t.} & A\,x \le y \end{array}$$

$$\begin{array}{ll} \max & z^T x \\ \text{s.t.} & (A + \sigma G)\,x \le y \end{array}$$

$G$ is Gaussian



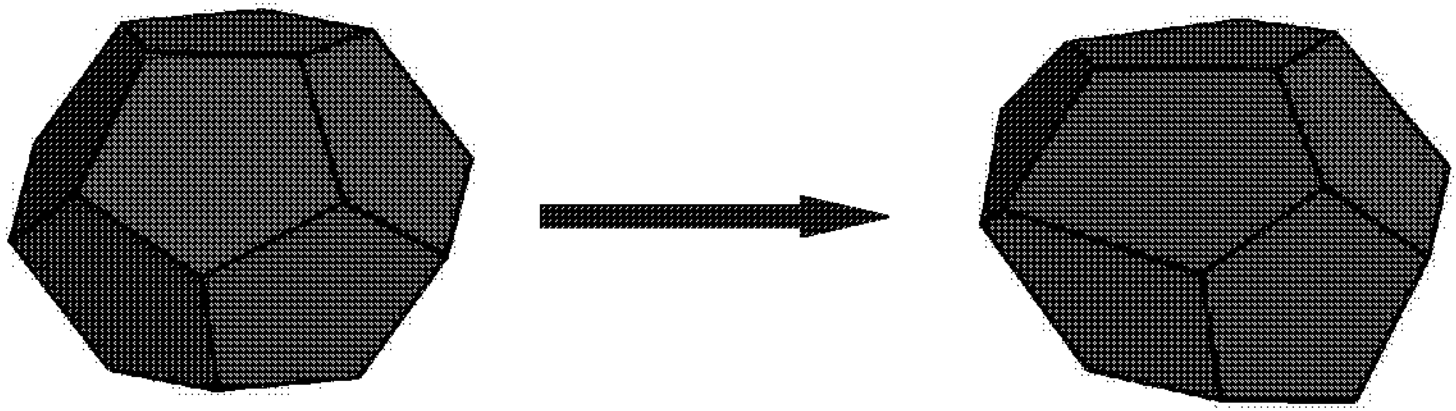


# Smoothed Analysis of Simplex Method

- Worst-Case: exponential
- Average-Case: polynomial
- Smoothed Complexity: polynomial

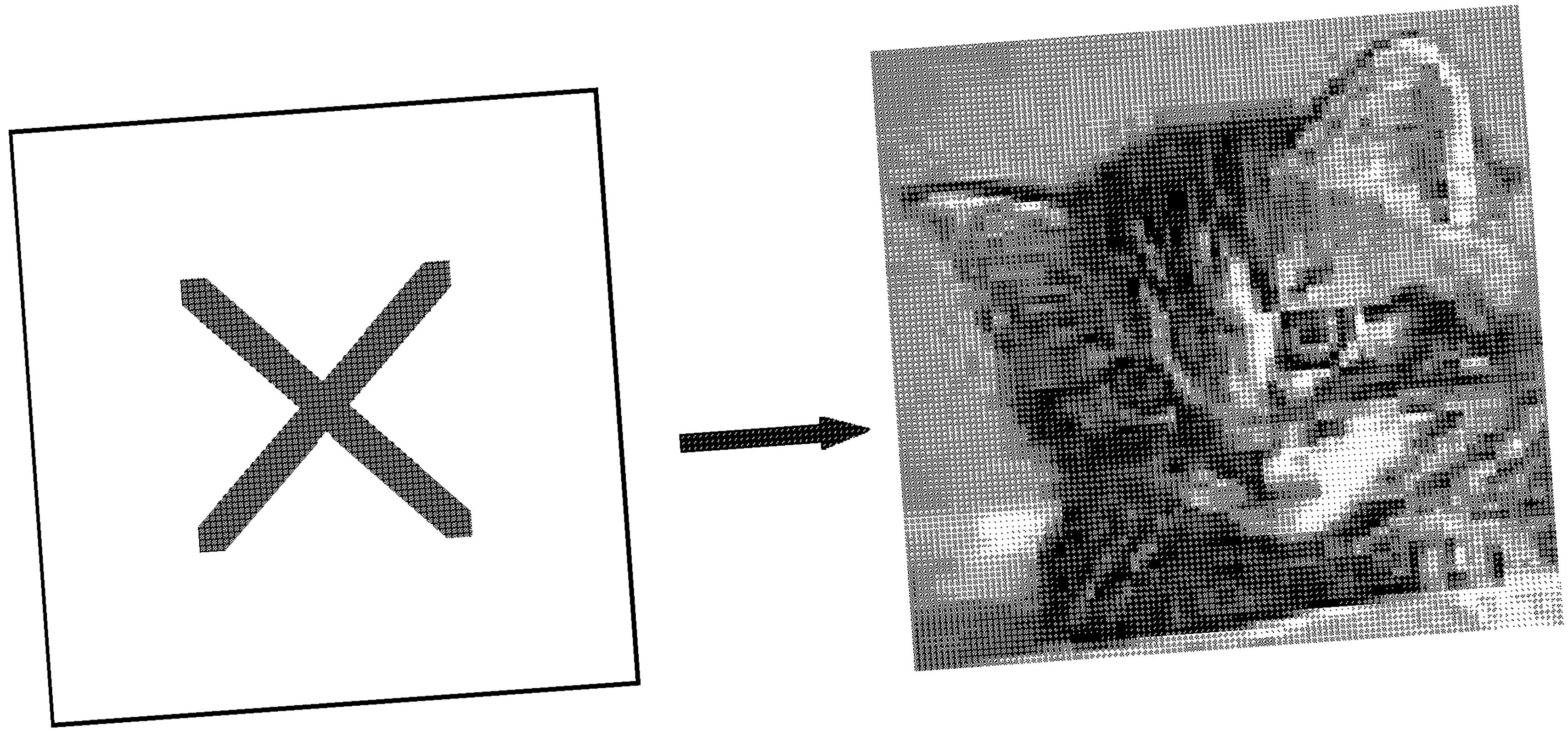
$$\begin{aligned} \max \quad & z^T x \\ \text{s.t.} \quad & \boldsymbol{a}_i^T x \le \pm 1, \\ & \|\boldsymbol{a}_i\| \le 1 \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} \max \quad & z^T x \\ \text{s.t.} \quad & (\boldsymbol{a}_i + \sigma \boldsymbol{g}_i)^T x \le \pm 1 \end{aligned}$$

# Perturbation yields Approximation

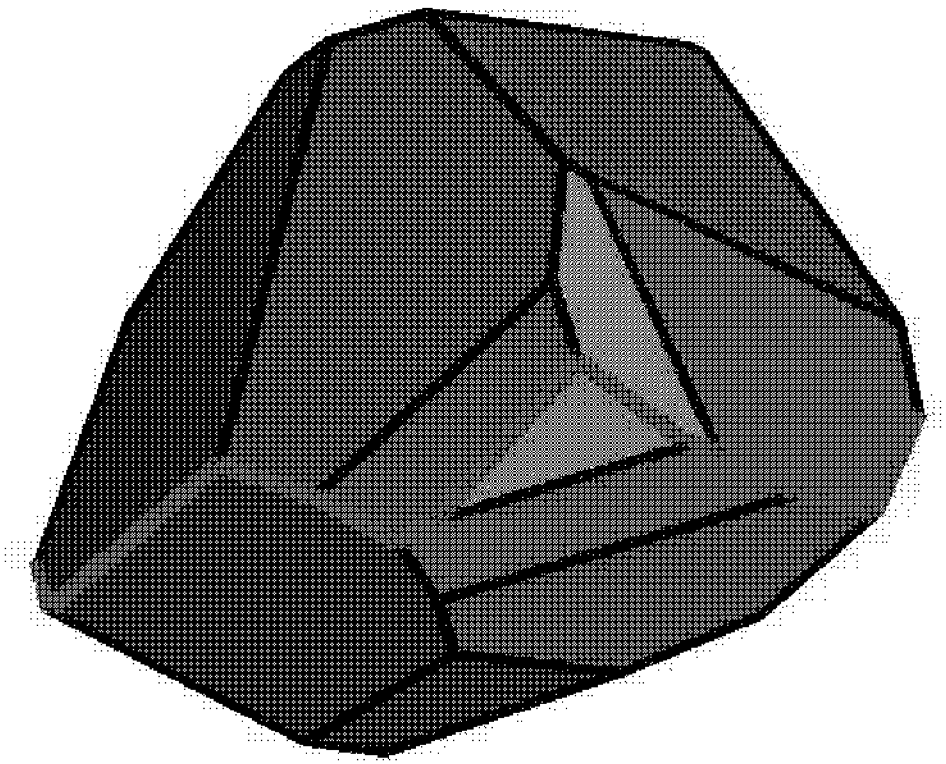


For polytope of good aspect ratio

# But, combinatorially



# The Simplex Method

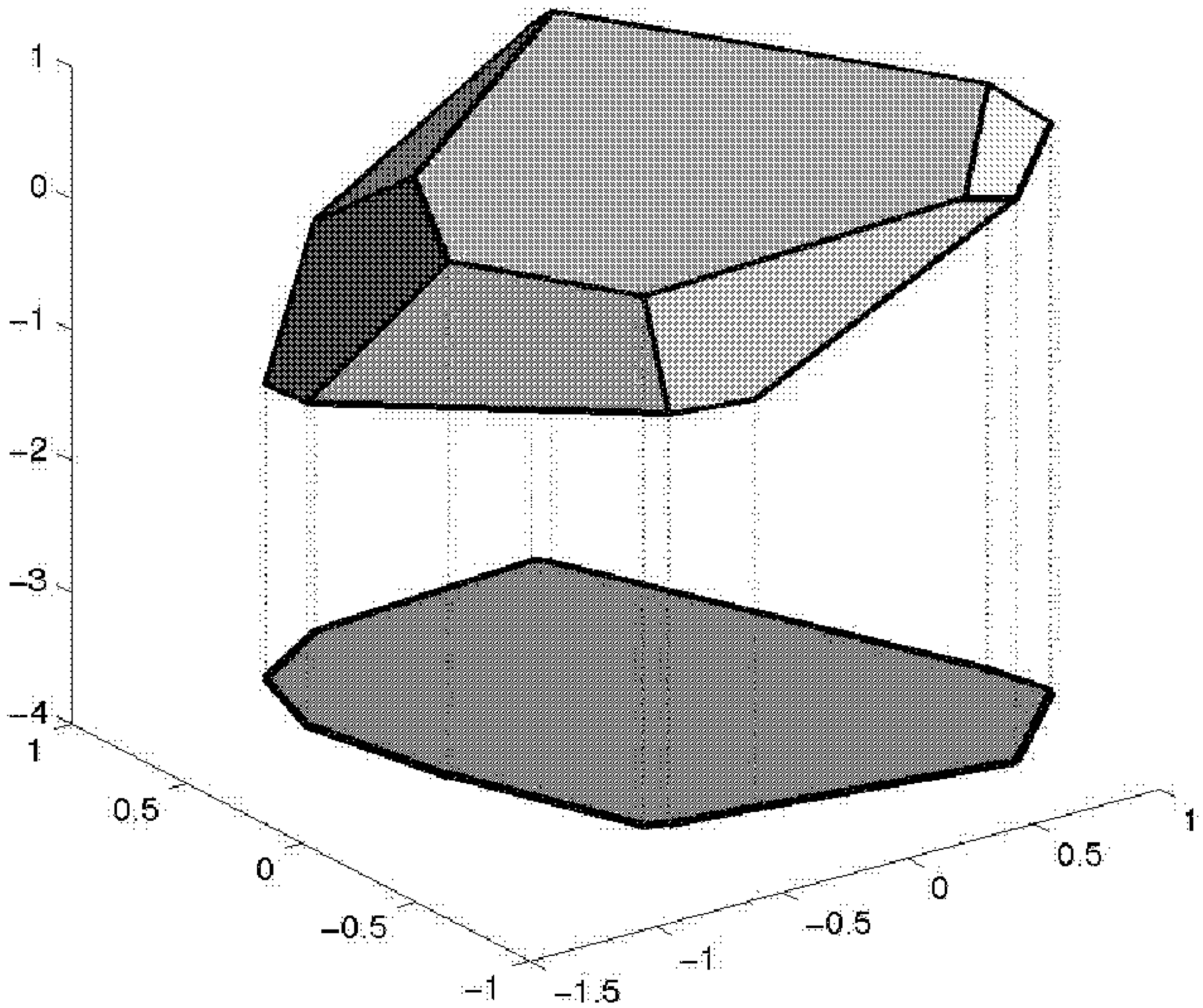


# History of Linear Programming

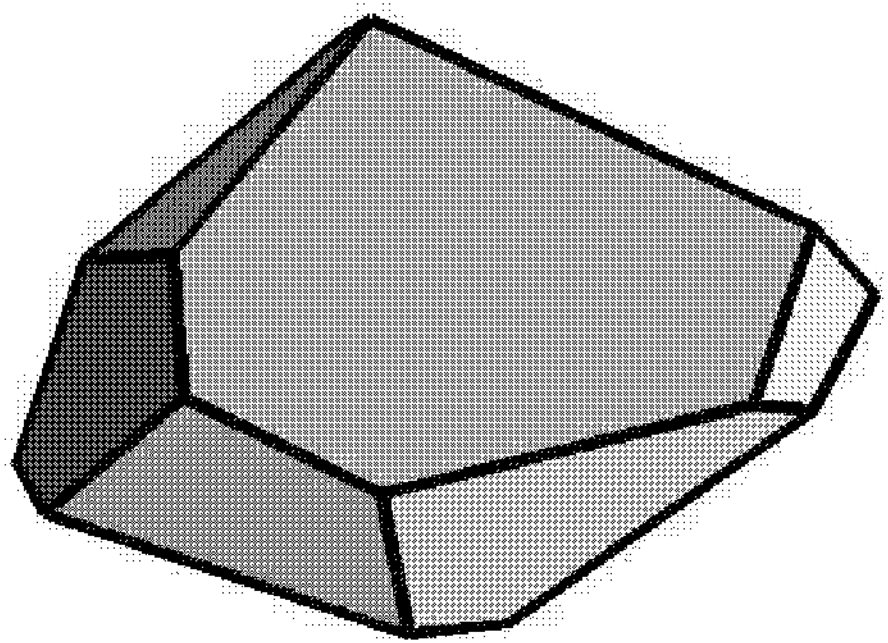
- Simplex Method (Dantzig, '47)
- Exponential Worst-Case (Klee-Minty '72)
- Avg-Case Analysis (Borgwardt '77, Smale '82, Haimovich, Adler, Megiddo, Shamir, Karp, Todd)
- Ellipsoid Method (Khaciyan, '79)
- Interior-Point Method (Karmarkar, '84)
- Randomized Simplex Method ($m^{O(\sqrt{d})}$ ) –

  (Kalai '92, Matousek-Sharir-Welzl '92)

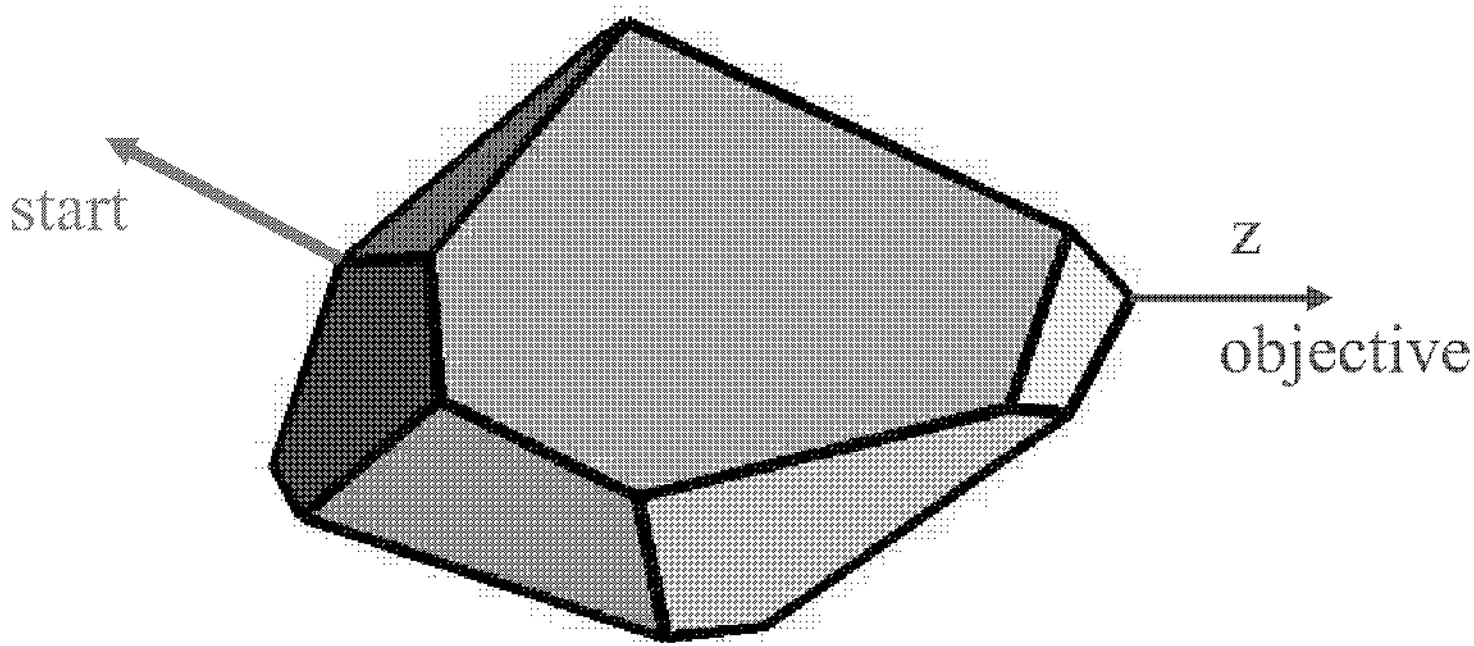
# Shadow Vertices



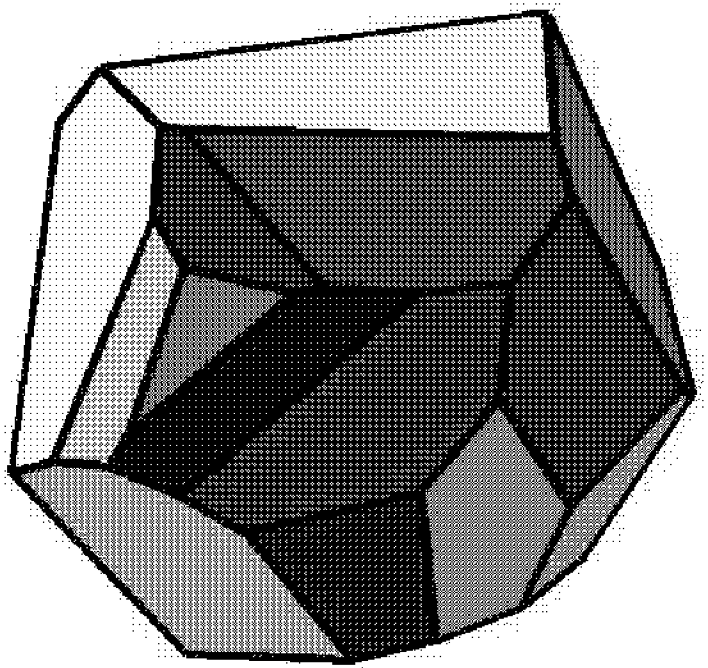
# Another shadow

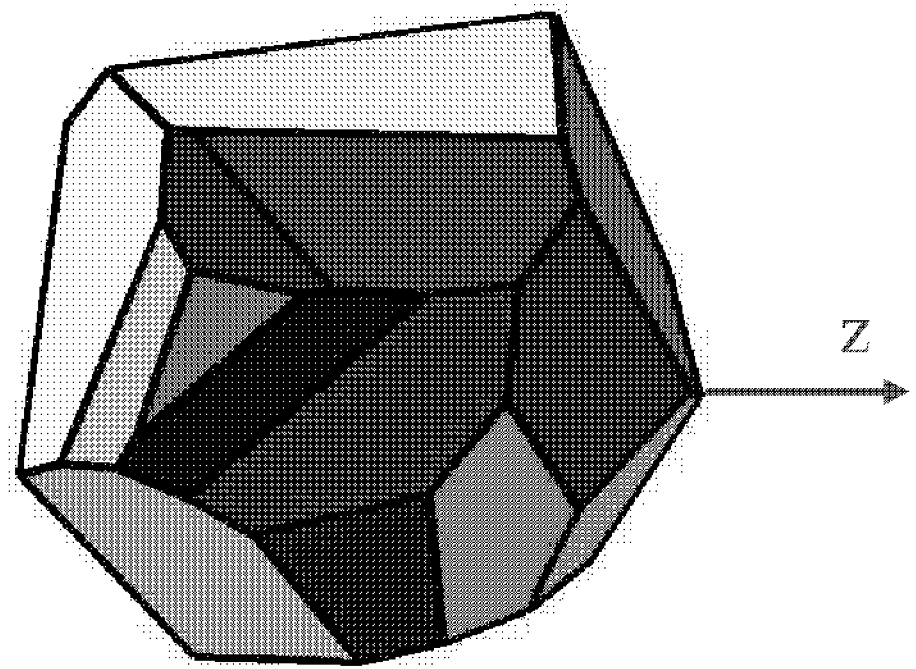


# Shadow vertex pivot rule



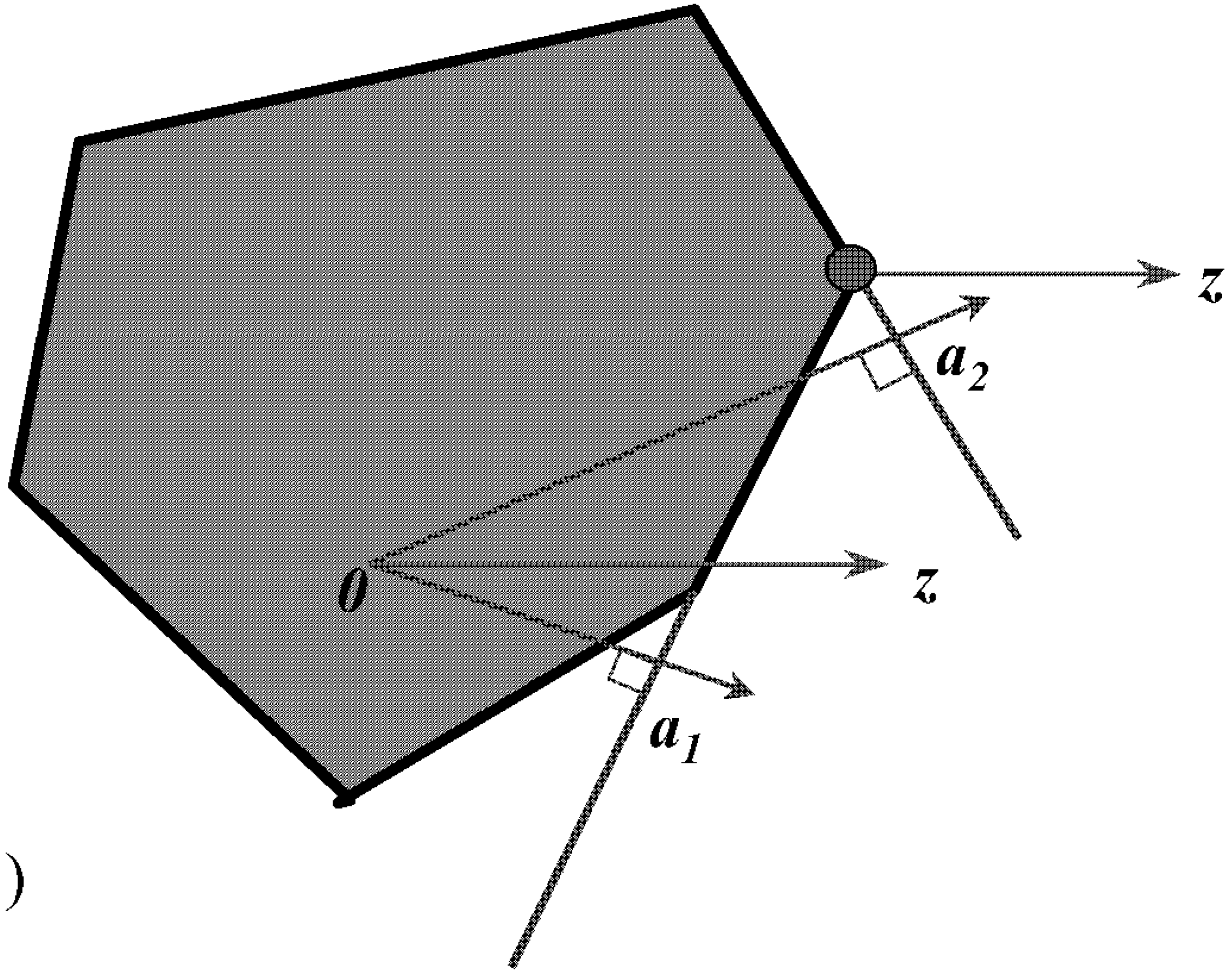
Theorem: For every plane, the expected size of the shadow of the perturbed tope is $poly(m,d,1/\sigma)$






Theorem: For every $z$, two-Phase Algorithm runs in expected time $poly(m, d, 1/\sigma)$

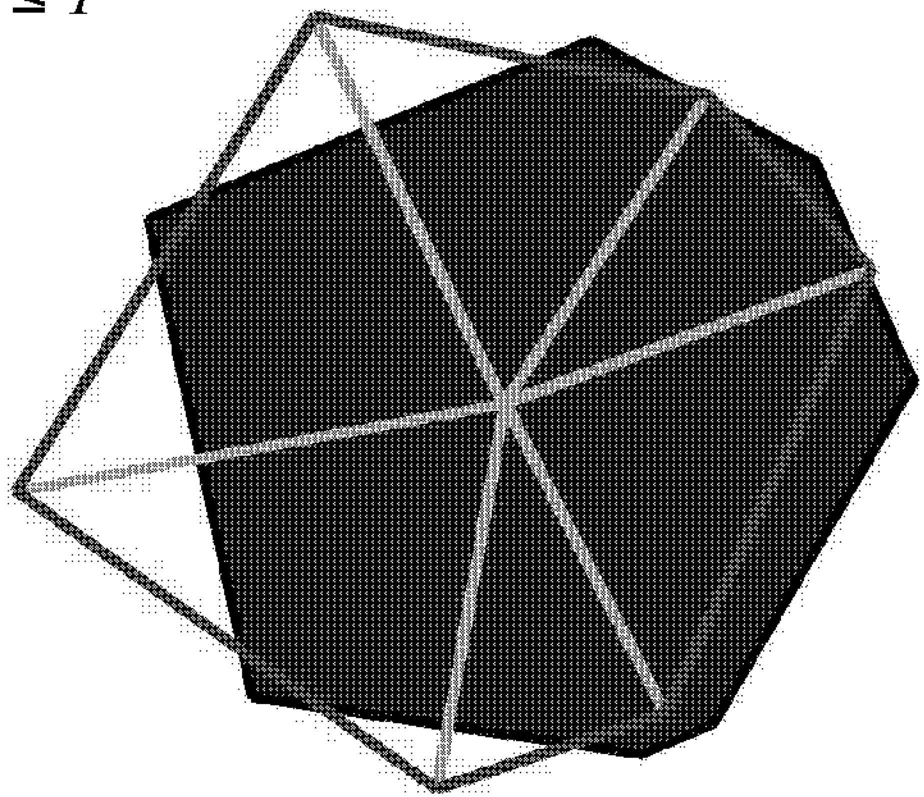
# A Local condition for optimality



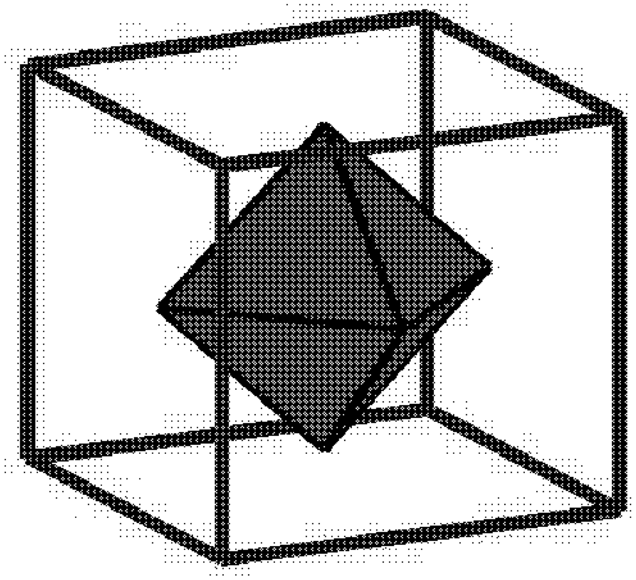

Vertex on $a_1, \ldots, a_d$ maximizes $z$ iff

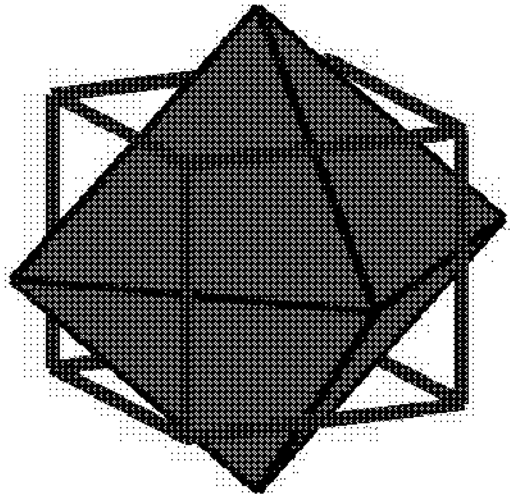
$z \in \text{cone}(a_1, \ldots, a_d)$

| **Primal** | | **Polar** |
| --- | --- | --- |
| $a_1^T x \le 1$ | | |
| $a_2^T x \le 1$ | → | ConvexHull($a_1$, $a_2$, ... $a_m$) |
| … | | |
| $a_m^T x \le 1$ | | |

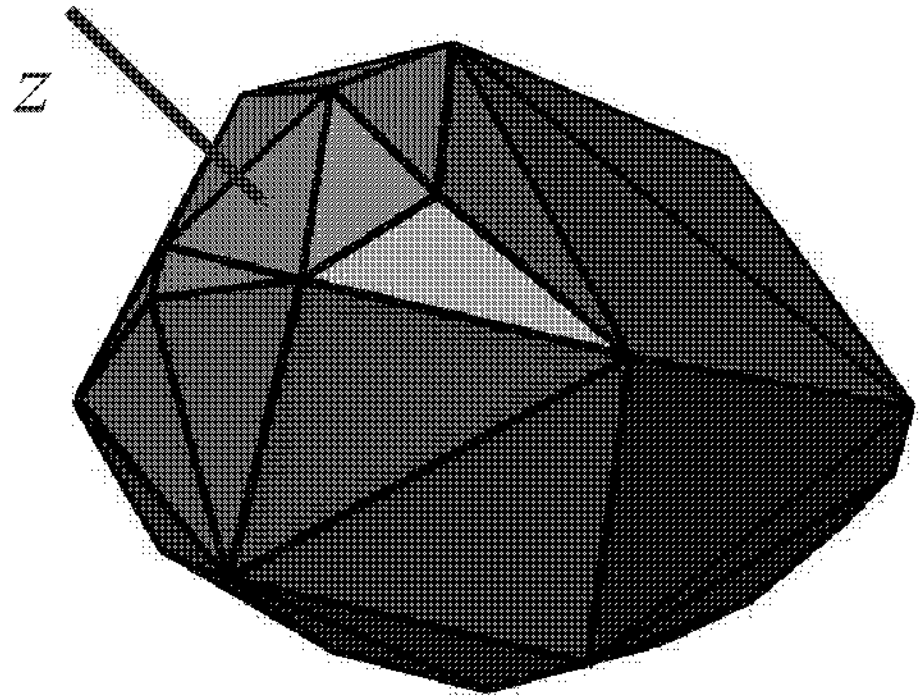


# Polar



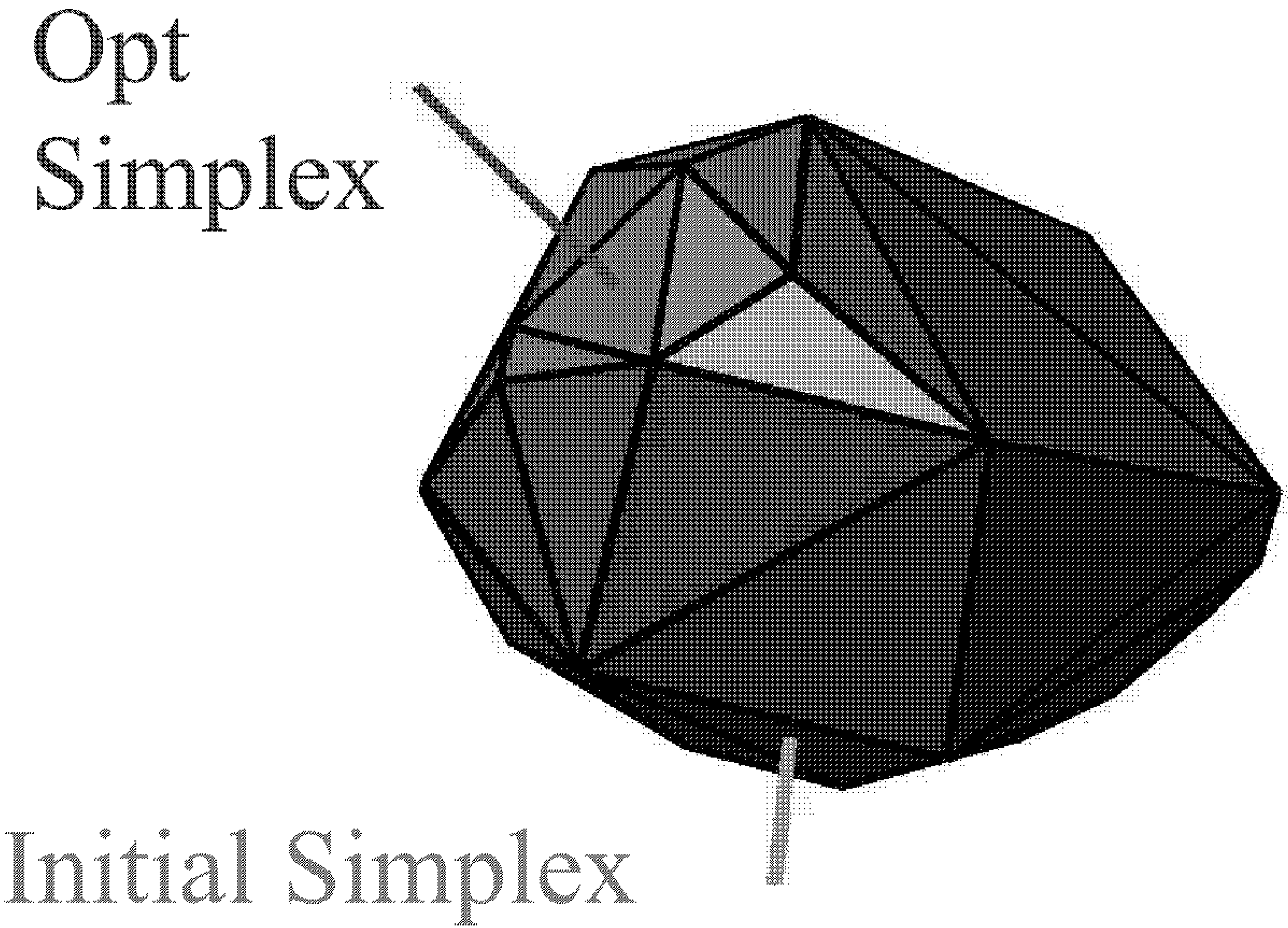


# Polar Linear Program

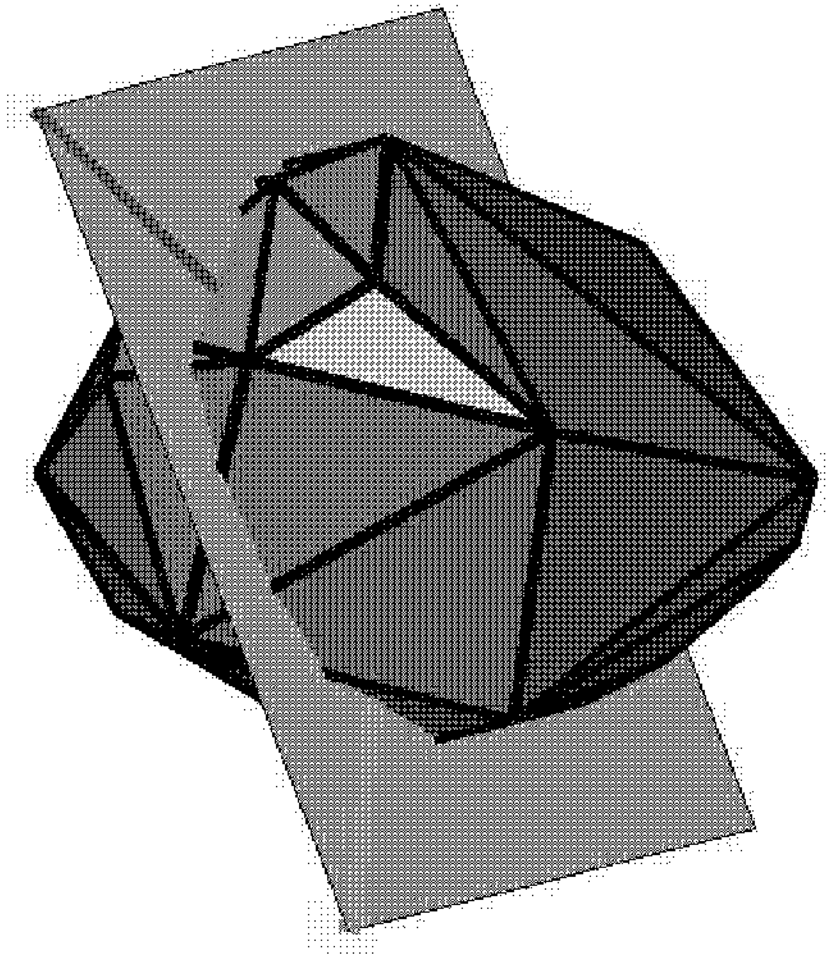


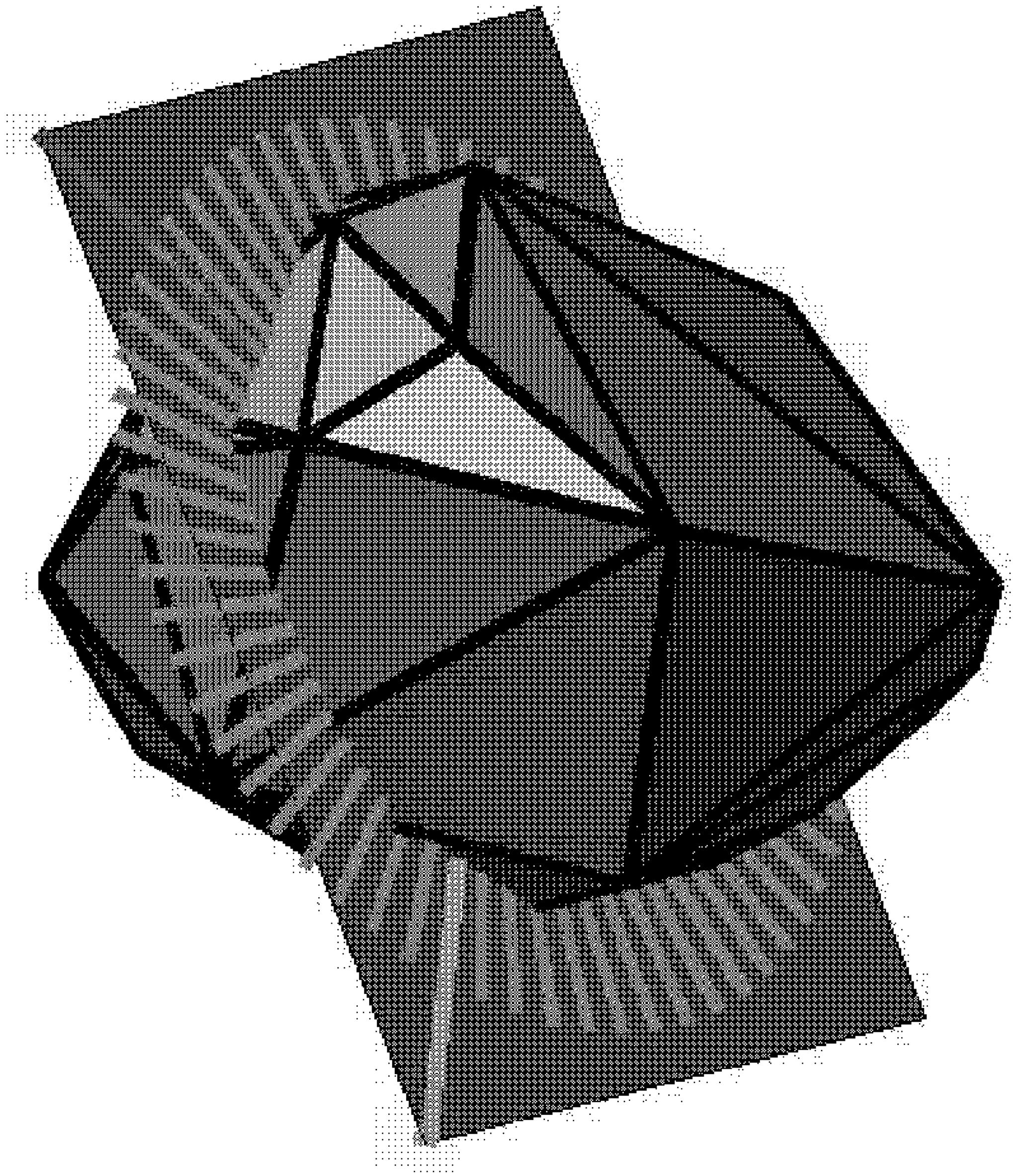

$$\max \quad \alpha$$

$$\alpha z \in \text{ConvexHull}(\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m)$$

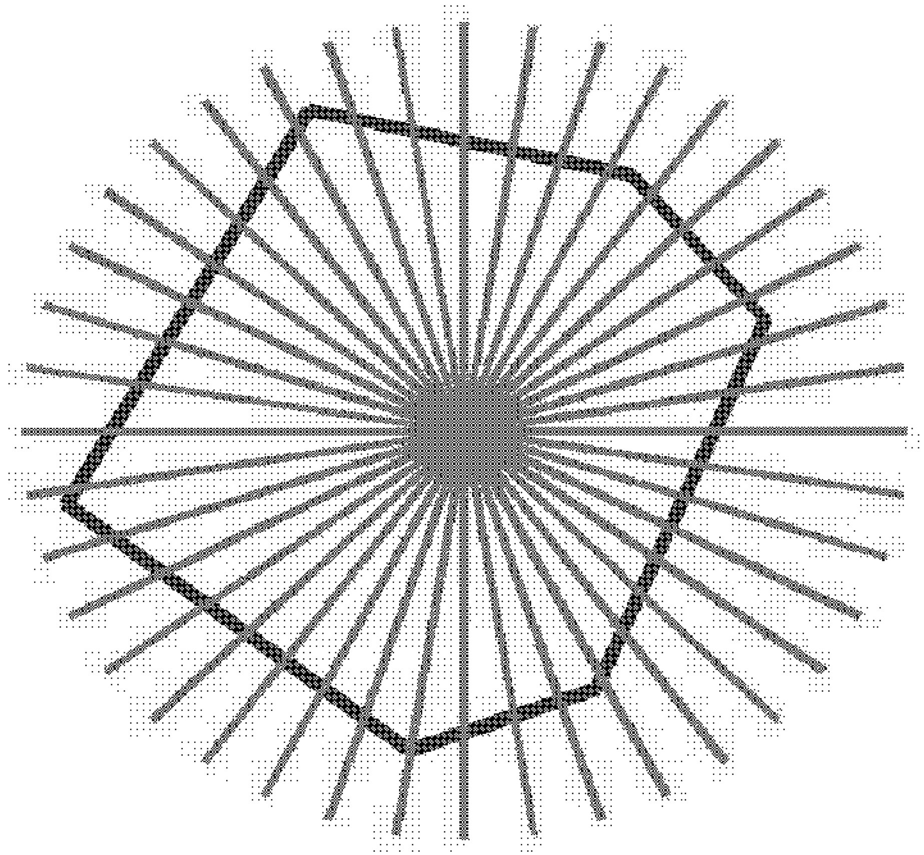

Opt
Simplex
Initial Simplex

# Shadow vertex pivot rule

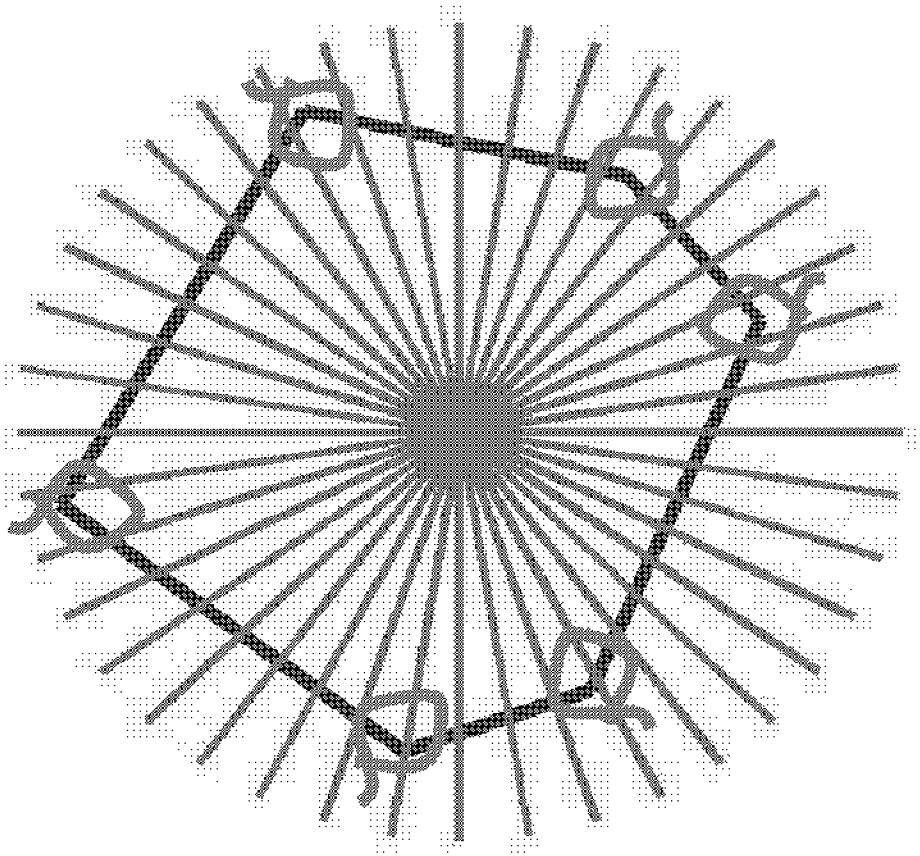




# Count facets by discretizing to $N$ directions, $N$ □ ∫



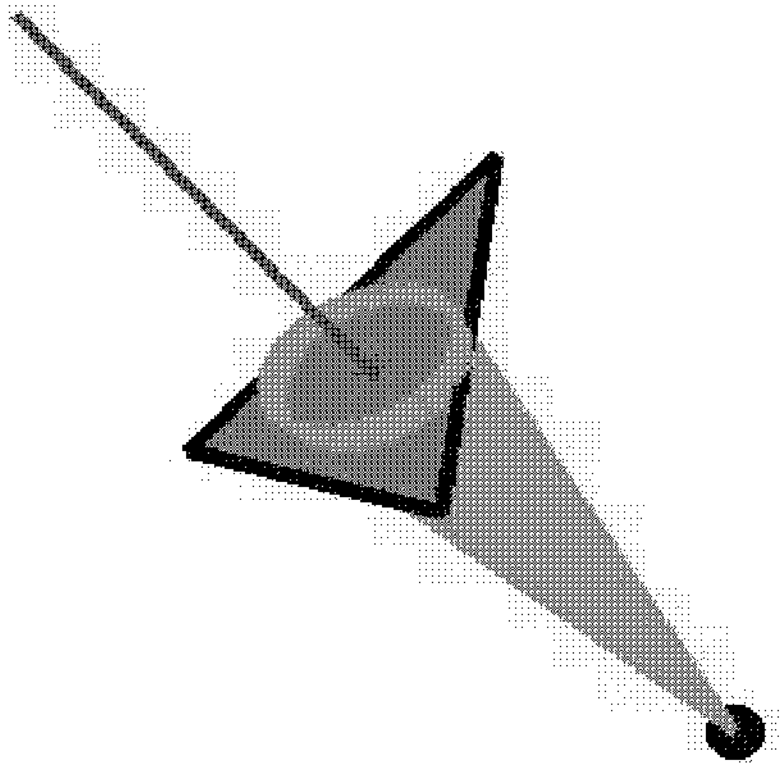
# Count pairs in different facets

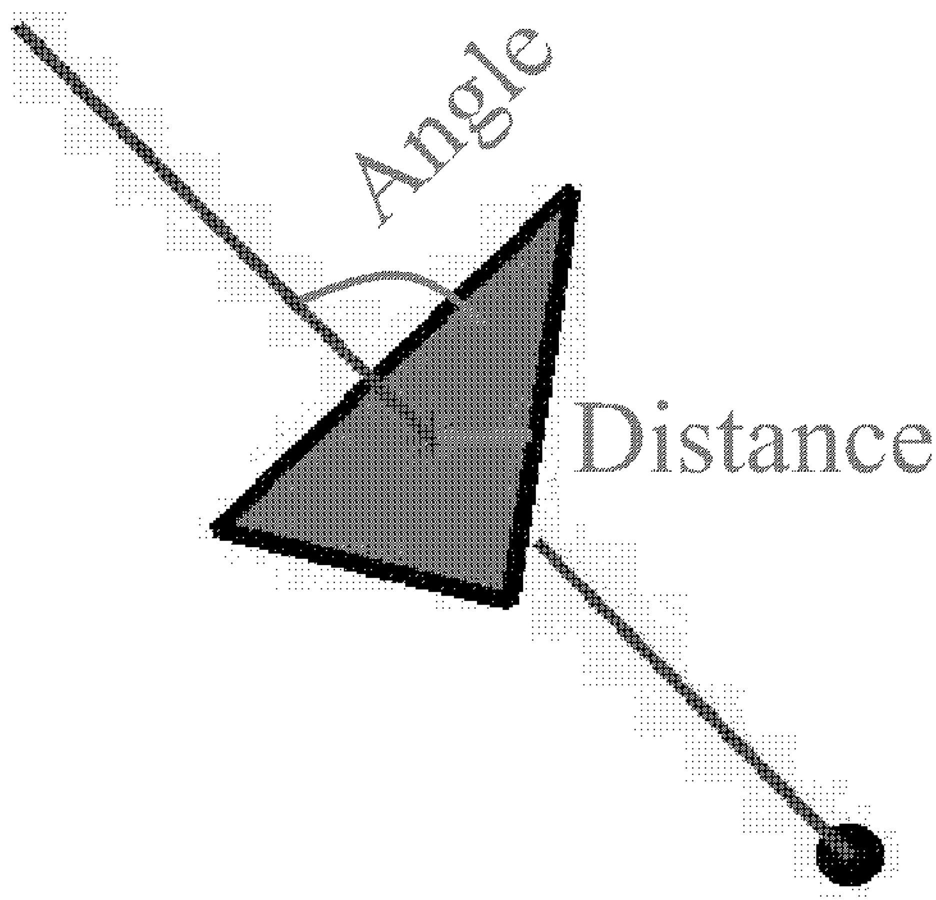


$$\Pr\left[\begin{matrix}\textbf{Different}\\ \textbf{Facets}\end{matrix}\right] < \boldsymbol{c/N}$$

So, expect ***c*** Facets

# Expect cone of large angle




Angle
Distance

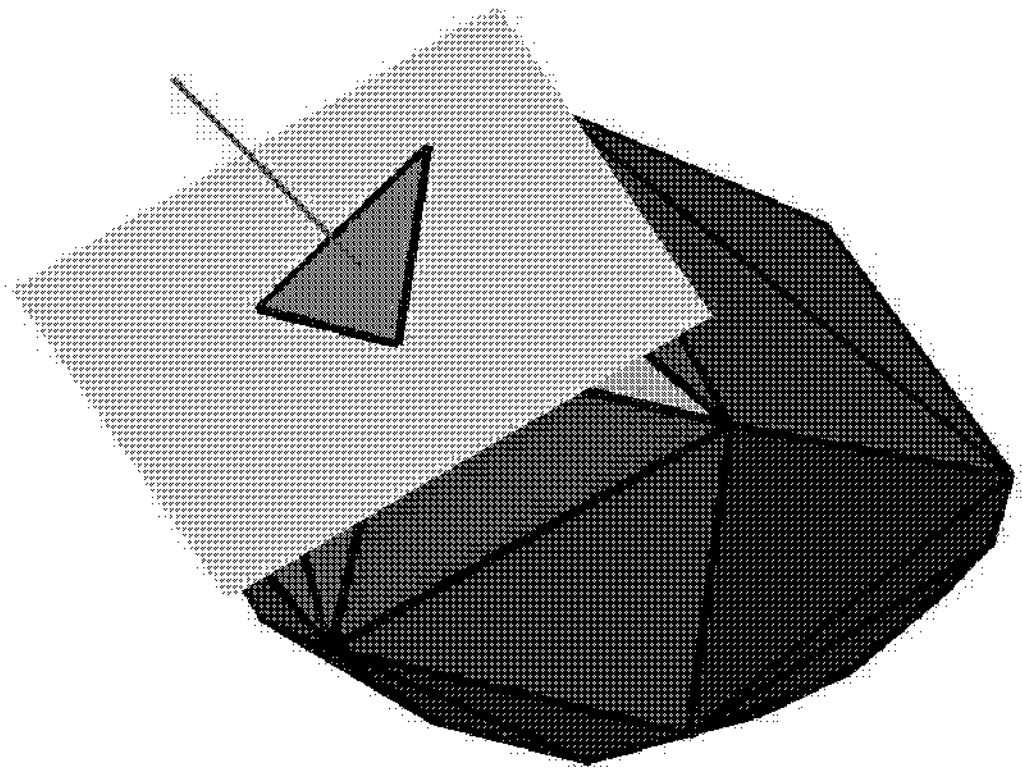
# Future Research – Simplex Method

- Smoothed analysis of other pivot rules
- Analysis under *relative* perturbations.
- Trace solutions as un-perturb.
- Strongly polynomial algorithm for linear programming?

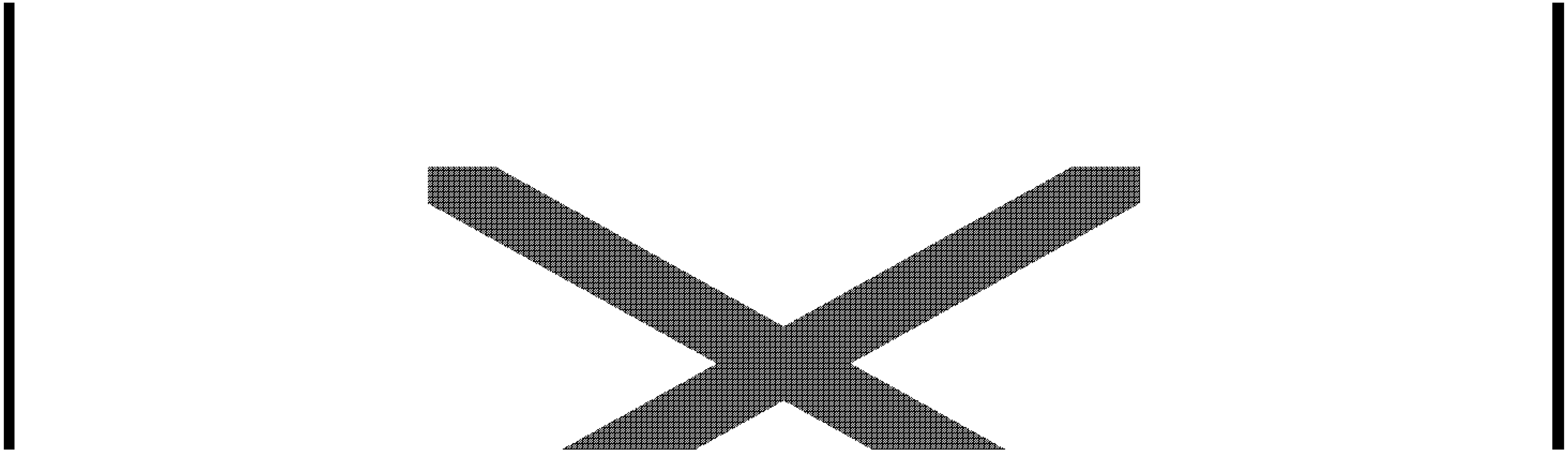
# A Theory Closer to Practice

- Optimization algorithms and heuristics, such as Newton's Method, Conjugate Gradient, Simulated Annealing, Differential Evolution, etc.
- Computational Geometry, Scientific Computing and Numerical Analysis
- Heuristics solving instances of NP-Hard problems.
- Discrete problems?
- Shrink intuition gap between theory and practice.

# Isolate on one Simplex



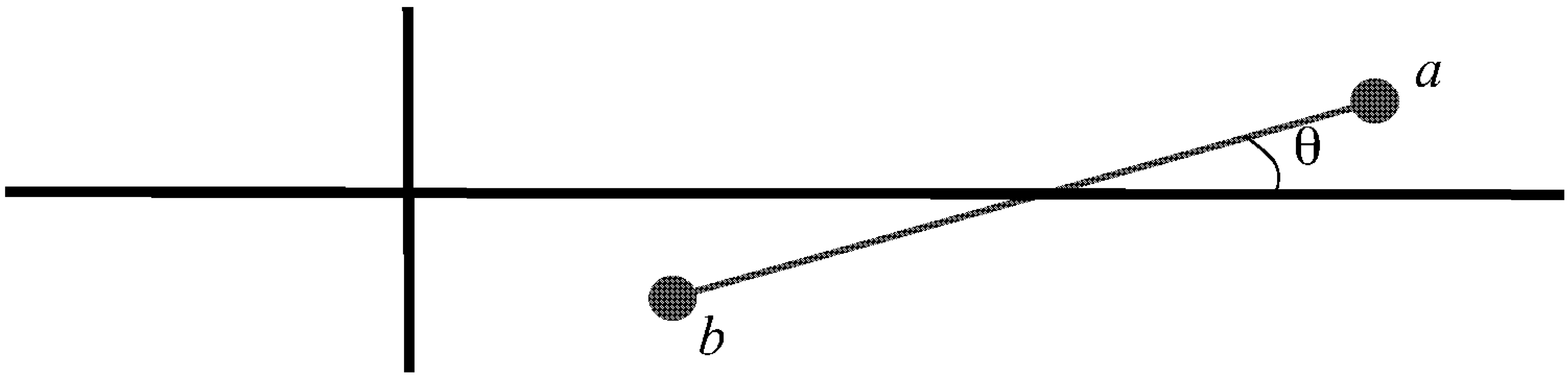
# Integral Formulation

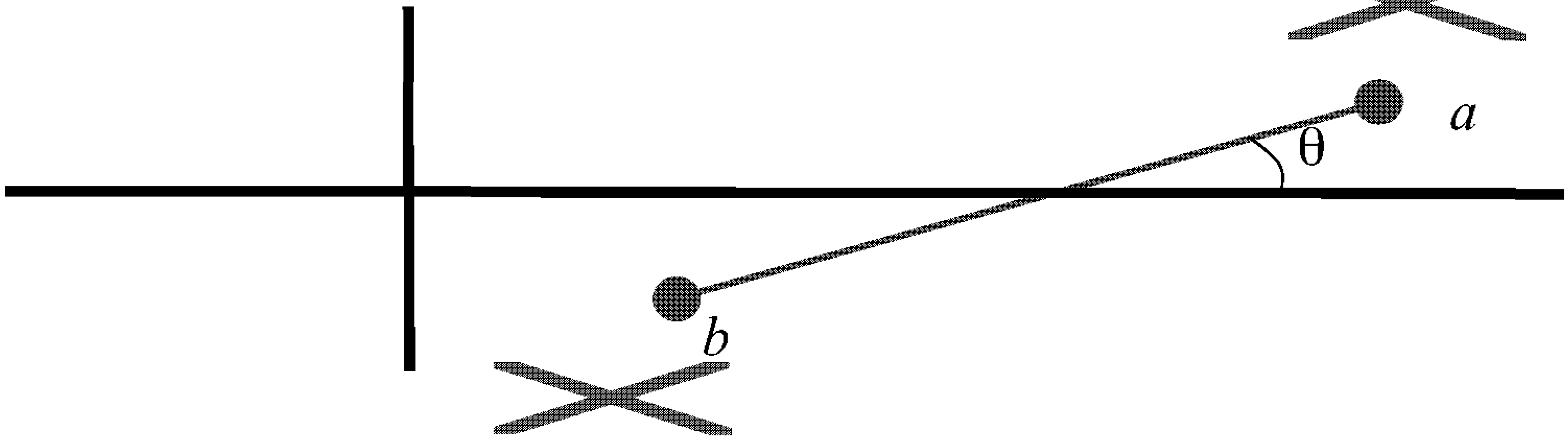


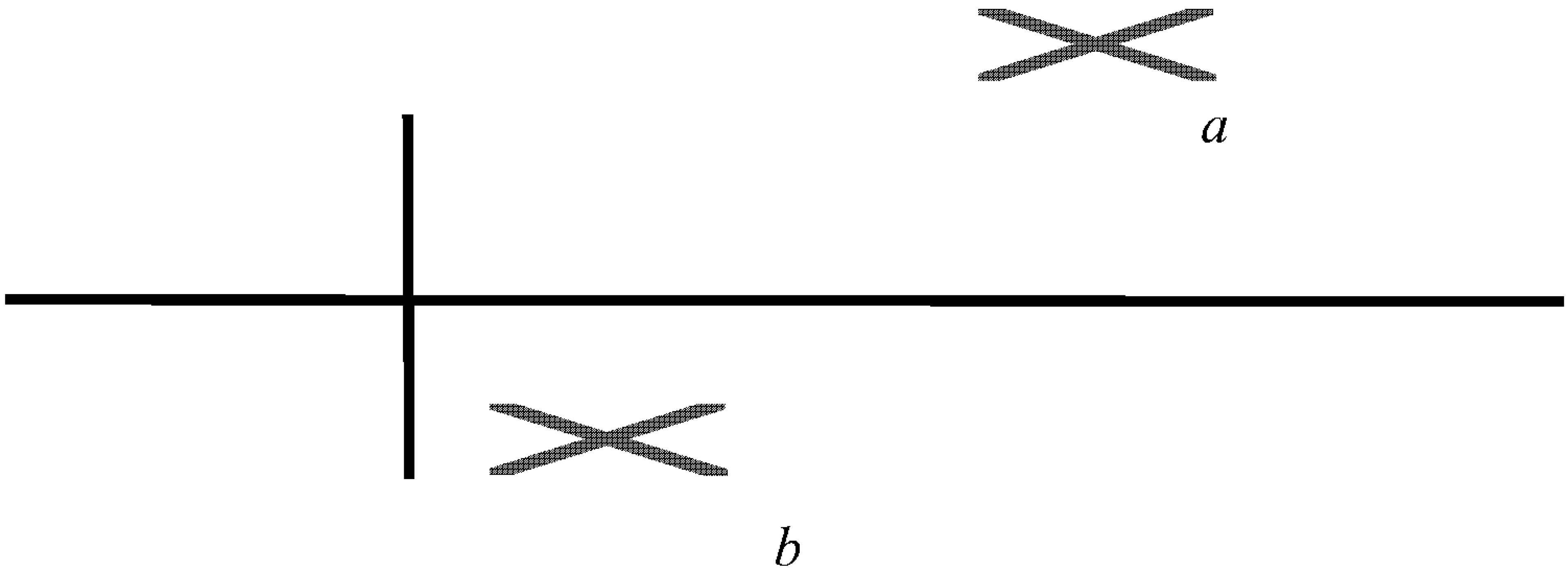
# Example:

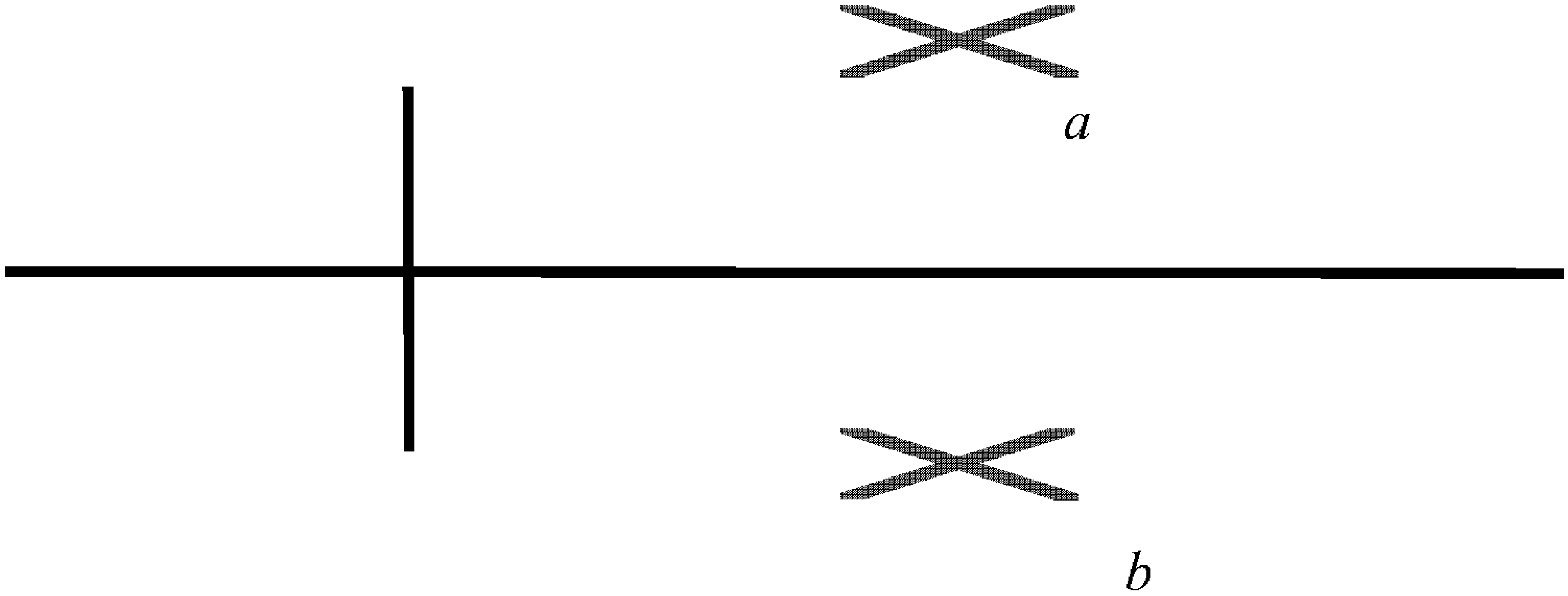
For $a$ and $b$ Gaussian distributed points, given that $\overline{ab}$ intersects x-axis

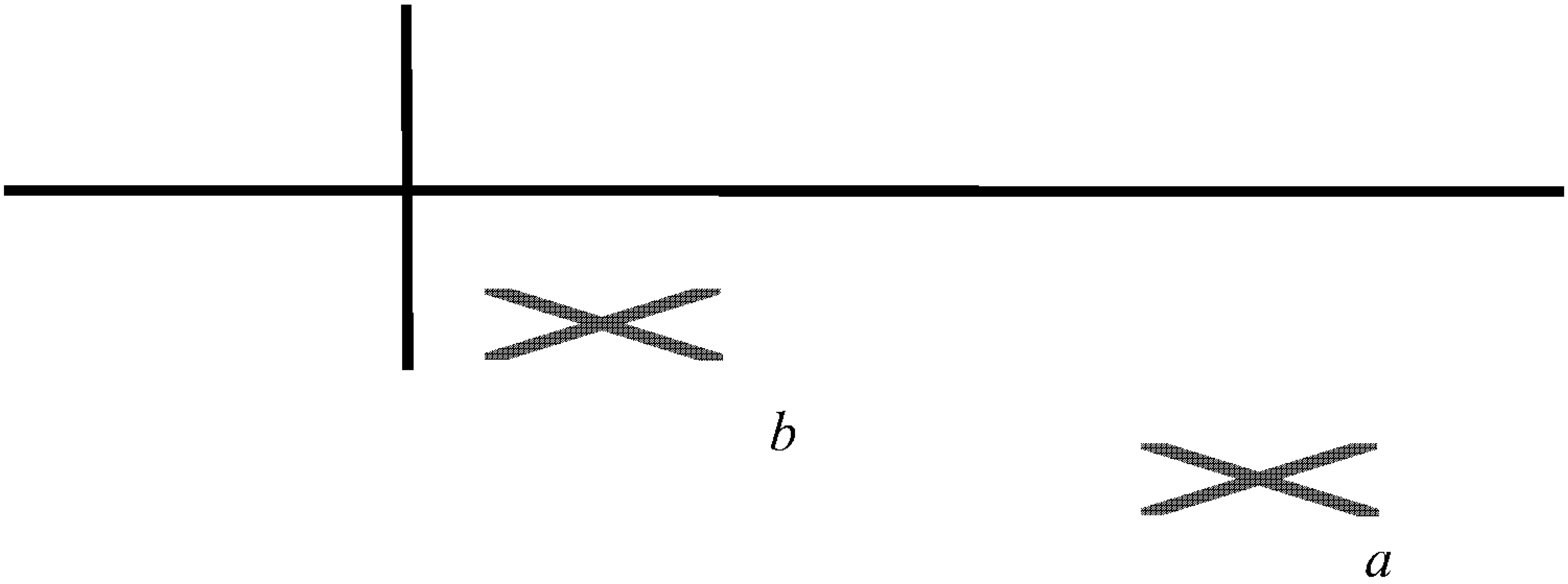
$$\mathrm{Prob}[\theta < \varepsilon] = O(\varepsilon^2)$$

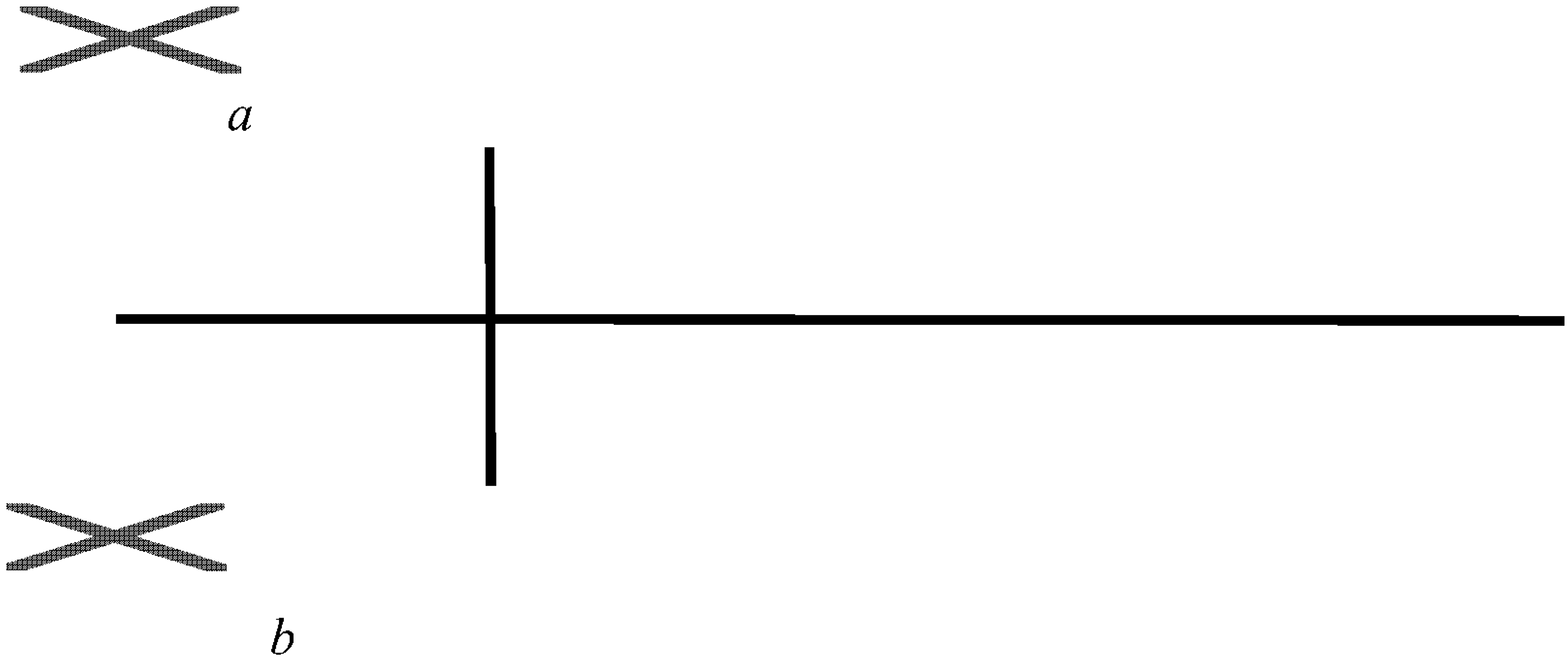


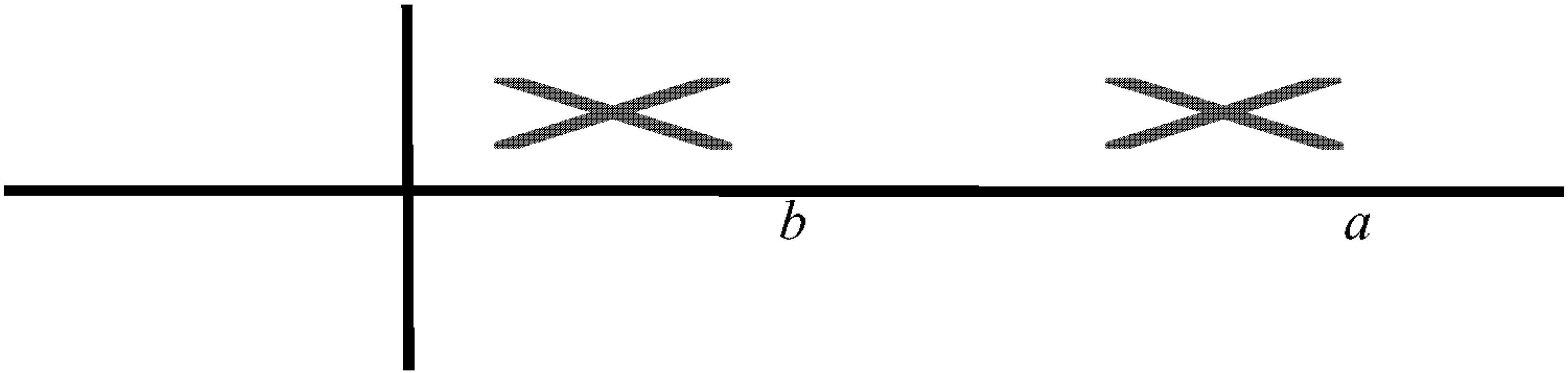

a
θ
b

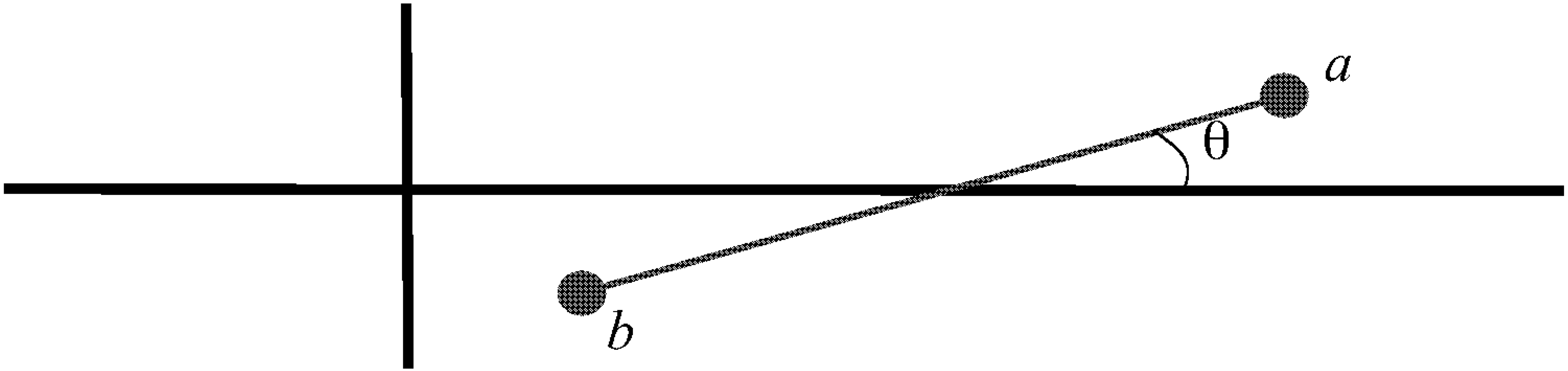

a
b

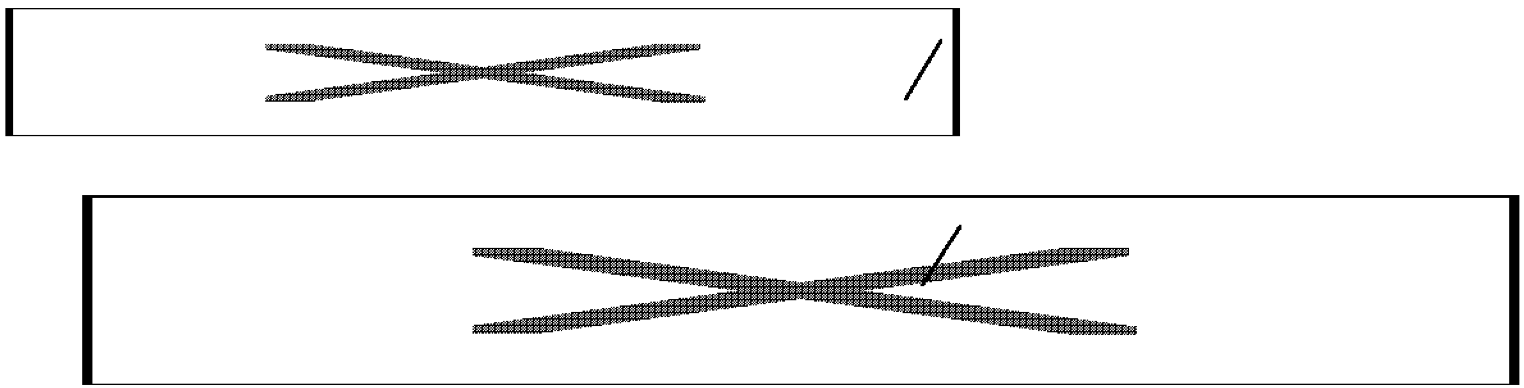

a
b


b
a


a
b

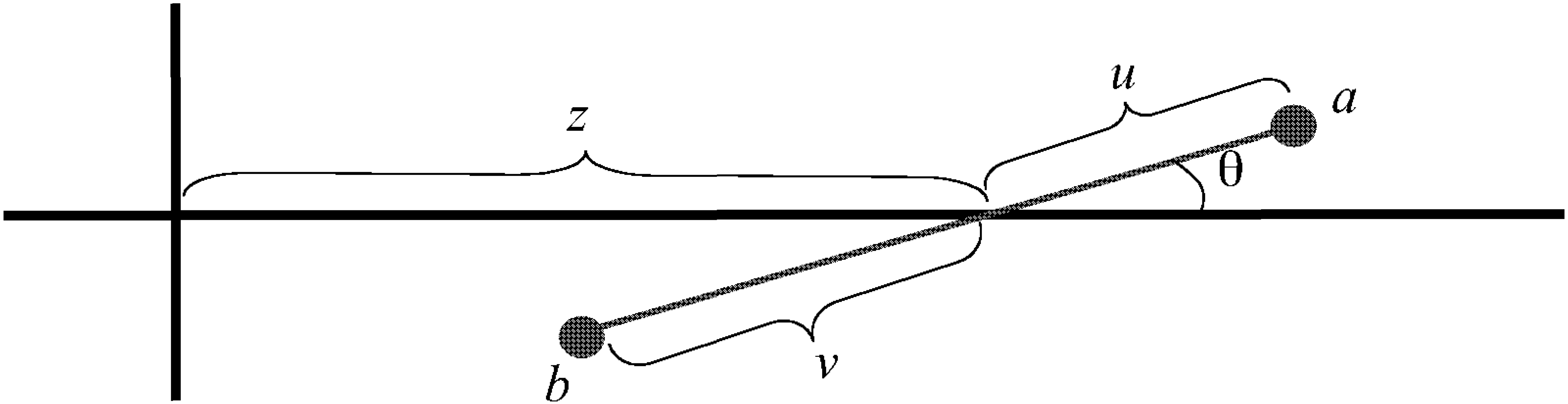

b
a



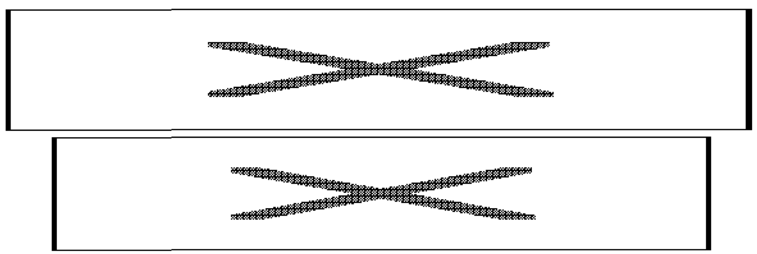



**Claim:** For $\varepsilon < \varepsilon_0$, $P_\varepsilon < \varepsilon^2$

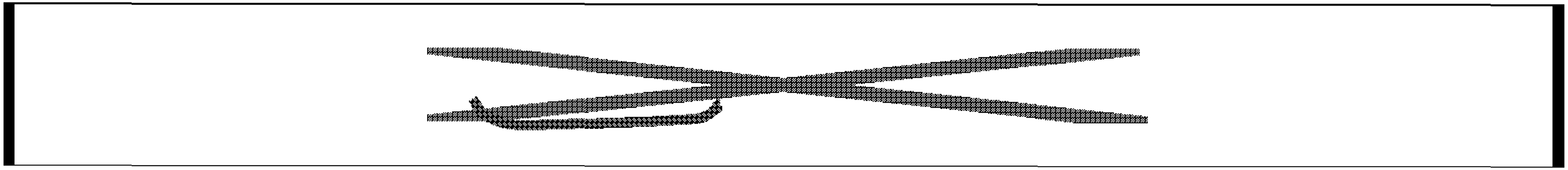
# Change of variables



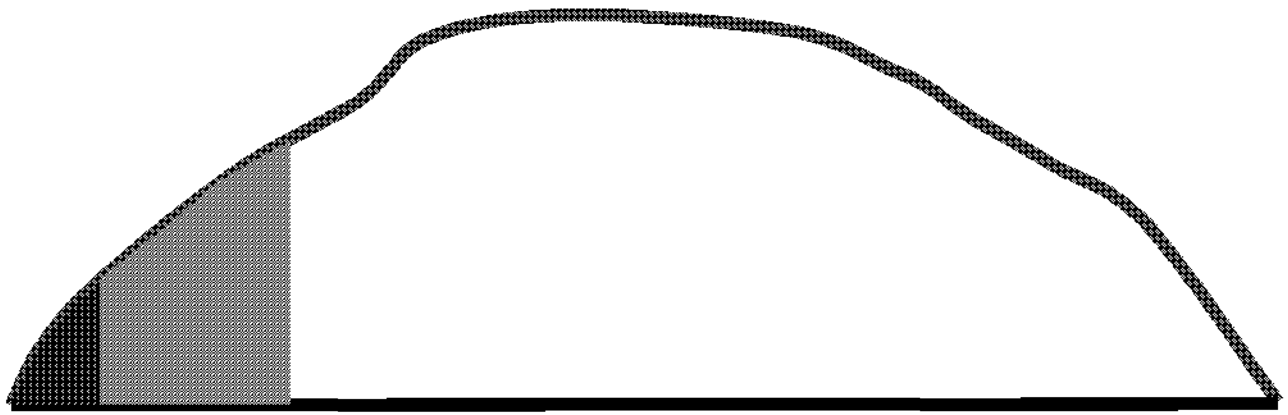

$$da\ db = \left|(u+v)\sin(\theta)\right|\ du\ dv\ dz\ d\theta$$



# Analysis: For $\varepsilon < \varepsilon_0$, $P_\varepsilon < \varepsilon^2$

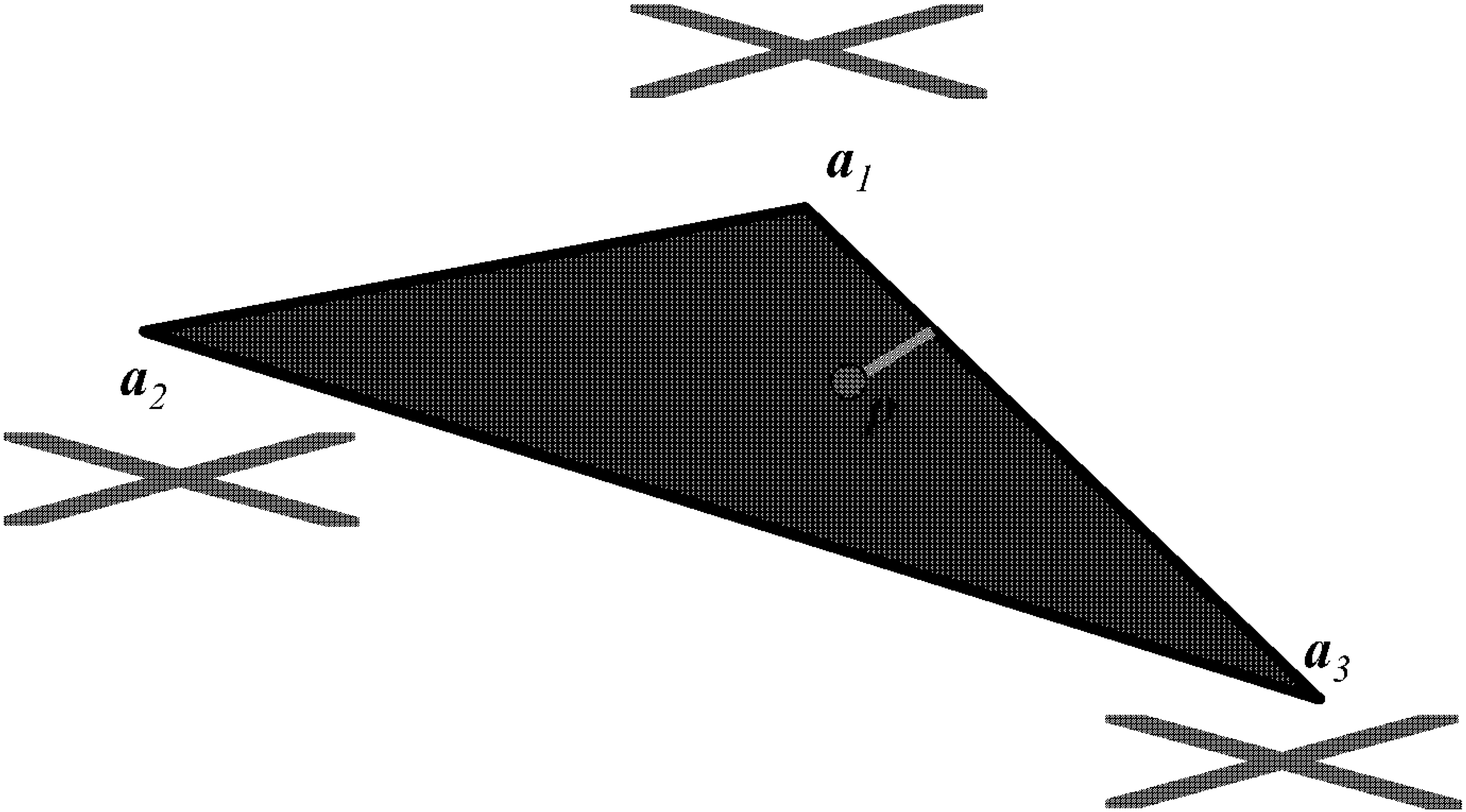


Slight change in $\theta$ has little effect on $v_i$
for all but very rare $u,v,z$

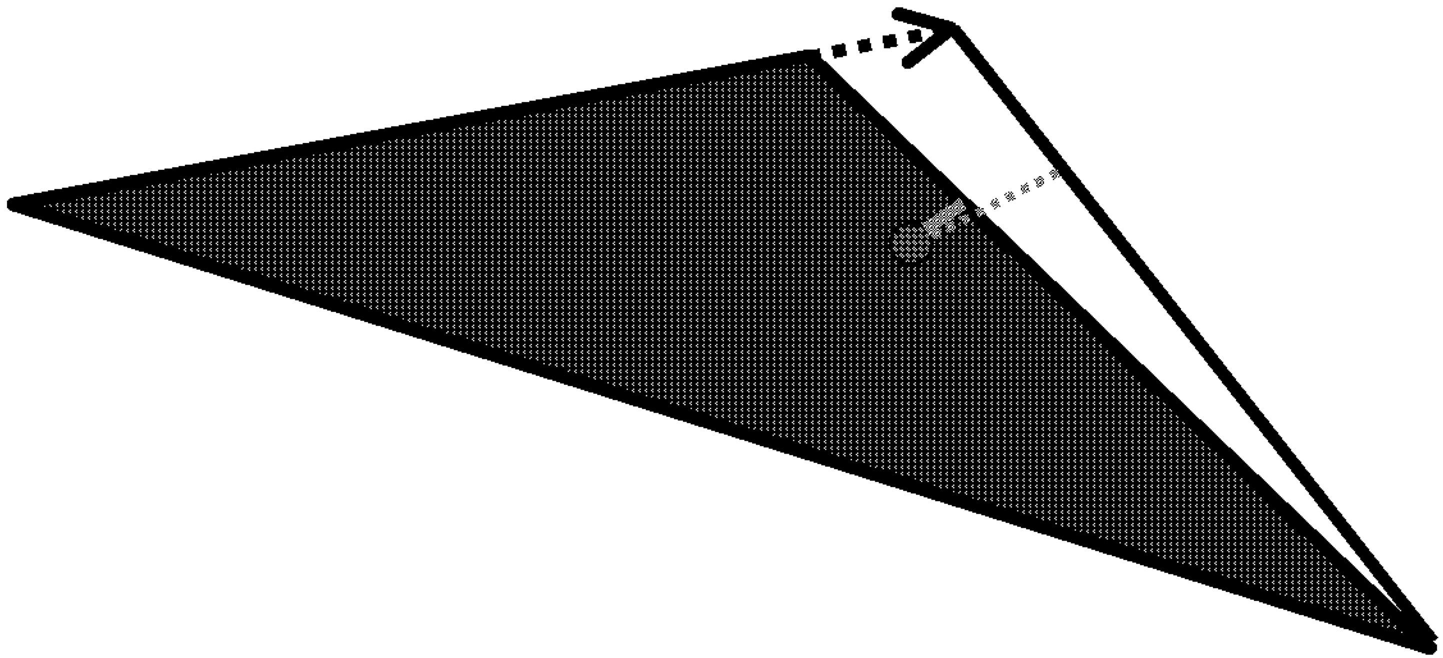


# Distance:

## Gaussian distributed corners



# Idea: fix by perturbation



# Trickier in 3d