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INTRODUCTION TO NONLINEAR SCIENCE

Mauro Rustici

Universita' di Sassari
Dipartimento di Chimica
Sassari, Italy

Introduction to nonlinear science

Mauro Rustici

Università di Sassari, Dipartimento di Chimica, Via Vienna 2, 07100 Sassari, Italia, rustici@ssmain.uniss.it

I. Introduction

In these lectures I would like developping an introductory course to nonlinear science in order to get some tools to tackle complexity.

Particularly we want deal with some nonlinear dynamics application to ecological systems.

First of all we can ask "what is nonlinearity". Our educational system is built on the idea that a natural system subjected to well defined external condictions will follow a unique course and that a slight change in these condictions will likewise induce a slight change in the system's responce. This idea, along its corollaries of reproducibility unlimited predictibility and semplicity, has long dominated our thinking and has led to the image of a "linear" world: a world in which the observed effects are linked to the underlying causes by a set of laws reducing for all pratical purposes to a simple proportionality.

This idea is now being challenged and provide only a partial view of the natural world.

The principal difference between linear and nonlinear laws is connected to superposition property. As everybody kwnow any two solutions of a linear equation can be added together to form a new solution; this is the *superposition principle.* This principle is used to solve, indipendent of other complexities, essentially any linear problem. Naively we can breaks the problem into many small pieces, then adds the separate solutions to get the solution to the whole problem.

In contrast two solutions of a nonlinear equation cannot be added together to form another solution. Superposition fails. Thus, one must consider a nonlinear problem *in toto*; one cannot break the problem into small subproblems and add their solutions. It is therefore perhaps not surprising that no general analytic approach exists for solving typical nonlinear equations.

Physically, the distinction between linear and nonlinear behavior is best abstracted from examples. For istance, when water flows through a pipe at low velocity, its motion is *laminar* and is characteristic of linear behavior: regular, predictable and descriVable in simple analytic mathematical terms. However, when the velocity overcomes a critical values, the motion becomes *turbolent*, with localized eddies moving in a complicated, irregular, and erratic way that typifies nonlinear behavior. By reflecting on this and other examples, we can isolate at liest three characteristics that distinguish linear and nonlinear phenomena.

- The motion itself is qualitatively different. Linear systems typically show motth, regular motion in space and time that can be described in terms of well-behaved functions. Nonlinear systems, however, often show transitions from smooth motion to chaotic, erratic or as will see later, even apparentely random behavior.

- The response of a linear system to small changes in this parameters or to external stimulation is usually smooth and in direct proportion to the stimulation. But for nonlinear systems, a small change in the parameters can produce an enourmous qualitative difference in the motion. Further, the response to an external stimulation itself: for example, a periodically driven nonlinear system may exhibit oscillations at, say, one half, one-quarter, or twice the period of the stimulation.

- A localized "lump" or pulse, in a linear system will normally decay by spreading out as time progressed. This phenomenon, known as dispersion, cause waves in a linear systems to lose their identity and dye out, such as when low-amplitude water waves disappears as the move from the original disturbance. In contrast, nonlinear systems can have highly coherent, stable localized structures -such as the eddies in turbolent flows- that persist either for long times or, in some idealized mathematical model, for all time.

To go beyond these qualitative distinctions, let me start with a very symple physical systems in order to show how nonlinearities arise in a very broad range of natural phenomena, from classical mechanics to biology.

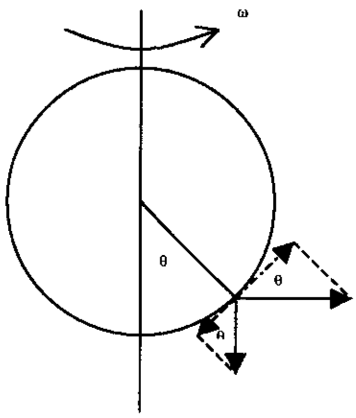

Figure 1: Schematic representation of the motion of a mass m on a vertical rotating hoop

A. Nonlinear behavior in classical mechanics

We consider a rigid vertical ring of radius r in the field of gravity. A mass m is initially placed at an angle θ_0 from the lower end of the vertical diameter and is allowed to move along the ring with no friction. As long as the ring as whole is at rest, it will perform a periodic motion around the position $\theta = \theta_0$ when $\theta_0 \neq 0$ or will remain fixed at *equilibrium* state for ever if $\theta = \theta_0$.

Suppose to rotate the ring around its vertical diameter with a costant angular velocity ω. Experiments shows that as long as ω is small the mass m still oscillates around the seme equilibrium position as before. But, beyond a *critical threshold* ω_c, one observes that the situation change completely and the mass oscillates around a new equilibrium position corrisponding to a nonzero value of the angle θ. Actually there exist two such equilibria, placed simmettrically around the vertical diameter. There is not preference for either of the equilibria to be chosen: the choice is dictated by the initial position and velocity, in many respect is governed by chance. Still, in a given experiment only one on these equilibria will be realized and the mass oscillate around it. To the observer, this will appear an asymmetric realization of a perfectly symmetric physical situation. We refer to this phenomenon as *symmetry breaking*, in this case the reflection symmetry around the vertical diameter has been broken.

We can organize this information on a digram (Fig.2) in which the equilib-

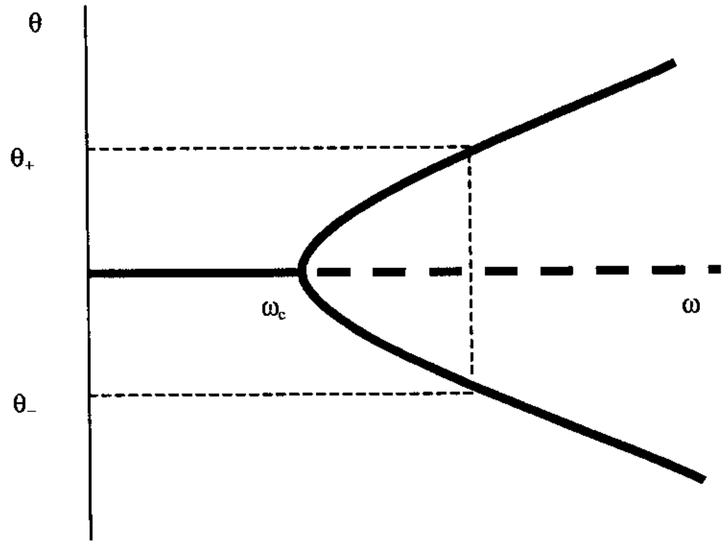

Figure 2: Bifurcation of new equilibria θ_+, θ_- as the angular velocity ω of the hoop exeeds the threshold value ω_c

rium position θ is plotted against the angular velocity ω. Below the threshold ω_c only one position ($\theta = 0$) is available. Beyond ω_c this state become unstable and we express this by dashed line. When $\omega > \omega_c$ two new equilibria become available. So we obtain to branches with merges at $\omega = \omega_c$ but separate when $\omega \neq \omega_c$. This phenomenon is known as *bifurcation.*

One of the manifestations of this nonlinear response is now the systems disposes of multiple solutions which it can choose. Physical systems can present different types of behavior as the value of characteristic parameter is varied.

B. Thermal convection

Imagine a thin fluid layer between two horizontal conducting plates in the field of gravity. The plates are mantained at fixed temperature T_0 and T_1. Suppose first that $T_0 = T_1$; $\Delta T = 0$. The layer sooner or later reach the state of thermodynamic equilibrium, characterized of absence of bulk motion

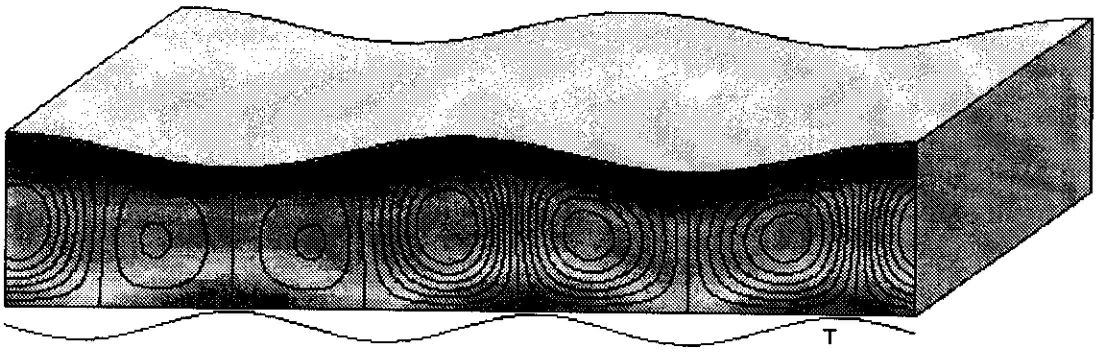

Figure 3: Pattern formation in thermal convection

and by a uniform temperature and density througout. Suppose now we gradually apply a $\Delta T > 0$. This *thermal constrain* displaces the system from equilibrium and gives rise to heat conduction from the lower (hot) plate to the upper (cold) one and a concomitant temperature distribution along the vertical. As long as ΔT is weak the behavior will be limited to this. In particular, the fluid will remain at rest and an observer moving along a horizontal plane will still perceive a uniform temperature and density enviroment.

This situation change completely when ΔT exceeds a *critical threshold* ΔT_c. The fluid ceases now to be at rest and begins to perform bulk movement organized in the form of well-structured convection cells, known as Bèrnard cells (Fig.3). In a given cell the fluid moves upward until reaches the upper boundary and follows it, then moves downward until reaches the lower boundary and follows it, and starts all over egain. If this motion occurs in the clockwise direction in the adiacent cells it will occur in the opposite, counterclockwise direction. The dimension of the cells is determined by the geometry of the system. Despite the fact that the fluid moves, from a macroscopic point view we have a stationary nonequilibrium state.

We have seen that two adjacent cells rotate in opposite directions (Fig.3). At a given point in space, therefore, a small volume element of the fluid can find itself at two distinct states in the sense that it can be part of a cell rotating clockwise or counterclockwise. It is also important to note that it is not possible to assign some preference for either of these two directions:

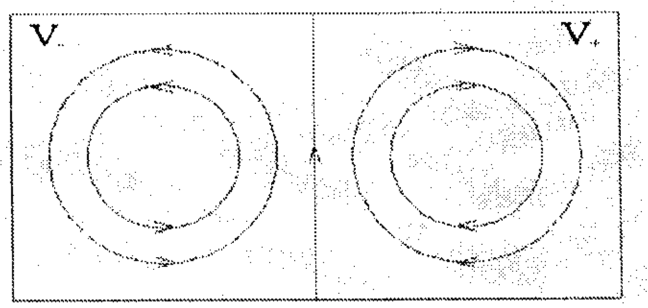

Figure 4: Rayleigh-Bernard convection

the particular direction that will be chosen will be dictated by the locally condictions at the moment of experiments which are determined by random elements such temperature fluctuations, dust particles and mechanical vibrations. An observer moving along the horizontal plane homogeneity will be broken. In other words the Bernard convection breaks the translational symmetry in the horizontal direction. The situations is somewaht analogous to a liquid-solid transiction, well the fully isotropy and rotational symmetry of the liquid phase are broken in favor of the less symmetric crystalline solid.

In analogy with the previous example we may organize the above information in the form of a *bifurcation diagram.* A convenient stable variable to be plotted agianst the constrained ΔT is now the vertical component of the velocity, W. Before the threshold ΔT_c the fluid is at rest $W = 0$. Beyond ΔT_c we observing an ascending ($W > 0$) or a descending ($W < 0$) movement. These two branches of states coalesce at ΔT_c with the state of rest, but bifurcate out of it for $\Delta T > \Delta T_c$.

Despite the completely different nature of the systems considered here we observe that in both cases nonlinear behavior, associates with multiplicity of states, emerges through a bifurcation mechanism when a constrain acting on the system exceeds a critical threshold.

One of the reasons that make the Bernard problem so important in nonlinear science is that the system can also undergo a whole series of successive transitions and practically the entire repertoire of nonlinear dynamics behavior can be observed.

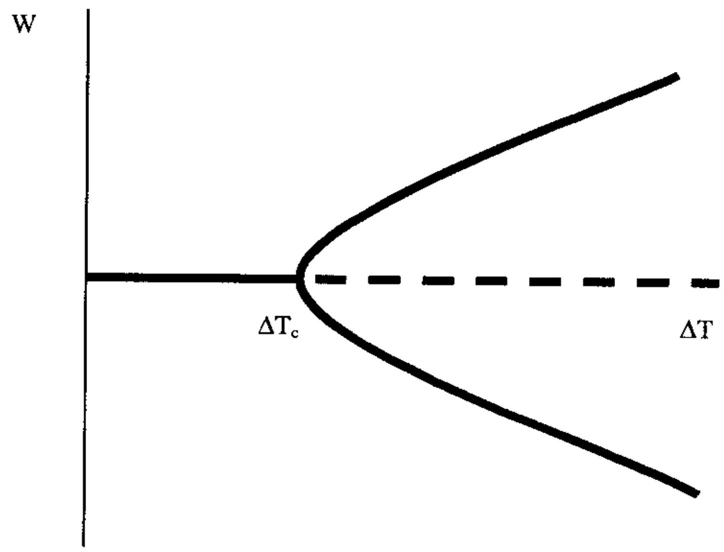

Figure 5: Bifurcation diagram for the onset of termal convection beyond a critical temperature difference

C. Nonlinear phenomena in chemistry

For a long time chemists thought that a homogeneous, time-indipendent state should enventually emerge from any chemical transformation. Nevertheless on 1951 an important oscillation reaction was discovery by Belousov. The paper was sent to a journal and rejected due to the referee comments "it is impossible". Why impossible? Obviously because chemical reactions should go to the thermodynamic equilibrium. These should go to the equilibrium smoothly - this was the conventional opinion of that time.

The study of this reaction was continued by Zhabotinsky and is now known as the Belousov-Zhabotinsky reaction or symply BZ reaction. The BZ reaction is now considered the prototype oscillator.

The simplest way to prepare the BZ reaction is to mix malonic acid, sodium bromate, sulfuric acid and ferroin. If add a droplet (5 ml) of the solution to a container (Petri dish, 6 cm in diameter) so that the thickness of the layer is 0.5-1 mm, there appears colorful spatio-temporal patterns (Fig

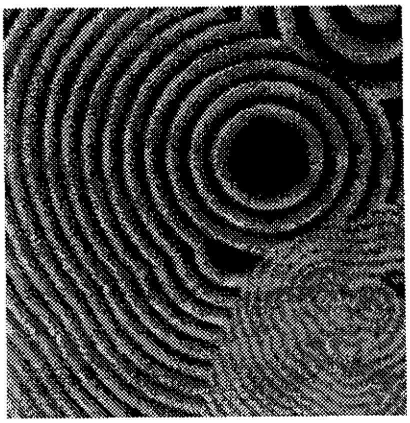

Figure 6: Wave propagation in a two-dimensional layer of BZ reagent

5). In thicker layers there is an interference of hydrodynamic flows with the reaction.

Thank to the design of open reactors the reaction can nowdays be carried out without interruption for long period of time. In a well-stirred opened reactor the rate at which the reactanct are pumped in the reactor can be utilized as control paramiter. Under slow pumping condictions the chemicals will remain in the reactor for a very long time and will reach the chemical equilibrium. Exists a critical threshold for the flow upon which stationary behavior is no longer possible and sustained oscillations are observed. The birth of this new regime is associated with the breacking of translational symmetry in the time domain.

In a closed reactor diffusion and convective motion can coupled ti local kinetcs and regulr patterns in space and time in the form of propagating wave front arises (see Fig.5). The waves appear primarly in two different forms: circular fronts displaying a roughly cylindrical cymmetry around an axis perpendicular to the layer; and spiral fronts rotating in the space clockwise or counterclockwise.

D. Conlusion

The above discussion showes the typical manifestation of nonlinearity in different field. The examples that I have showed forms a set of very simple systems in which is possible to observing the characteristic complex systems. These systems can be well studied by numerical and esperimental tests and for that reason form a very intersting groups in the research field. Nevertheless non linear dynamics is now applied in every field of nature also where is not so simple to verify the model proposed.

It is intersting to note that one of the first model that it has been introduced in the world of complexity is an ecological model. Vito Volterra first proposed a simple model for the predation of one species by another to explain the levels of certain fish in the Adriatic.

II. Continous Models for Interacting Population

When species interact the population dynamics of each species is affected. In general there is a whole web of interacting species, called *trophic web*, which makes for structurally complex communities. I would like to consider systems involving two species. There are three main types of interaction.

- If the growth rate of one population is decreased and the other increased are in a *predator-pray* situation.
- If the growth rate of each population is decreased then it is *competition*
- If each population's growth rate enhanced then it is called *mutualism* or *symbiosis*.

A. Lotka-Volterra Systems

The model consists of three irreversible steps. X is the population of rabbits (pray), which reproduce autocatalytically. a is the amount of grass, which we assume to be constant, or at least in great excess compared with the comsumption by the rabbits. Y is the population of lynxes, and P represents dead lynxes.

$$A + X \rightarrow 2X \tag{1}$$
$$X + Y \rightarrow 2Y \tag{2}$$
$$Y \rightarrow P \tag{3}$$

As indicated, each steps is irreversible: rabbits will never turn back into grass, nor dead linxes into lives ones. We can write down a system of differential equations to describe the behavior of the predator and pray species:

$$\frac{dX}{dt} = kX - sXY \quad (4)$$
$$\frac{dY}{dt} = sXY - fY \quad (5)$$

or

$$\frac{dX}{dt} = X(k - sX) \quad (6)$$
$$\frac{dY}{dt} = Y(sX - f) \quad (7)$$

The assumption of this model are

- The pray in the absence of any predation grows in a Malthusian way; this is the kX term in (6).
- The effect of predationis to reduce the pray's per capita growth rate by a term proportional to the pray and predator populations; this is the $-sXY$ term.
- In the absence of any pray for sustenance the predator's death rate results in exponational decay, thats is the $-fY$ term.
- The pray's contribution to the predator's growth rate is sXY

This model is know as the *Lotka-Volterra model* since the same equation were also derived by Lotka from a hypothetical chemical reaction which he said could exibit periodic behavior in the chemical concentration.

In order to understand the model showed above we can ask the following question

1 Under what circumstances might this model be reasonable?

2 How do we solve the resulting system of equations?

3 Under what conditions (if any) are the populations in equilibrium?

4 Do the equilibrium point imply stability?

In the linear case, we are able to develop a generalized solution procedure that we could apply to any system of 2 coupled, linear, homogeneous, ordinary differential equations. However, such a generalized procedure is not possible for nonlinear equations, even those a simple as the ones presented here. We are forced to either look at specific solution points only (e.g. equilibrium points), or develop numerical approximations to the solution(s). There are many approaches to numerically solving systems of nonlinear differential equations.

The equations (6,7) forms a set of differentialy ordinary nonlinear equations that in general and in the more compact form can be written as

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X}, \lambda) \tag{8}$$

In writing eq (8) I have also accounted for the fact that the system involves a number of parameters λ, referred to as *control parameters.*

We study the evolution of our system, as described by eq. (8) into the abstract n-dimensional space spanned by the full set of variables $(X_1 \cdots, X_n)$. We refer to the space as the *phase space.* The initial condiction can not define with a infinite precision. In the phase space rapresentation this means that the experimentally accessible state will not be given by a point but rather by a volume $\Delta\Gamma_0$ surrounding such point and represents the finite ingnorance that is present at the start of every experiment or sequence of observations. Every point in this volume element is a initial condition of (8), and so from each phase point starts a trajectory as t increases. The collection of such trajectories in phase space forms a flow analogous to the flow of a fluid. The question that we can ask is how the size of the volume element of $\Delta\Gamma_0$ change with the time? It is possible showing that the size of the volume depends by $\nabla\mathbf{F}$. If $\nabla\mathbf{F} = 0$ the system is conservative and the size does not change with the time; if $\nabla\mathbf{F} < 0$ the system is dissipative and the volume contract during the flow. Therefore, whe shall introduce the classical attractors: stable equilibria, stable limit cicles, and stable periodic or quasiperiodic orbits. Stable equilibria are zero-dimensional, while limit cycles are closed, one dimensional curves in phase space.

As the time goes to infinity in a dissipative system, the asymptotic motion takes place on a set of lower dimension respect the phase space. We can also ask what happens when the motion in the phase space is bounded, but the orbits is condemned to wander for ever because all the available equilibria point, limit cycles and quasiperiodics orbit are unstable? The answer to this question leads to the study of point sets that are called *strange* attractors, because they have fractal dimension. These attractors have some eccentric dynamics properties and a lot of these lead to a chaotic dynamics. A defining attribute of an acctractor on which the dynamics is chaotic is that it displays exponentially sensitive dependences on initial conditions. The exponantial sensitivity of chaotic solution means that, as the time goes on, small errors in the solution can grow very rapidly with time. Hence, after some time, effects such as noise and computer roundoff can totally change the solution from what it would be in the absence of these effects. Thus, even the smallest perturbation, such as a butterfly flapping its wing, eventually has a large effect. Long term prediction becomes impossible. Chaos describes the dynamics on attractor, while strange refers to the geometry of the attractor. For most cases involving differential equations, strangeness and chaos communly occur together.

For the case illustrated in our initial example (Lotka-Volterra model) chaos can not be occurs. Because we need at least a threedimensional phase space. Nevertheless the bidimensional phase space defined by the variable preys and predators could contain equilibria points or close orbites. Such orbits describe a periodic dynamics that could explain the oscillatory levels of certain ecological species. In order to make this type of study we need to evaluate the stability of the equilibrium point. At an equilibrium point, the velocity vector $\mathbf{F}$ vanishes. If $\bar{\mathbf{X}}$ represent such point this is stable if it attracts nearby trajectories, unstable if it repels them. In order to analyze the local stability, we expand $\mathbf{F}$ in power series of $\mathbf{x}$ referred to as a perturbation around $\bar{\mathbf{X}}$. If we keep only the linear term the (8) can be written as

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathcal{L}(\lambda) \cdot \mathbf{x} \tag{9}$$

were $\mathcal{L}(\lambda)$ is the Jacobi matrix. It follows that eq.(9) admits solutions that depend on time exponantially

$$\mathbf{x} = \mathbf{u}\, \mathrm{e}^{\omega t} \tag{10}$$

Substituting into (9) one find that $\mathbf{u}$ and ω must satisfy the relations

$$\mathcal{L}(\lambda) \cdot \mathbf{u} = \omega \mathbf{u} \tag{11}$$

In other words **u** and ω are, respectively, eigenvectors and eigenvalues of $\mathcal{L}(\lambda)$ and the stability is thus reduced to an eigenvalue problems. An important point is that indipendentely of the properties of **u**, which takes into account the structure of **x** as a vector in phase space, knowledge of the eigenvalue ω provides one with a full solution of the problem of stability. Separating ω into real and imaginary parts we have from (10)

$$|\mathbf{x}| \approx e^{\mathrm{Re}\omega t} e^{i(\mathrm{Im}\omega t)} \tag{12}$$

It follows that
if $\mathrm{Re}\omega < 0$, $|\mathbf{x}|$ is exponentially decreasing and hence the reference state $\mathbf{x} = 0$ (or $\mathbf{X} = \bar{\mathbf{x}}$ is *asymptotically stable*;
if $\mathrm{Re}\omega > 0$ the perturbations grow exponentially and hence the reference state is *unstable*.

If the system involving two variable the $\mathcal{L}$ assume the following form

$$\mathcal{L} = \begin{pmatrix} \partial\dot{X}/\partial X & \partial\dot{X}/\partial Y \\ \partial\dot{Y}/\partial X & \partial\dot{Y}/\partial Y \end{pmatrix}_{X^*,Y^*} = \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix}_{X^*,Y^*} \tag{13}$$

where X and Y are the variable. We can evaluated the eigenvalue ω resolving the following determinant

$$\begin{vmatrix} l_{11} - \omega & l_{12} \\ l_{21} & l_{22-\omega} \end{vmatrix}_{X^*,Y^*} \tag{14}$$

The equation (14) can be expanded to give the characteristic equation that takes the explicit form

$$\omega^2 - T\omega + \Delta = 0 \tag{15}$$

where T and Δ are recspectively, the trace and the determinant of the matrix $\mathcal{L}$.

$$\begin{aligned} T &= l_{11} + l_{22} \\ \Delta &= l_{11}l_{22} - l_{12}l_{21} \end{aligned} \tag{16}$$

We can applies the linear stability analysis to our pray-predator systems eq. (6,7).

We can determine points in phase space where equilibrium occurs in the Lotka-Volterra relationships by setting derivative equal to zero and solving the resulting algebraic relationships as follows:

$$X(k - sX) = 0 \tag{17}$$
$$Y(sX - f) = 0 \tag{18}$$

Hence the equilibria are $(X^*, Y^*) = (0,0)$ and $(k/s, f/s)$. To classify these fixed points we compute the Jacobian $\mathcal{L}$ evaluated on the equilibria. For the first equilibria $(0,0)$ we obtain

$$\mathcal{L} = \begin{pmatrix} k - sy & -sx \\ sy & sx - f \end{pmatrix}_{X^*,Y^*} = \begin{pmatrix} k & 0 \\ 0 & -f \end{pmatrix} \tag{19}$$

the characteristic equation become

$$\omega^2 - (k - f)\omega - kf = 0 \tag{20}$$

Hence we obtain two eigenvalues with opposit sign (k,-f). The steady state is a saddle point and is unstable. For the non trivial solution $(k/s, f/s)$ the $\mathcal{L}$ matrix is

$$\begin{pmatrix} 0 & -f \\ k & 0 \end{pmatrix} \tag{21}$$

the equation characteristic become

$$\omega^2 + kf = 0 \tag{22}$$

because the trace is 0. This particular case is referred to a center and the trajectories are close line surrounding the steady state. A center singularity since the eigenvalues are purely imaginary. Because the $\mathrm{Re}\,\omega = 0$ the steady is neutrally stable. So the solutions of the singular point are periodic in time.

A major inadequacy of the Lotka-Volterra model is derive from the fact that the solutions are not structurally stable. In fact any small perturbation will move the solution onto another trajectory which does not lie everywhre

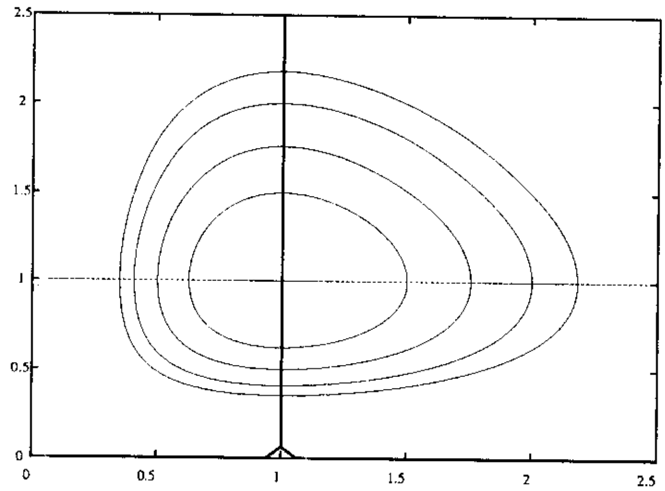

Figure 7: Closed phase plane trajectory starting from different inititial condictions.

close to the original one. Thus a small perturbation can have a very marked effect on the amplitude of the oscillation. They are called conservative systems. They are usually of little use as models for real interacting populations. However the method of analysis of the steady states is typical.

There have been many attemps to apply the Lotka-Volterra model to real-world oscillatory phenomena. In view of the systems structural instability, they must essentially all to fail to be of quantitative pratical use.

B. Realistic Predator-Pray Models

A lotka-Volterra model show that simple predator-pray interactions can give a oscillatory behavior of the populations. Depending on the detailed system such oscillations can grow or decay or go into a limit cycle oscillation or even exhibit chaotic behavior, although in the latter case there must be at least three interacting species, ore the model has to have some delay terms.

A limit cycle solution is a closed trajectory in the predator-prey space which is not a member of a continuous family of closed trajectories suche as the solutions of the Lotka-Volterra model. A stable limit cycle trajectories is such that any small perturbation from the trajectory decay to zero. In other words a limit cycle is an isolated close trajectory. Isolated means that neighboring trajectories are not close; they are spiral either towards or away from the limit cycle. If all neighboring trajectories approcach the limit cycle, we say the limit cycle is stable or attracting. Otherwise the limit cycle is unstable.

One of the unrealistic assumptions in the Lotka-Volterra models is that the pray grow in absence of predations by a malthusian growth. In absence of any predaction we might expect the prey to satisfy a logistic growth. In this model the birth rate is dependent of the population of the pray.

This means that the costant k for the equation (4) can be express as $a - bX$ where a and b are constants. So the eq. (4) becomes

$$\frac{\mathrm{d}X}{\mathrm{d}t} = (a - bX)X - sXY \tag{23}$$

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = sXY - fY \tag{24}$$

To obtain the steady state(s) of this model, we set eqs 23-24 equal to zero and solve for X and Y. We find the solutions $(0, 0)$ and $(f/s, (as-bf)/s^2$. To analyze the stability of this state, we must calculate the matrix $\mathcal{L}$ obtaining

$$\mathcal{L} = \begin{pmatrix} a - 2bx - sy & -sx \\ sy & sx - f \end{pmatrix}_{X^*,Y^*} \tag{25}$$

for the trivial solution $(0, 0)$ we obtain

$$\begin{pmatrix} a & 0 \\ 0 & -f \end{pmatrix} \tag{26}$$

where the eigenvalues are $a, -f$. The steady state is unstable (saddle point).

For the solution $(f/s, (as - bf)/s^2$ the $\mathcal{L}$ matrix becomes

$$\begin{pmatrix} -bf/s & -f \\ (as - fb)/s & 0 \end{pmatrix} \tag{27}$$

Hence $T = -bf$ and $\Delta = f(as - fb)/s$ and the characterisctic equation will be

$$\omega^2 + \frac{\omega bf}{s} + \frac{f}{s}(as - fb) \tag{28}$$

we observe that the trace (T) is always negative (dissipative system), while the determinant (Δ) can be either positive or negative. The stability of the steady state will depend of the sign of the determinat in this case. As we vary a, s, f or b the determinant passes trhough zero. The eignevalues of the steady state will change; a bifurcation will occur. If $as > bf$ the determinat is positiv and will obtain two negative eigenvalues. The perturbations decreases and the steady state is stable. When $as < bf$ will obtain two positive eigenvalues and the perturbation increases with the time, hence the solution become unstable. Nevertheless the condition $as > bf$ must be satisfied so that the steady state predator levol y is positive. So the trace is always negative and the determinat is always positive, so that the steady state is always stable. In other words, its neutral stability has been lost.

The lesson to be learned from this example is that a relatively minor change in uquations has a major influence on the predictions. In particular this means that neutral stability, and thus also the oscillations that accompany a neutrally stable steady state, tend to be somewaht ephemeral. This is a serious criticism of the realism of the Lotka-Volterra model.

In order to improve the model we can correct the predation term. The last is the functional response of the predator to change in the prey density, generally shows some saturation effect. Instead of a predator responce sXY, as in the Lotka-Volterra model, we take for example $cS(X)Y$ where $S(X) = nX/(k + X)$. In this way the predator-prey interaction is saturated: each predator maximally kills n prey per time unit. In this way we can rewrite the model as

$$\frac{\mathrm{d}X}{\mathrm{d}t} = (a - bX)X - n\frac{X}{k + X}Y \tag{29}$$

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = n\frac{X}{k + X}Y - fY \tag{30}$$

If we perform the stability analysis of this model we can prove that this system contain a limit cycle. Fig.8 shows the corrisponding periodic behavior of the prey and predator. Parameter values are $a = 1$, $f = 0.5$, $k = 25$, $b = 0.01$, $c = 2$ and $n = 0.5$.

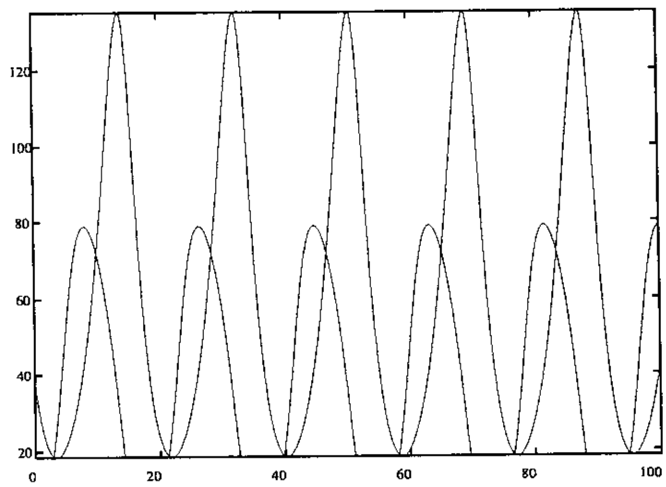

Figure 8: Periodic behavior of the pray and predator pupulation.

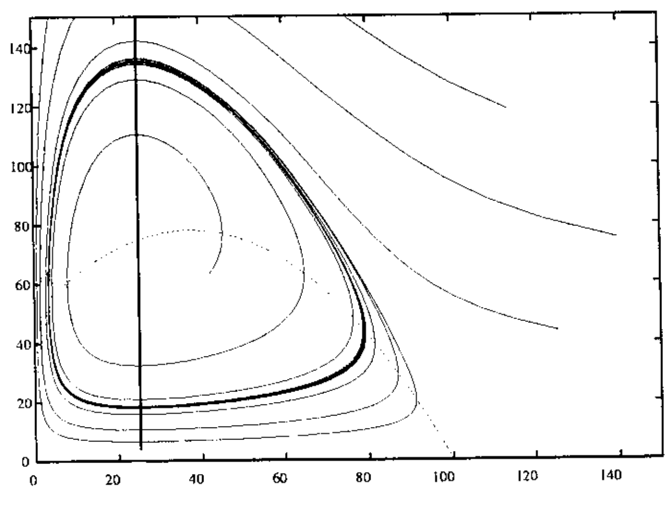

Figure 9: Limit cycle solution for the predator-prey sistem.