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POSITRONIUM AND MUONIUM HYPERFINE STRUCTURE AND A NEW DETERMINATION OF α. BETHE-SALPETER BOUND-STATE PERTURBATION THEORY *

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Dr. Picasso mentioned our work[1] in the opening talk at this Conference when he gave Fulton, Owen and Repko's latest theoretical result for the muonium hyperfine splitting. Actually, this calculation was an afterthought and at the time hardly seemed worth the effort, as the uncertainty in both the measurement and theory for this quantity was so large because of the Ruderman effect[2]. We primarily were interested in calculating the hyperfine structure of positronium which did not suffer from this limitation. Our interest in positronium was motivated by the following considerations:-

(1) The positronium ground state triplet-singlet splitting affords, potentially, a very accurate test of quantum electrodynamics. This follows since to our order of approximation, the bound electron-positron system is free from contamination by hadronic effects or unusual leptonic corrections. Thus agreement between theoretical and experimental determinations of the triplet-singlet splitting is necessary in any systematic check of the predictive powers of quantum electrodynamics.

(2) Furthermore, positronium is the only experimentally accessible bound state system which can and must be described by using the relativistic two-body equation for interacting fermions. The external field approximation used in treating the Lamb shift, etc., is not applicable here as both particles must be treated on an equal footing.

When we, (Fulton, Repko and myself) started our calculation, quantum electrodynamics was in a state of disarray[3,4]. Different values for α and the Lamb shift were "floating around". The situation for positronium was equally unclear.

The positronium hyperfine splitting had first been measured by Dr. Martin Deutsch in 1952[5], and subsequently several times by Hughes and his collaborators (their latest result came out in 1970)[6]. For the positronium hyperfine frequency, ν, they obtained

$$\nu_{exp} = 2.03403(12) \times 10^5 \text{ MHz} \quad . \tag{1}$$

Their experimental arrangement is illustrated in Figure 1, where the Na^{22} is the positron source. By varying the static magnetic field he determined the field intensity which gave the greatest number of two photon decays.

This magnetic field at which the increase of two photon-positronium decays is observed is related to the Zeeman pattern and the frequency of the r-f field (see Figure 2), from which (via the easily derived Breit-Rabi formula, $\lambda_{3,4} = \bar{E} \pm \frac{\Delta W}{2}\sqrt{1 + \chi^2}$, $\chi^2 = \frac{2\mu_o \, g \, Hz}{\Delta W}$; Hz = magnetic field. $\lambda_{3,4}$ are the energy eigenvalues of the Zeeman substates while ΔW is the triplet-singlet splitting in the absence of a magnetic field)[7,8] the triplet-singlet splitting can be determined.

The previous theoretical value obtained by Karplus and Klein[9] for the positronium hyperfine splitting was

$$\nu_{Th}^{KK} = 2.03381(36) \times 10^5 \text{ MHz} \tag{2}$$

for $\alpha = 137.03608$. The "theoretical error" is our estimate of the expected order of magnitude of the uncalculated diagrams[10]. We see that the theoretical uncertainty limited the usefulness of this quantity in providing a test of quantum electrodynamics and the Bethe-Salpeter equation.

With the result of our calculation we have been able to improve the Karplus and Klein result so that the present theoretical value reads[10]

$$\nu_{Th}^{FOR} = \alpha^2 R_\infty [7/6 - (16/9 + \ell n\ 2)\frac{\alpha}{\pi} + \frac{3}{4}\alpha^2 \ell n\ \alpha^{-1}]$$

$$= 2.03415(7) \times 10^5 \text{ MHz} \quad . \qquad (3)$$

Comparing this with (1) we find that there is agreement between theory and experiment within quoted errors.

We extended our calculation to muonium which gave an additional theoretical contribution $\Delta\nu_m$[1]

$$\Delta\nu_m = 24 \left(\frac{m_e}{m_\mu}\right)^2 [1 + \frac{m_e}{m}]^{-5} c\ \alpha^4 R_\infty \ell n\ \alpha^{-1}$$

$$= 0.0252 \text{ MHz } (5.6 \text{ ppm}) \quad . \qquad (4)$$

The Ruderman correction which was the major source of error and limited the muonium hyperfine structure as a "good" test for quantum electrodynamics had just been shown not to exist by the recent experiments of Hague[11] and Telegdi[12], who found for the ratio of the muon to proton magnetic moment, $\frac{\mu_\mu}{\mu_p}$

$$\frac{\mu_\mu}{\mu_p} = 3.183347(9) \qquad \text{(Hague)}$$

$$\frac{\mu_\mu}{\mu_p} = 3.183340(11) \qquad \text{(Telegdi)} \qquad (5)$$

respectively. Since Dr. Picasso described this last experiment, I shall not discuss it.

Using these values, equation (4) with the Brodsky and Erickson[13] result gives for the theoretical predictions

$$(\nu_m^{+h})_{wash/LRL} = 4463.323(19)\ MHz \qquad (6)$$

$$(\nu_m^{+h})_{chi/Ill.} = 4463.313(21)\ MHz \qquad (7)$$

This is to be compared with the experimental determination of ν_m[14]

$$\begin{aligned} \nu_m^{exp} &= 4463.3022(89)\ MHz \qquad \text{Chicago} \\ \nu_m^{exp} &= 4463.310(30)\ MHz \qquad \text{Yale} \end{aligned} \qquad (8)$$

Thus with the major source of uncertainty associated with the chemical correction due to the muon's environment eliminated we have another precise test of quantum electrodynamics.

Basides which, comparison of this quantity with the hydrogen hyperfine structure gives an experimental upper limit on the proton polarization[1], i.e.

$$\frac{\nu_m}{\nu_H} = \frac{\mu_\mu}{\mu_p}\left(1 + \frac{m_e}{m_p}\right)^3\left(1 + \frac{m_e}{m_\mu}\right)(1 - \delta_p - \delta_{p'} + \delta_\mu) \qquad (9)$$

where δ_p is the proton recoil correction [(-35 ± 1)ppm], $\delta_{p'}$ is the proton polarization correction, and δ_μ is the muon recoil correction. Assuming $\delta_{p'} = 0$, the left- and right-hand sides of equation (8) are, respectively, 3.1422727 and 3.1422737, where we have temporarily suppressed the uncertainties in ν_m, $\frac{\mu_\mu}{\mu_p}$ and δ_p. The difference between the above numbers leads to an upper limit on $\delta_{p'}$ of (0.34 ± 5.0)ppm which is in accord with current estimates of proton polarizability.

Yet perhaps more important, the muonium hyperfine splitting affords, using the simultaneous high precision measurements of ν_m and $\frac{\mu_\mu}{\mu_p}$ by the Chicago-Illinois group with our latest, theoretical, result in (4), an independent determination of α with accuracy comparable to that obtained from the Josephson effect[14]. We find

$$\alpha^{-1} \text{ (muonium hyperfine structure)} = 137.03617(30) \quad (10)$$

which is in good agreement with

$$\alpha^{-1} \text{ (Josephson effect)} = 137.03608(21) \quad . \quad (11)$$

The foregoing calculations are based on the Bethe-Salpeter equation[13] which is the relativistic two-body equation derived from quantum field theory. I wish to take this opportunity to briefly describe how this equation is used for these calculations and sketch our positronium calculation.

To begin with, the Bethe-Salpeter equation is the equation for the two-particle Green's function obtained by selectively summing the diagrams when the Green's function is expanded in the interaction picture or in terms of in-states[15]. A compact way of writing the Bethe-Salpeter equation for two $\frac{1}{2}$-spin particles is

$$G(34;12) = S_F'^{(1)}(3,1)S_F'^{(2)}(4,2) + S_F'^{(1)}(3,5)S_F'^{(2)}(4,6)I(56;78)G(78;12) \quad (12)$$

where

$G(34;12)$ is the two-particle propagator,

$S_F'^{(1)}(3,1)$ is the full one-particle propagator,

$I(56;78)$ is the irreducible kernel which is the set of two-particle Feynmann diagrams

and where we have replaced $x_i \to i$ and are integrating over repeated arguments.

For positronium the two-particle propagator $G(34;12)_{\ell^+\ell^-}$ is

$$G(34;12)_{\ell^+\ell^-} \equiv \langle 0|T\{\psi_H(3)\psi_H^{(c)}(4)\bar{\psi}_H(1)\bar{\psi}_H^{(c)}(2)\}|0\rangle_{NE}$$

$$\equiv \frac{\langle 0|T\{\psi(x_3)\psi^{(c)}(x_4)\bar{\psi}_H(x_1)\bar{\psi}_H(x_2)S\}|0\rangle}{\langle 0|S|0\rangle} \tag{13}$$

$$- \frac{1}{\langle 0|S|0\rangle}\langle 0|T\{\psi(x_3)\psi^{(c)}(x_4)S\}|0\rangle \frac{1}{\langle 0|S|0\rangle}$$

$$\times \langle 0|T\{\bar{\psi}(x_1)\bar{\psi}^{(c)}(x_2)S\}|0\rangle$$

where the second term arises because of the absence of exchange diagrams for non-interacting fermion and anti-fermion.

The $\psi^{(c)}$ and $\bar{\psi}^{(c)}$ in equation (13) are given by

$$\psi^{(c)} = C[\bar{\psi}]^T \quad \text{and} \quad \bar{\psi}^{(c)} = \psi^T C \quad \text{with} \quad C = i\gamma_2\gamma_0 . \tag{14}$$

Equation (12) derived from equation (13) yields for $I(x_3x_4;x_1x_2)$ (assuming the usual electromagnetic coupling)

$$I(x_3x_4;x_1x_2) = I_B(x_3x_4;x_1x_2) + I_A(x_3x_4;x_1x_2) + \ldots \tag{15}$$

where

$$I_B(x_3x_4;x_1x_2) = -ie^2\,\delta(x_3-x_1)\delta(x_4-x_2)\,\gamma^{\mu(1)}\gamma^{\nu(2)}\,D_F(x_1;x_2)_{\mu\nu}\ ,$$

$$I_A(x_3x_4;x_1x_2) = -ie^2\,\delta(x_3-x_4)\delta(x_1-x_2)\,(\gamma^{\nu}C)(C^{-1}\gamma^{\mu})\,D_F(x_1,x_3)_{\mu\nu}\ , \tag{16}$$

etc.

To obtain an equation from (12) describing bound states[16] consider the propagator in (13) for the case $(x_3)_o$, $(x_4)_o > (x_1)_o = (x_2)_o = t$, then

$$G(34;12)_{\ell^+\ell^-} \to \langle 0|T\{\psi_H(3)\psi_H^{(c)}(4)\}\,\bar{\psi}_H(1)\bar{\psi}_H(2)|0\rangle_{NE}$$

$$= \sum_n \langle 0|T\{\psi_H(3)\psi_H^{(c)}(4)\}|n\rangle\langle n|\bar{\psi}_H(1)\bar{\psi}_H^{(c)}(2)|0\rangle_{NE}$$

$$\equiv \sum_n f_n^{(pos)}(3,4)\ f_n^{(pos)\dagger}(1,2)_{t_1=t_2}\ . \tag{17}$$

Equation (12) becomes

$$\sum_n f_n(3,4)f_n^{\dagger}(1,2)\Big|_{t_1=t_2} = \sum_n f_n^{(0)}(3,4)f_n^{\dagger}(1,2)\Big|_{t_1=t_2}$$

$$+ S_F(3,5)\ S_F^{(c)}(4,6)\ I(56;78) \sum_m f_m(7,8)\ f_m^{\dagger}(1,2)_{t_1=t_2} \tag{18}$$

Taking out the centre of mass of motion in $f_n^{\dagger}(1,2)_{t_1=t_2}$ it can be written as

$$f_n^{\dagger}(1,2)_{t=t} = \phi_n^{\dagger}(\vec{x})\ e^{iK_nX}\ .$$

Multiplying equation (18) by $f_q(\vec{x})\ e^{-iK_qX}$ where K_q is the four-vector of a bound state, and taking the limit of both sides as $t \to -\infty$ in the sense

that all oscillating terms are taken to approach zero, equation (18) becomes

$$f_K^{pos}(3,4) = S_F'(3,5)S_F'^{(c)}(4,6)\ I_{\ell^+\ell^-}(56;78)f_K^{pos}(7,8) \quad (19)$$

as the inhomogeneous term in equation (18) must vanish. In the case of two-particles of unequal mass an equation of the same form as equation (19) can be derived, i.e.

$$f_K(3,4) = S_F'^{(1)}(3,5)S_F'^{(2)}(4,6)\ I_{m_1,m_2}(56;78)\ f_K(7,8) \quad (20)$$

where $I_{m_1,m_2}(56;78)$, besides being dependent on the masses m_1 and m_2 does not contain any annihilation diagrams. In terms of self-energy parts for the fermion propagators equation (20) can be written as

$$\begin{aligned} f_K(3,4) = \{&S_F^{(1)}(3,5)S_F^{(2)}(4,6) + S_F^{(1)}(3,5)S_F^{(2)}(4,4')\Sigma^{(2)}(4',6')S_F'^{(2)}(6',6) \\ &+ S_F^{(1)}(3,3')\Sigma^{(1)}(3',5')S_F'^{(1)}(5',5)S_F^{(2)}(4,6) \\ &+ S_F^{(1)}(3,3')\Sigma^{(1)}(3',5')S_F'^{(1)}(5',5)S_F^{(2)}(4,4')\Sigma^{(2)}(4',6') \\ &S_F'^{(2)}(6',6)\}\ I(56;78)\ f_K(7,8) \quad . \end{aligned} \quad (21)$$

Solving equation (21) exactly appears entirely hopeless. One of the many problems is $I(56;78)$ cannot be expressed in closed form. It is for this reason we start with a simpler equation

$$f_{K_c}(3,4) = S_F^{(1)}(3,5)S_F^{(2)}(4,6)\ I_c(56;78)\ f_{K_c}(7,8) \quad (22)$$

where

$$I_c(56;78) = -i\alpha\,\delta(x_5-x_7)\delta(x_6-x_8)\,\gamma_o^{(1)}\,\gamma_o^{(2)}\,\frac{\delta(t_5-t_6)}{r} \tag{23}$$

is the Coulomb interaction.

Although equation (23) cannot be solved exactly it can be solved using standard perturbation techniques. We shall develop the relationship between the energy eigenvalues of equations (21) and (22).

Defining

$$\begin{aligned}\tilde{I}(56;78) \equiv \{&\delta(55')\delta(66') + \delta(55')\,\Sigma^{(2)}(6,6'')S_F'^{(2)}(6'',6') \\ &+ \Sigma^{(1)}(5,5'')\,S_F'^{(1)}(5'',5')\delta(6,6') \\ &+ \Sigma^{(1)}(5,5'')S_F'^{(1)}(5'',5')\Sigma^{(2)}(6,6'')S_F'^{(2)}(6'',6')\}I(5'6';78)\end{aligned} \tag{24}$$

equation (21) can be written in the form

$$f_K(3,4) = S_F^{(1)}(35)\,S_F^{(2)}(46)\,\tilde{I}(56;78)\,f_K(7,8) \tag{25}$$

Introducing centre of mass coordinates

$$X = \eta_1 x_3 + \eta_2 x_4 \qquad \text{with} \qquad \eta_1 + \eta_2 = 1$$

and

$$x = x_3 - x_4$$

each of the factors in equation (25) may be written in the CM system.

Thus

$$S_F^{(1)}(3,5)S_F^{(2)}(4,6) = \frac{1}{(2\pi)^8}\int \frac{d^4K'\ d^4p\ \ell^{-iK'(X-X')}\ \ell^{-ip(x-x')}}{[(p+\eta_1 K')\gamma^{(1)}-m_1][(-p+\eta_2 K')\gamma^{(2)}-m_2]} \tag{26}$$

where

$$K' = p_1+p_2 \qquad \text{and} \qquad p = \eta_2 p_1 - \eta_1 p_2$$

While $\tilde{I}(56;78)$ can only depend on x_5-x_6, x_7-x_8 or the difference of the centre of mass coordinates $X-X'$ and thus can be written as

$$\tilde{I}(56;78) = \frac{1}{(2\pi)^4}\int d^4K''\ \ell^{-iK''(X-X'')}\ I_{K''}(x,x') \tag{27}$$

while it is easily seen from the definition of $f_K(3,4)$ that it can be written as

$$f_K(3,4) = \phi_K(x)\ \ell^{-iKX} \ . \tag{28}$$

Equation (25) in centre of mass can be written as

$$\phi_K(x) = G_K(x-x')\ I_K(x',x'')\ \phi_K(x'') \tag{29}$$

where

$$G_K(x-x') = \frac{1}{(2\pi)^4}\int \frac{d^4p\ \ell^{-ip(x-x')}}{[(p+\eta_1 K)\gamma^{(1)}-m_1][(-p+\eta_2 K)\gamma^{(2)}-m_2]} \tag{30}$$

$G_K(x-x')$ can be written as the product of two factors, i.e. in the frame where $\vec{K} = 0$ we have

$$G_K(x-x') = \frac{1}{(2\pi)^4}\int \frac{d^4p\ e^{-ip(x-x')}}{[H^{(1)}(\bar{p})+H^{(2)}(-p)-K]}\left\{\frac{1}{H^{(1)}(\bar{p})-(n_1K+p_o)}\right.$$

$$\left. + \frac{1}{H^{(2)}(-\bar{p})-(n_2K-p_o)}\right\}\gamma_o^{(1)}\gamma_o^{(2)} \qquad (31)$$

$$= g_K(x,x'')\ \Lambda_K(x'',x')\ \gamma_o^{(1)}\gamma_o^{(2)}$$

so here

$$g_K(x,x') \equiv \frac{1}{(2\pi)^4}\int \frac{d^4p\ e^{-ip(x-x')}}{[H^{(1)}(\bar{p})+H^{(2)}(-\bar{p})-K]} \qquad (32)$$

$$H^{(i)}(\bar{p}) = \bar{\alpha}^{(i)}.\bar{p} + \beta^{(i)}\ m_i$$

and

$$\Lambda_K(x_1x') \equiv \frac{1}{(2\pi)^4}\int d^4p\ e^{-ip(x-x')}\left[\frac{1}{H^{(1)}(\bar{p})-(n_1K+p_o)} + \frac{1}{H^{(2)}(-\bar{p})-(n_2K-p_o)}\right] \qquad (33)$$

$$= \frac{i}{(2\pi)^4}\int d^3p\ e^{i\bar{p}.(\bar{x}-\bar{x}')}\{\theta(t-t')\Lambda_+^{(1)}(\bar{p})\ e^{-i|t-t'|(E(p)-n_1K)}$$

$$- \theta(t-t')\Lambda_-^{(2)}(-\bar{p})\ e^{-i(E(p)+n_2K)|t-t'|}$$

$$- \theta(t'-t)\Lambda_-^{(1)}(\bar{p})\ e^{i|t-t'|(-E(p)-n_1K)}$$

$$\Lambda_+^{(2)}(-\bar{p})\theta(t'-t)\ e^{i|t-t'|(-E(p)+n_2K)}\}$$

with

$$\Lambda_{\pm}^{(i)}(\bar{p}) = \frac{E^{(i)}(p) \pm H^{(i)}(\bar{p})}{2E^{(i)}(p)} .$$

Writing

$$I_K(x,x') = - \frac{i\alpha\, \delta(x-x')\gamma_o^{(1)}\gamma_o^{(2)}\delta(t)}{r} + I'_K(x,x') \qquad (35)$$

and noting that $\Lambda_K(\bar{x},\bar{x}')$ is independent of K we have from (29)-(32) that

$$\phi_K(x) = g_K(x,x'')\, \Lambda_K(x',x')\, \gamma_o^{(1)}\gamma_o^{(2)}\, I_K(x',y)\, \phi_K(y)$$

or

$$g_K^{-1}(x,x')\, \phi_K(x') = \Lambda_K(x,x')\, \gamma_o^{(1)}\gamma_o^{(2)} \left[- \frac{i\alpha\, \delta(x'-y)\gamma_o^{(1)}\gamma_o^{(2)}\delta(t')}{r'} + I'_K(x',y) \right] \phi_K(y) \qquad (36)$$

More explicitly equation (36) is

$$[H^{(1)}(\bar{p}) + H^{(2)}(-\bar{p}) - K]\, \phi_K(x) = - i\alpha \int \frac{d^3y\, \Lambda_K(x,\bar{y})\, \phi_K(\bar{y})}{r_y} + \int d^4x'd^4y\, \Lambda_K(x,x')\gamma_o^{(1)}\gamma_o^{(2)}\, I'_K(x',y)\, \phi_K(y) \qquad (37)$$

with

$$r_y = |\bar{y}| .$$

For t = 0, equation (37) becomes

$$[H^{(1)}(\bar{p}) + H^{(2)}(-\bar{p}) - K]\ \phi_K(\bar{x}) = \int d^3x'\ \Lambda(\bar{x},\bar{x}')\ \frac{\alpha}{r'}\ \phi_K(\bar{x}')$$

$$+ \int d^4x'\ d^4x''\ \Lambda_K(\bar{x},x')\ \gamma_o^{(1)}\gamma_o^{(2)}\ I_K'(x',x'')\ \phi_K(x'') \tag{38}$$

where

$$-i\ \Lambda_K(\bar{x},\bar{x}') \equiv \Lambda(\bar{x},\bar{x}') \quad .$$

We note that the non-relativistic limit of equation (38) is just the Schrödinger equation for two particles interacting through their mutual coulomb force.

We define $\phi_{K_c}(x)$ to be the solution of the equation

$$[H^{(1)}(\bar{p}) + H^{(2)}(-\bar{p}) - K_c]\ \phi_{K_c}(x) = -i\int d^3x'\ \Lambda_{K_c}(x,\bar{x}')\ \frac{\alpha}{r'}\ \phi_{K_c}(\bar{r}')$$

or

$$[H^{(1)}(\bar{p}) + H^{(2)}(-\bar{p}) - \frac{\alpha}{r} - K_c]\ \phi_c(\bar{x}) = \int d^3x'\ \Omega(\bar{x},\bar{x}')\ \frac{\alpha}{r'}\ \phi_{K_c}(\bar{r}') \tag{39}$$

where

$$\Omega(\bar{x},\bar{x}') = \Lambda(\bar{x},\bar{x}') - \delta(\bar{x},\bar{x}') \quad .$$

Equation (39), the Bethe-Salpeter equation with a coulomb interaction does not in the limit of $m_2 \to \infty$ give the Dirac equation. This is because the Dirac equation effectively contains other coulomb contributions which are in $I_K'(x,x')$ - namely, the crossed coulomb graphs.

It is useful in developing the perturbation theory to define the hybrid wave function, $\phi_K'(x)$ by

$$\phi'_K(x) \equiv -i \int d^3x' \, \Lambda_K(x;0,\bar{r}') [H^{(1)}(\bar{p}) + H^{(2)}(-\bar{p}) - K_c]^{-1} \frac{\alpha}{r'} \phi_{K_c}(\bar{x}') \tag{40}$$

$$= -i \int d^3x' \, \Lambda_K(x;0,\bar{r}') \, \chi(\bar{r}')$$

with

$$\chi(\bar{r}) \equiv [H^{(1)}(\bar{p}) + H^{(2)}(-\bar{p}) - K_c]^{-1} \frac{\alpha}{r} \phi_{K_c}(\bar{x}) \quad . \tag{41}$$

Applying the operator $G^{-1}(x,x') - I^c_K(x,x')$ on $\phi'_K(x')$ gives

$$[G_K^{-1}(x,x') - I^c_K(x,x')] \, \phi'_K(x') = G_K^{-1}(x,x')(-i)\Lambda_K(x';o,\bar{r}'')\chi(\bar{r}'')$$

$$- I^c_K(x,x')(-i) \, \Lambda_K(x',0,\bar{r}'') \, \chi(\bar{r}'')$$

$$= -i \, \gamma_o^{(1)}\gamma_o^{(2)} [g_K^{-1}(x;0,\bar{r}'') \, \chi(\bar{r}'') - \frac{\alpha}{r} \phi_{K_c}(\bar{r})]$$

$$= i \, \gamma_o^{(1)}\gamma_o^{(2)} (K-K_c) \, \chi(\bar{r}) \tag{42}$$

where we have used

$$\Lambda(\bar{x},\bar{x}') \, \chi(\bar{r}') = \phi'_K(\bar{x}) = \phi_{K_c}(\bar{x})$$

and

$$\frac{\alpha}{r} \phi_{K_c}(r) = [H^{(1)}(p) + H^2(-p) - K_c] \, \chi(\bar{r}) \quad .$$

Rewriting equation (29) as

$$[G_K^{-1}(x,x') - I^c_K(x,x')] \, \phi_K(x') = I'_K(x,x') \, \phi_K(x') \tag{43}$$

and using equation (42) gives

$$\Delta E \equiv K-K_c = - i \bar{\phi}_K(x)\, I'_K(x,x')\phi'_K(x') \quad . \tag{44}$$

We express the exact wavefunction $\phi_K(x)$ in terms of $\phi'_K(x)$ as this wavefunction is similar in form to the Coulomb wavefunction by writing equation (43) as

$$[G_K^{-1}(x,x') - I_K^c(x,x') - I'_K(x,x')][\phi'_K(x') + (\phi_K(x') - \phi'_K(x'))]$$

$$= i\, \gamma_o^{(1)}\gamma_o^{(2)}\, \Delta E\, \chi(\bar{r}) - I'_K(x,x')\, \phi'_K(x') + \bar{G}_K^{-1}(x,x')$$

$$\times [\phi_K(x') - \phi'_K(x')] = 0$$

or

$$\phi_K(x) = \phi'_K(x) + \bar{G}(x,x')\, I'_K(x',x'')\, \phi'_K(x'') - i\, \Delta E\, \bar{G}_K(x,x')$$

$$\times\, \gamma_o^{(1)}\gamma_o^{(2)}\, \chi(\bar{r}')\, \delta(x'_o) \tag{45}$$

where

$$\bar{G}_K = \frac{1}{[G_K^c]^{-1} - I'_K} = G^c + G_K^c\, I'_K\, G_K^c + \ldots \tag{46}$$

G_K^c being the Coulomb propagator for two particles. Using equation (45) our result (44) becomes

$$\Delta E = - i\, \bar{\phi}'_K(x) I'_K(x,x')[\phi'_K(x') + \bar{G}_K(x',x'')I'_K(x'',x''')\phi'_K(x''')$$

$$- i\, \bar{G}_K(x',x'')\, \gamma_o^{(1)}\gamma_o^{(2)}\, \chi(\bar{r}'')\, \Delta E]$$

$$= - i\, \{\bar{\phi}'_K(x) I'_K(x,x')\phi'_K(x') + \bar{\phi}'_K(x) I'_K(x,x')\bar{G}_K(x',x'')I'_K(x'',x''')$$

(47)

To order less than α^8 we may approximate $\phi_K'(x)$ by $\phi_{K_c}(x)$. $\phi_{K_c}(x)$ may be calculated to the desired accuracy by iterating the Pauli wavefunction $\phi_p(\bar{x})$ where for the lowest S-state

$$\phi_P(x) = \frac{1}{(2\pi)^3}\int d^3p \; \ell^{\,i\bar{p}\bar{x}} \frac{1}{(p^2+\gamma^2)^2}\phi_P(0)$$

with

$$\phi_P(0) = \chi_1\chi_2\left(\frac{m\alpha}{2}\right)^{3/2}\frac{1}{\sqrt{\pi}}, \qquad \gamma = \frac{m\alpha}{2}, \tag{48}$$

$$\chi_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \qquad \chi_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

Thus

$$\phi_K'(x) \simeq \phi_{K_c}(x) \simeq G_K(x,x')\; I_c(x',x'')\phi_p(\bar{x}'') \tag{49}$$

In our calculations of the $\alpha^2\ell n\alpha^{-1}$ corrections to positronium and muonium we considered the one-and two-photon graphs in $I_K'(x,x')$ as the three-photon graphs lowest order contribution is $m\alpha^6$ (see Figure 3).

As an example of the foregoing perturbation theory we sketch the calculation of the $\alpha^2\ell n\alpha^{-1}$ contribution to positronium. The transverse photon and the single annihilation diagram lead to the lowest order splitting as well as higher order corrections. The two-photon diagrams contribute an α correction to the lowest order correction and a three-photon diagram an α^2 correction, etc. So to calculate the $\alpha^2\ell n\alpha^{-1}$ correction all one-and two-photon diagrams (Figure 3) must be considered.

The contribution from the single transverse photon is

$$(\Delta E)_B = - i \bar{\phi}_{K_c}(x)\, I_B(x,x')\, \phi_{K_c}(x') \tag{50}$$

with

$$I_B(x,x') = - 4\pi\, i\, \alpha\, \delta^4(x-x')\, \gamma_o^{(1)}\gamma_o^{(2)}\, \frac{\alpha_i^{(1)}\, \alpha_j^{(2)}}{(2\pi)^4}$$

$$\int \frac{d^4q\, e^{-iqx}}{q^2+i\epsilon}\, (\delta_{ij} - \frac{q_i\, q_j}{\bar{q}^2}) \tag{51}$$

where $\phi_{K_c}(x)$ is obtained from (49) and can be written as

$$\phi_{K_c}(x) = \phi_c(x) + \delta\, \phi_c(x) + \delta\, \phi_c'(x) \tag{52}$$

$$\phi_{K_c}(x) = \frac{2\alpha}{(2\pi)^2} \int \frac{d^3k\, e^{i\bar{k}.\vec{r}}}{(\bar{k}^2+\gamma^2)}\, m\, F_k(t)\, \phi_p(0)$$

with

$$F_k(t) = F_k^{(o)}(t) + \delta\, F_k(t) \tag{53}$$

$$F_k^{(0)}(t) = \frac{1}{2E}\, [(1 + \frac{\bar{\alpha}^{(1)}\bar{k}}{2m})(1 - \frac{\bar{\alpha}^{(2)}\bar{k}}{2m})]\, f_k^{(0)}(t) \tag{54}$$

$$f_k^{(0)}(t) = [m+E]\, e^{-i(E-m)|t|} + (m-E)\, e^{-i(E+m)|t|}$$

$$\delta\, F_k(t) = \frac{\bar{k}^2}{8m^2E}\, \delta\, f_k^{(1)}(t) - \frac{\bar{k}^2}{8m^2E}\, (\bar{\alpha}^{(1)} + \bar{\alpha}^{(2)}) \cdot \vec{k}\, \delta\, f_k^{(2)}(t)$$

$$\delta\, f_k^{(1)} = - (m-E)\, e^{-i(E-m)|t|} + (m+E)\, e^{-i(E+m)|t|}$$

$$\delta\, f_k^{(2)} = \frac{t}{|t|}\, \{e^{-i(E-m)|t|} - e^{-i(E+m)|t|}\} \quad .$$

$$(\Delta E)_B = - i\, \bar{\phi}_c(x)\, I_B(x,x')\, \phi_c(x') - i\, \delta\, \bar{\phi}_c(x)\, I_B(x,x')\, \phi_c(x') \tag{55}$$

$$- i\, \bar{\phi}_c(x)\, I_B(x,x')\, \delta\, \phi_c(x')$$

since the other terms do not contribute.

We find

$$[(\Delta E)_B]_{\alpha^6 \ell n \alpha^{-1}} = \frac{8}{3}\, \frac{\pi^2\, \alpha^2 |\phi(0)|^2 \langle \bar{\sigma}(1)\, .\, \bar{\sigma}(2) \rangle}{m\, (2\pi)^3} \int \frac{d^3p\, \bar{p}^2\, \tan^{-1} \frac{p}{\gamma}}{(p^2+\gamma^2)^2\, p\, E(p)}$$

$$= \frac{1}{12}\, m\, \alpha^6 \ell n \alpha^{-1}\, \langle \bar{\sigma}(1)\, .\, \bar{\sigma}(2) \rangle \quad . \tag{56}$$

In a similar fashion the Coulomb transverse diagrams may be calculated. They contribute $(\Delta E)_{CT}$ where

$$(\Delta E)_{CT} = - i \int d^4x\, d^4x' \, [\bar{\phi}_p(0)\, I_{CT}(x,x')\, \phi_c(x')$$

$$+ \bar{\phi}_c(x)\, I_{CT}(x,x')\, \phi_p(0)]$$

$$= - \frac{1}{12}\, m\, \alpha^6 \ell n \alpha^{-1}\, \langle \bar{\sigma}(1)\, .\, \bar{\sigma}(2) \rangle \quad . \tag{57}$$

This cancellation between the single transverse photon and the Coulomb transverse diagrams is not unexpected. Such a cancellation also occurs in the lower order, Karplus and Klein calculation.

The two transverse photon is the only remaining diagram which leads to an $\alpha^2 \ell n \alpha^{-1}$ correction. For this we find

$$[(\Delta E)_{TT}] = -i \int d^4x\, d^4x' \{\bar{\phi}_P(0)\, [\hat{I}_K(x,x') + I_B(x,x'')\, G_K(x'',x''')\, I_B(x''',x')]\, \phi_c(x') + \bar{\phi}_c(x)\, [\hat{I}_K(x,x') + I_B(x,x'')\, G_K(x'',x''')\, I_B(x''',x')]\, \phi_P(0)\} \quad (58)$$

Neglecting wavefunction retardation which does not contribute to this order we have

$$[(\Delta E)_{TT}]^{T-S}_{\alpha^6 \ell n \alpha^{-1}} = \frac{3}{8} m \alpha^6 \ell n \alpha^{-1} \quad (59)$$

In summary, the $\alpha^6 \ell n \alpha^{-1}$ contribution to the hyperfine splitting in positronium is

$$\Delta E = \frac{3}{8} m \alpha^6 \ell n \alpha^{-1}$$

Taking

$$\alpha^{-1} = 137.03608$$

gives

$$\Delta\nu(\alpha^6 \ell n \alpha^{-1}) = 34 \text{ MHz}$$

comparing previous and present theoretical values with the current experimental measurement we see

$$\nu^{th}(\text{Karplus and Klein}) = 2.03381 \times 10^5 \text{ MHz}$$

$$\nu^{th}(\text{present calculation}) = 2.03415 \times 10^5 \text{ MHz}$$

$$\nu^{exp}(1970) = 2.03403(12) \times 10^5 \text{ MHz}$$

Thus by examining the following table which includes an estimate of the uncalculated diagrams we see the agreement is quite good.

The situation in quantum electrodynamics has changed rapidly in the last several years from a state of confusion to one in which there is excellent agreement between theory and experiment[17].

Since physicists find their challenge when experiments and theory do not fit too well, we have mixed feelings about our contribution to the scene.

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† The latest value (3.2 ± 3.6)ppm has just been brought to my attention. See R.S. Cole and W.W. Repko, Bull. Amer. Phys. Soc. 16, 849 (1971).

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TABLE I

	Order	$\Delta \nu$ 10^5 MHz	Percentage of contribution of last order	
			Actual	Expected
Schroedinger Level (ionization frequency)	α^2	16449.2	--	--
Triplet-singlet	α^4	2.04386	0.012	0.005
ground state	α^5	-0.01005	0.49	0.73
splitting	$\alpha^6 \ell n \alpha^{-1}$	0.00034	3.4	3.6
	α^6 (est)	±0.00007	--	20

Theoretical Contributions to Positronium Frequencies

Figure Captions

Figure 1: r-f cavity with a static magnetic field. The γ-ray counters, which are 180° apart are used in coincidence to detect 2γ-annihilation of the singlet state.

Figure 2: In the presence of an external magnetic field the m=0 sub-level of $^3S_1$ includes a mixture of singlet and triplet states. The singlet state decays rapidly into two photons. At the observed magnetic field this process is subtained by r-f induced transitions between m=0 and m=1 substates of $^3S_1$.

Figure 3: Lowest Order Diagrams in $I'_K(x,x')$.

(a) One-transverse photon exchange.

(b) Single annihilation.

(c) Single iteration of the one-transverse photon exchange (i.e. I'_B G I'_B)*.

(d) and (e) Coulomb-transverse.

(f) Crossed Transverse.

(g) Crossed Coulomb.

(h) and (i) Double annihilation.

*See second term of equation (58).

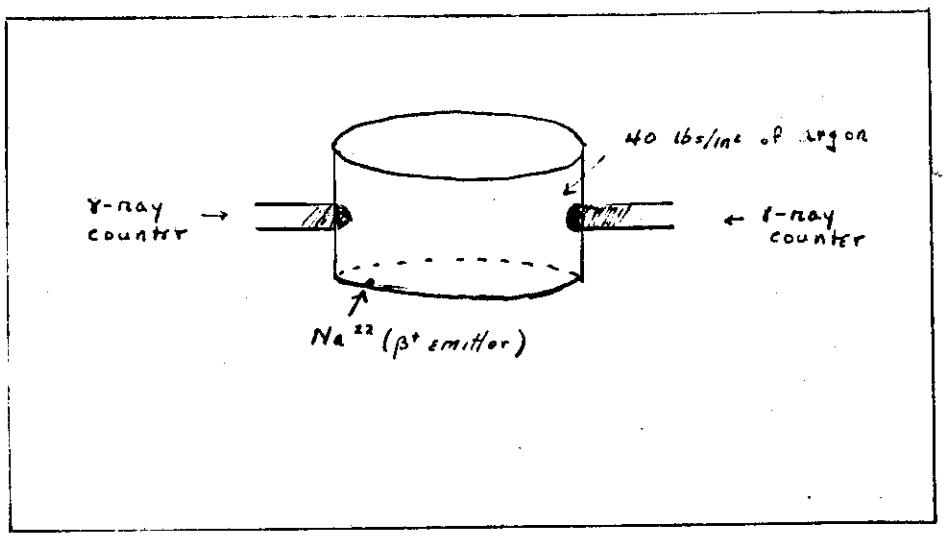


FIGURE 1

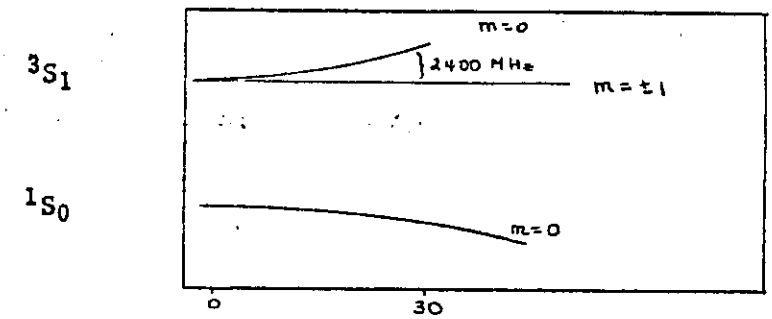
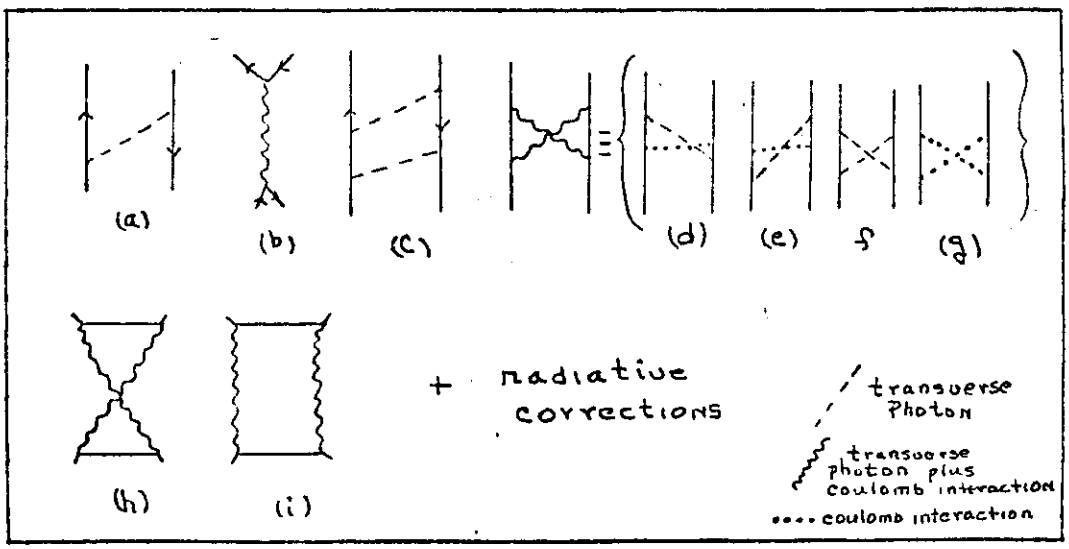


FIGURE 2



One and two photon diagrams for $I_K^1(x,x')$

FIGURE 3