IC/71/137

TOPICAL SEMINAR
ON ELECTROMAGNETIC INTERACTIONS

ICTP, Trieste, 21-26 June 1971.

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

SMALL-DISTANCE BEHAVIOUR ANALYSIS AND WILSON EXPANSIONS *

K. Symanzik

Deutsches Elektronen-Synchrotron DESY, Hamburg, Fed. Rep. Germany.

MIRAMARE - TRIESTE
October 1971

* A rapporteur talk given at the above Topical Seminar. This talk is based on the report DESY 71/39, July 1971 (Commun. Math. Phys. 23, 49 (1971)) and was also given at the Meeting on Renormalization Theory, CRNS, Marseille, June 1971.

In the narrow sense, small-distance behaviour is the behaviour of matrix elements $\langle\alpha|O_1(x_1)\cdots O_n(x_n)|\beta\rangle$ of products of local operators as $x_i \to x$, $\forall i$, for fixed states α, β. In a wider sense, one means thereby the large-momenta behaviour of Fourier transforms of Green's functions, whereby squares of momenta, or of partial sums of momenta, do not become absolutely large at the rate they would if all momenta were euclidean (zero-components imaginary, other components real).

Obviously, the large-momenta behaviour of scattering amplitudes and formfactors is not covered hereby. However, small-distance behaviour in the stated sense is nevertheless interesting, since its comparison with results of other considerations, using, e.g., locality and, in particular, unitarity, yields consistency checks on particular lagrangian field theories. Although we have no reason to assume any simple lagrangian field theory (that must, for the technique to be described to be applicable, even be renormalisable in the conventional perturbation-theoretical sense) to be a picture of nature, such checks are desirable from a theoretical point of view. Note that the large-distance, or small-momenta behaviour of Green's functions is constrained so strongly by analyticity, nearby singularities, exact and partial symmetries, etc., that it cannot possibly be sensitive to special features of particular lagrangian theories, provided these are set up so as to be consistent with those symmetries and the desired mass spectra.

We consider the theory of one hermitian scalar field:

$$\begin{aligned} L = {} & \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{24}g\,\phi^4 + \\ & + \frac{1}{2}(Z_3-1)\,(\partial_\mu\phi\,\partial^\mu\phi - m^2\phi^2) + \\ & + \frac{1}{2}Z_3\,\delta m^2\,\phi^2 - \frac{1}{24}(Z_1-1)\,g\,\phi^4 . \end{aligned} \tag{1}$$

The technique to be described, however, is applicable to all renormalizable theories, in particular QED goes completely parallel. We write (the functions with an odd number of arguments are identically zero)

$$\langle T\phi(x_1)\ldots\phi(x_n)\rangle_{\text{conn.}} = G(x_1\ldots x_{2n})$$

and, of these Green's functions, denote the amputated one-particle irreducible parts, the vertex functions, as $\Gamma(x_1\cdots x_{2n})$, with $\Gamma(x_1x_2) = -G^{-1}(x_1x_2)$

in the convolution sense. The Fourier transforms are also written $\Gamma(p_1 \cdots p_{2n})$, with $\Sigma p = 0$ understood.

The behaviour of $\langle\alpha|\phi(x)\,\phi(y)|\beta\rangle$, $x \to y$, can be obtained from the large λ behaviour of

$$\Gamma((\lambda p + r_1)\ (-\lambda p + r_2)\ r_3 \cdots r_{2n}), \qquad \Sigma r = 0 \tag{2}$$

which is a special case of

$$\Gamma((\lambda p_1 + r_1)\ \ldots\ (\lambda p_{2n} + r_{2n})), \qquad \Sigma p = \Sigma r = 0 \ . \tag{3}$$

Equal-time commutation relations of currents $j^{\mu}_{\alpha}(x)$ and $j^{\nu}_{\beta}(y)$ could be investigated on the basis of the large-λ behaviour, to accuracy λ^{-1}, of vertex functions

$$\Gamma^{\mu\nu}_{\alpha\beta}(\lambda p + r_1)\ (-\lambda p + r_2),\ \ r_3 \cdots r_{2n}\ , \qquad \Sigma r = 0 \tag{4}$$

with p a fixed timelike vector, whereby the first two arguments in (4) relate those of the current operators.

Writing Eq. (3) as $f(\lambda p, r)$, one finds in renormalised perturbation theory •

$$f(\lambda p, r) = \lambda^{\alpha} \sum_{K=0}^{\infty} \sum_{L=0}^{\infty} \lambda^{-K} (\ln \lambda)^{L} f_{KL}(p,r) \tag{5}$$

the f_{KL} being power series in g , where for general momenta p , Weinberg's [1] asymptotic exponent α equals $4-2n$, the superficial divergence of the contributing Feynman graphs, and $f_{0L}(p,r)$ r-independent, while for particular momenta p , as,e.g., in the case (2) for $n \geqslant 3$, α may be larger, and also the r-independence of the $f_{0L}(p,r)$ violated. Weinberg [1] gave the interpretation of these phenomena. For general momenta p , the dominant routing of the (large parts of the) external momenta involves the entire graph, while for particular sets of those momenta, the dominant routing may avoid some part of the graph such that α does not depend on details of the avoided parts of the graph while the coefficients f_{0L} do. In the case of euclidean momenta Weinberg only considered that this occurs only if some (in the model (1), even) non-trivial partial sum of the momenta p_i vanishes.

The tool to make the idea of routing large momenta through only parts of a graph quantitative are Wilson expansions [2], which we will use here in the precise form given by Zimmermann [3], whose notation we adopt. We give one example. For $p^2 \neq 0$,

$$\Gamma\big((\lambda p + r_1)\ (-\lambda p + r_2)\ r_3 \ldots r_{2n}\big) =$$

$$= \Gamma\big(\lambda p(-\lambda p)00\big)\ \tfrac{1}{2} \left\langle TN_2\{\phi(0)^2\}\ \tilde{\phi}(r_3)\ldots\tilde{\phi}(r_{2n})\right\rangle +$$

$$+ O(\lambda^{-1}(\ln\lambda)^{\beta}) \tag{6}$$

where the estimate of the remainder, and the one $O((\ln\lambda)^{\beta'})$ for the left-hand side and first term on the right-hand side, are valid to arbitrarily high orders in perturbation theory, for β and β' sufficiently large. Eq. (6) shows how the p-dependence, to the stated accuracy, factors out completely, and that we do have in Eq. (2) a set of momenta p_i in Eq. (3) of particular type, violating the conditions and results for general momenta p_i stated after Eq.(5).

We call the sum in Eq. (5) over L for $\kappa = 0$ the asymptotic form of $f(\lambda p, r)$. Eq. (6) shows that for obtaining the asymptotic form of (2) we need only the one of $\Gamma(\lambda p(-\lambda p)00)$. Similarly, Wilson expansions also allow in other cases the asymptotic forms of complicated functions to be obtained from those of simpler functions. Furthermore, for $\lambda \to \infty$, more accurate versions of (6) exist [3], such as would be needed for ETCRs in the case (4), as indicated there.

The other technique we need is mass vertex insertion [4]. We replace (1) by

$$L^s(\phi,\partial\phi) = L(\phi,\partial\phi) - \tfrac{1}{2}\, s\, Z_3\, \Delta m_0^2 \phi^2 \quad . \tag{7}$$

The Green's functions will now be $G^s(x_1 \cdots x_{2n}; m^2, g)$, and by Bogolubov-Parasiuk-Hepp renormalization theory for the Fourier transforms of the vertex functions one must have

$$\Gamma^s(p_1 \ldots p_{2n}; m^2, g) = Z(s)^{-n}\ \Gamma(p_1 \ldots p_{2n}; m^2(s),\ g(s)) \quad . \tag{8}$$

Differentiating (8) at $s = 0$ gives [5]

$$\left[m^2 \frac{\partial}{\partial m^2} + \beta(g) \frac{\partial}{\partial g} - 2n\, \gamma(g) \right] \Gamma(p_1 \ldots p_{2n}; m^2, g) \equiv$$

$$\equiv Op_{2n}\Gamma = \Delta\Gamma(p_1 \ldots p_{2n}; m^2, g) \tag{9}$$

where the Feynman graphs for $\Delta\Gamma$ are obtained from those for Γ by inserting an extra mass vertex in all graphs for Γ in all possible ways and summing up. More precisely, $\Delta\Gamma(p(-p); m^2, g)$ is calculated from a once-subtracted Bethe-Salpeter equation, the normalization

$$\Delta\Gamma(p(-p); m^2, g)\Big|_{p^2 = m^2} = -im^2 \tag{10}$$

thereby imposed, and $\Delta\Gamma$ for $n \geq 2$ is then obtained by quadrature. An explicit form for $\Delta\Gamma$ is, in Zimmerman's [3)] notation,

$$\Delta\Gamma(p_1 \ldots p_{2n}; m^2, g) =$$

$$= -\frac{1}{2}\, im^2\, \varphi(g) \left\langle TN_2\{\phi(0)^2\}\, \tilde{\phi}(p_1) \ldots \tilde{\phi}(p_{2n}) \right\rangle^{prop} \tag{11}$$

where

$$\varphi(g) = (-im^2)^{-1}\, \Delta\Gamma(00; m^2, g) = 1 + O(g^2)$$

such that comparison with the definition via Eq. (7) gives

$$Z_3\, \Delta m_0^2 :\phi(x)^2: = m^2\, \varphi(g)\, N_2\{\phi(x)^2\} \quad .$$

In Eq. (9), the functions $\beta(g)$ and $\gamma(g)$ are obtained by requiring (9) to be consistent with the renormalization conditions

$$\Gamma(p(-p); m^2, g)\Big|_{p^2 = m^2} = 0 \tag{12a}$$

$$(\partial/\partial p^2)\, \Gamma(p(-p); m^2 g)\Big|_{p^2 = m^2} = i \tag{12b}$$

$$\Gamma(p_1 \ldots p_4; m^2, g)\Big|_{symm.pt.} = -ig \tag{12c}$$

where the symmetry point is characterized by $p_i p_j = 1/3\, (4\delta_{ij} - 1) m^2$, and one finds

$$\beta(g) = b_0\, g^2 + b_1\, g^3 + \ldots\,, \qquad b_0 = 3(32\pi^2)^{-1} \tag{13a}$$

$$\gamma(g) = c_0\, g^2 + c_1\, g^3 + \ldots, \qquad c_0 = (2^{11}\, 3\pi^4)^{-1}\ . \tag{13b}$$

The usefulness of (9) for large-momenta behaviour analysis follows from

$$\Gamma(\lambda p_1 \ldots \lambda p_{2n}; m^2, g) = \lambda^{4-2n}\, \Gamma(p_1 \ldots p_{2n}; m^2 \lambda^{-2}, g) \tag{14}$$

and the analogous relation for $\Delta\Gamma$. For large λ and general momenta,

$$\Gamma(\lambda p_1 \ldots \lambda p_{2n}; m^2, g) \sim \lambda^{4-2n} \tag{15a}$$

$$\Delta\Gamma(\lambda p_1 \ldots \lambda p_{2n}; m^2, g) \sim \lambda^{2-2n} \tag{15b}$$

up to powers of $\ln \lambda$, by Weinberg's [1] arguments, the reduction for $\Delta\Gamma$ being due to the extra denominator. Consequently, in this case, in Eq. (9) $\Delta\Gamma$ is negligible for large momenta relative to the separate terms on the left-hand side.

To discuss Eq. (9) in more detail [6], we introduce

$$\rho(g) = \int_{g_0}^{g} \beta(g')^{-1}\, dg' = -\, b_0^{-1}\, g^{-1} - b_0^{-2}\, b_1 \ln g + \\ + \ldots g + \ldots g^2 + \ldots \tag{16}$$

which increases monotonically from $-\infty$ for g running from 0 through positive values until $\rho(g)$ ceases to be positive, and we will assume that actually $\rho(g) \to \infty$ as $g \to g_\infty$, which may either be positive finite, or $+\infty$. Then

$$g(\lambda) = \rho^{-1}(\ln \lambda^2 + \rho(q)) = \\ = g + \sum_{n=1}^{\infty} (n!)^{-1} (\ln \lambda^2)^n\, (\beta(g)\, d/dg)^{n-1}\, \beta(g) \tag{17}$$

is defined for all $0 \leqslant \lambda < \infty$, and increases monotonically from 0 to g_∞ for λ in this interval. The expansion given in Eq. (17) is useless except for perturbation-theoretical purposes. But for $\lambda \to 0$, from (16),

$$g(\lambda) = g[1 + b_0\, g\, \ln \lambda^{-2} + b_0^{-1}\, b_1\, g\, \ln(1 + b_0\, g\, \ln \lambda^{-2}) + \tag{18}$$

$$+ \text{power series in } g \text{ without constant term} + \text{terms that vanish for } \lambda \to 0]^{-1}$$

while the behaviour for $\lambda \to \infty$ is dependent on assumptions, e.g. if $\beta(g_\infty) = 0$ is a simple zero, and thus $\beta'(g_\infty) < 0$, then as $\lambda \to \infty$

$$g(\lambda) = g_\infty - (g_\infty - g)\lambda^{2\beta'(g_\infty)} + \text{smaller terms.}$$

Now, explicitly, from Eq. (9),

$$\Gamma(p_1 \dots p_{2n}; m^2\lambda^{-2}, g) = \Gamma_{as}(p_1 \dots p_{2n}; m^2\lambda^{-2}, g) +$$

$$+ \int_{\lambda^2}^{\infty} \lambda'^{-2}\, d\lambda'^2\ \Delta\Gamma(p_1 \dots p_{2n}; m^2\lambda'^{-2}, g(\lambda'^{-1}\lambda)) \cdot$$

$$\cdot \exp\left[2n \int_{g(\lambda'^{-1}g)}^{g} dg'\ \beta(g')^{-1}\ \gamma(g')\right] \tag{19}$$

where Γ_{as} is (for $\lambda = 1$) a solution of the homogeneous equation to (9), provided the λ' - integral in Eq. (19) exists. Momenta p_i such that this happens (cf. (14) and (15b)) are called <u>nonexceptional momenta</u>, otherwise, <u>exceptional momenta</u>. In the first case, Eq. (19) and the property

$$\Gamma_{as}(p_1 \dots p_{2n}; m^2\lambda^{-2}, g) =$$

$$= \Gamma_{as}(p_1 \dots p_{2n}; m^2, g(\lambda)) \exp\left[-2n \int_{g}^{g(\lambda)} dg'\ \beta(g')^{-1}\ \gamma(g')\right] \tag{20}$$

imply

$$\Gamma_{as}(p_1 \dots p_{2n}; m^2, g) =$$

$$= \lim_{\lambda \to \infty} \left\{ \Gamma(p_1 \dots p_{2n}; m^2\lambda^{-2}, g(\lambda^{-1})) \cdot \right.$$

$$\left. \cdot \exp\left[2n \int_{g(\lambda^{-1})}^{g} dg'\ \beta(g')^{-1}\ \gamma(g')\right] \right\} \tag{21}$$

This formula, valid to all orders in perturbation theory, can be interpreted as follows: The Γ_{as} are the vertex functions of a zero-mass theory obtained from the finite-mass theory by a certain limiting process. For irreducible graphs, the zero-mass limit can (for nonexceptional momenta) be taken directly. For reducible graphs and such momenta, infrared divergences arise due to renormalization subtractions on the insertions being made at the respective mass-shell momenta. The ensuing infrared divergences (in the zero-mass limit) are in Eq. (21) cancelled by the $\ln\lambda$-terms arising from using Eq. (17) for the $g(\lambda^{-1})$ in the Γ-argument and in the exponential, which are related to infrared divergences due to four-vertex insertions and to full-propagator insertions, respectively. That, for the limit (21) to exist, exceptional momenta must be avoided simply means that at such momenta the zero-mass-theory functions Γ_{as} are infrared singular, while the finite-mass-theory functions Γ are not.

Eq. (20) shows that the zero mass theories defined through the Γ_{as} constitute a one-parameter family, with all members related to each other by dilatation (and physically irrelevant normalization change), a fact also directly interpretable in terms of renormalization conditions appropriate for zero-mass theories. Eqs. (14), (19), and (20) give

$$\Gamma(\lambda p_1 \ldots \lambda p_{2n}; m^2, g) =$$

$$= \lambda^{4-2n} \exp\left[-2n \int_g^{g(\lambda)} dg'\, \beta(g')^{-1}\, \gamma(g')\right] \Gamma_{as}(p_1 \ldots p_{2n}; m^2, g(\lambda)) + + O(\lambda^{2-2n}(\ln\lambda)^{\beta}) \qquad (22)$$

which exhibits that the first term on the right-hand side, which equals $\Gamma_{as}(\lambda p_1 \cdots \lambda p_{2n}\ ;\ m^2, g)$, indeed is the asymptotic form of Γ in the sense spelled out in connection with Eq. (5), and Eq. (22) shows, using Eqs. (13) and (17), how the logarithms in the $K = 0$-sum in (5) can be rearranged in a suggestive manner. To this we return later.

For <u>exceptional momenta</u>, by definition, the integral in Eq. (19) does not converge and Γ_{as} does not exist, the reason being failure of the estimate (15b). This comes about, if (15a) holds, by the (or a) dominant routing of the p_i through the $\Delta\Gamma$ graph sparing the extra mass vertex, or, expressed differently, by a quadratic UR-divergence of $\Delta\Gamma$ for mass $\to 0$. Then the correct asymptotic form of Γ, $\underline{\Gamma_{as}}$, satisfies

$$\mathrm{Op}_{2n}\,\Gamma_{\underline{\underline{as}}} = \Delta\Gamma_{\underline{\underline{as}}} \tag{23a}$$

in place of

$$\mathrm{Op}_{2n}\,\Gamma_{as} = 0 \tag{23b}$$

for nonexceptional momenta. $\Delta\Gamma_{\underline{\underline{as}}}$ is the (correct) asymptotic form of $\Delta\Gamma$ and is obtained from a Wilson expansion.

As an example, we consider the four-point vertex $\Gamma(p_1 \cdots p_4)$, $\sum p = 0$. For euclidean momenta, $p_1+p_2 = -p_3-p_4 = 0$, etc., characterizes exceptional sets, $p_1 = -p_2-p_3-p_4 = 0$, etc., in general not leading to exceptionality in this model. From Eqs. (9) and (11)

$$\mathrm{Op}_4\,\Gamma(p(-p)00) = \Delta\Gamma(p(-p)00) =$$

$$= -\frac{1}{2}\,im^2\,\varphi(g)\left\langle TN_2\{\phi(0)^2\}\tilde{\phi}(p)\;\tilde{\phi}(-p)\;\tilde{\phi}(0)\;\tilde{\phi}(0)\right\rangle^{prop} =$$

$$= -\frac{1}{2}\,im^2\,\varphi(g)\;\Gamma(p(-p)00)\,\cdot$$

$$\cdot\;\frac{1}{2}\int dx\left\langle TN_2\{\phi(x)^2\}\;N_2\{\phi(0)^2\}\;\tilde{\phi}(0)\;\tilde{\phi}(0)\right\rangle^{prop} + \text{negligible for } p\to\infty$$

where a formula similar to Eq. (6) is used, such that (23a) becomes

$$[\mathrm{Op}_4 - \eta(g)]\;\;\Gamma_{\underline{\underline{as}}}(p(-p)00) = 0 \tag{24}$$

with

$$\eta(g) = -\frac{1}{4}\,im^2\,\varphi(g)\int dx\left\langle TN_2\{\phi(x)^2\}\;N_2\{\phi(0)^2\}\;\tilde{\phi}(0)\;\tilde{\phi}(0)\right\rangle^{prop}$$

and Eq. (20) is replaced by

$$\Gamma_{\underline{\underline{as}}}(\lambda p(-\lambda p)00;m^2,g) =$$

$$= \Gamma_{\underline{\underline{as}}}(p(-p)00;m^2,g(\lambda))\;\exp\left[-\int_g^{g(\lambda)} dg'\;\beta(g')^{-1}\;(4\gamma(g') + \eta(g'))\right] \;. \tag{25}$$

Also $\Gamma(p(-p)q(-q))$ for p and q large simultaneously, and for $-p^2 \gg -q^2 \gg m^2$, $p^2q^2 > (pq)^2$, and $\Delta\Gamma(p(-p))$ for large p, have similarly been analysed. In all cases, the $\underline{\Gamma_{as}}$ can be defined directly by limiting processes from the Γ_{as} by removing the UR-singularity in a sometimes multiplicative, sometimes subtractive manner; this fact we shall see later to be significant.

We finally consider asymptotic behaviour. Eq. (22) shows that no statement concerning $\lambda \to \infty$ can be made without information about the behaviour of $\Gamma_{as}(\cdots;m^2,g(\lambda))$ for $g(\lambda) \to g_\infty$. It is suggestive to assume [7] that these functions have limits hereby if g_∞ is finite, since then the limit can be taken termwise in the known expansions of these functions in powers of $g(\lambda)$. Then the evaluation

$$\int_g^{g(\lambda)} dg'\ \beta(g')^{-1}\ \gamma(g') = \gamma(g_\infty)\ \ln\lambda^2 + \int_g^{g(\lambda)} dg'\ \beta(g')^{-1}\ [\gamma(g') - \gamma(g_\infty)] \tag{26}$$

whereby the last integral may or may not converge for $g(\lambda) \to g_\infty$ but will not be as strongly λ-dependent as $\ln\lambda^2$, gives in Eq. (22) the factor $\lambda^{4-2n-4n\gamma(g_\infty)}$, indicating an anomalous dimension $1+2\gamma(g_\infty)$ in the sense of Wilson [2].

Actually, Eq. (26) points precisely to the weakness of the argument concerning the limit of $\Gamma_{as}(\cdots;m^2,g(\lambda))$, as $g(\lambda) \to g_\infty$, based on the power series expansion alone: Eq.(26) is also a power series in $g(\lambda)$ minus the same power series in g, and only our assumptions concerning $\beta(g)$ and $\gamma(g)$ lead to the conclusion that the power series diverges for argument g_∞. There is also no direct physical argument for the existence of the $g(\lambda) \to g_\infty$ limit in Γ_{as} since Eq. (20) shows that this limit is actually one of infinite dilatation!

In this situation, we may look back to Eq. (19) and write it as

$$\Gamma_{as}(p_1\ldots p_{2n};m^2,g(\lambda)) = \Gamma(p_1\ldots p_{2n};m^2,g(\lambda)) - \int_1^\infty d\mu^2\ \mu^{-2}\ \Delta\Gamma(p_1\ldots p_{2n};m^2\mu^{-2},g(\mu^{-1}\lambda)) \cdot \exp\left[2n\int_{g(\mu^{-1}\lambda)}^{g(\lambda)} dg'\beta(g')^{-1}\ \gamma(g')\right] \tag{27}$$

The value g_∞ for the coupling constant is not expected to be a peculiar one for the finite mass theory, such that the Γ, and also the $\Delta\Gamma$ which are related to the Γ by integral equations not involving any asymptotics, can confidently be assumed to exist for that coupling constant, such that (27) takes for $\lambda \to \infty$ the form

$$\Gamma_{as}(p_1 \ldots p_{2n}; m^2, g_\infty) = \Gamma(p_1 \ldots p_{2n}; m^2, g_\infty)$$
$$- \int_1^\infty d\mu^2\, \mu^{-2-4n\,\gamma(g_\infty)}\, \Delta\Gamma(p_1 \ldots p_{2n}; m^2\mu^{-2}, g_\infty) \qquad (28)$$

and the problem is that of the convergence of the integral, in contrast to (19), a question of asymptotics rather than infrared divergence. So we are back to a question similar to the one we started from, and the best we can so arrive at is a consistency argument. If the large-μ behaviour of $\Delta\Gamma$ in Eq. (28) is obtained on the basis of assumptions analogous to the one discussed in connection with Eq. (22), and if then the integral in (28) converges, those assumptions will have passed a simple consistency test. The asymptotic form of $\Delta\Gamma$ in (28), which is a problem involving exceptional momenta due to the extra vertex at zero momentum, see (11), has been obtained for $n = 1$, and the integral in (28) for that $\Delta\Gamma_{as}$ converges under relatively mild conditions.

Thus, adopting that set of assumptions, from Eq. (22) a certain power law for $\lambda \to \infty$ is obtained. For cases with exceptional momenta, in general also power laws are obtained with, however, different exponents, as follows, e.g., for $\Gamma(\lambda p(-\lambda p)00)$ from Eq. (25) by a reasoning similar to that used for Eq. (22), whereby the definability of $\Gamma_{as}(p(-p)00; m^2, g_\infty)$ from functions $\Gamma_{as}(\cdots; m^2, g_\infty)$, alluded to before in the discussion of UR-singularities in zero-mass theories, is seen to be crucial. All exponents in these power laws are g-independent, as they involve $\gamma(g_\infty)$ and, in the case of exceptional momenta, $\eta(g_\infty)$, and similar functions taken at g_∞.

From Eq. (20) for $g \to g_\infty$ one finds

$$\Gamma_{as}(\lambda p_1 \ldots \lambda p_{2n}; m^2, g_\infty) =$$
$$= \lambda^{4-2n-4n\,\gamma(g_\infty)}\, \Gamma_{as}(p_1 \ldots p_{2n}; m^2, g_\infty) \quad , \qquad (29)$$

i.e., an exact anomalous scaling law for the limiting asymptotic (zero-mass) theory. Hereby, $\gamma(g_\infty) \geqslant 0$ must hold for Eq. (22) for $n = 1$ not to contradict positive-definiteness [8]. $\gamma(g_\infty) = 0$ would imply [9] from Eq. (29) a free asymptotic theory, which means that in Eq. (22) for $\lambda \to \infty$ the largest asymptotic term would be absent if $n \geqslant 2$.

Concerning bare coupling constants [10], we first sketch what happens in QED [4),6)]. There the analogue of g in Eq. (16) is e^2 . So $g(\lambda) \to g_\infty$ for $\lambda \to \infty$ corresponds to $e^2(\lambda) \to e^2_\infty$ for $\lambda \to \infty$. The bare charge squared is

$$e_b^2 = e^2 Z_3^{-1} Z_2^{-2} Z_1^{+2} = e^2 Z_3^{-1} =$$

$$= e^2 \lim_{p^2 \to -\infty} \frac{ip^2}{\Gamma(p(-p);m^2,e^2)} = e^2_\infty \frac{ip^2}{\Gamma_{as}(p(-p);m^2,e^2_\infty)}$$

independent of p^2 , whereby

$$\tilde{D}^1_{F\mu\nu}(p) = \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) [\Gamma(p(-p);m^2,e^2)]^{-1}$$

in Landau gauge, and thus assumptions [11] analogous to those made before in ϕ^4 theory lead to a finite, e^2-independent bare charge. In ϕ^4 theory, from the discussed assumptions,

$$g_b = g Z_1 Z_3^{-2} =$$

$$= i\,\Gamma(p_1 \cdots p_4;m^2,g)\Big|_{p_i \to \infty} \left(\frac{ip^2}{\Gamma(p(-p);m^2,g)}\right)^2\Bigg|_{p \to \infty} =$$

$$= i\,\Gamma_{as}(p_1 \cdots p_4;m^2,g_\infty) \left(\frac{ip^2}{\Gamma_{as}(p(-p);m^2,g_\infty)}\right)^2$$

with no prescribed relation between p and the $p_1 \cdots p_4$, such that g_b is ambiguous, unless $\gamma(g_\infty) = 0$ which yields $g_b = 0$.

In comparison with the <u>renormalization group approach</u> (see, e.g., Ref. 7) to large-momenta behaviour, the one presented here has the advantage that statements true to all orders in renormalized perturbation theory are separated from those that depend on assumptions not verifiable in perturbation theory. For example, certain results obtained here on infrared singularities in zero-mass theories are valid beyond perturbation theory, as $g(\lambda) \to 0$ as $\lambda \to 0$ as shown in Eq. (18), while asymptotic behaviour statements, in contrast to the results on asymptotic forms, so far require assumptions. It also seems that the effect of exceptional momenta on asymptotic forms (and thence, by virtue of assumptions, on asymptotic behaviour) is here more transparent than in the renormalization group approach. We have, however, no results proved yet involving minkowskian exceptional momenta, and, thus, concerning scattering amplitudes and form factors. Their treatment is related to the analysis of light-cone singularities, which is more difficult than short-distance behaviour analysis.

ACKNOWLEDGMENTS

The author is grateful to Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

REFERENCES

1. S. Weinberg, Phys. Rev. 118, 838 (1960).

2. K.G. Wilson, Phys. Rev. 179, 1499 (1969).

3. W. Zimmermann, Brandeis Lectures 1970.

4. K. Symanzik, Commun. Math. Phys. 18, 227 (1970).

5. Cf. also C.G. Callan, Jr., Phys. Rev. D2, 1541 (1970).

6. K. Symanzik, Springer Tracts in Modern Physics, 57, 222 (1971).

7. K.G. Wilson, Phys. Rev. D3, 1818 (1971).

8. H. Lehmann, Nuovo Cimento 11, 342 (1954).

9. K. Pohlmeyer, Commun. Math. Phys. 12, 204 (1969).

10. M. Gell-Mann and F.E. Low, Phys. Rev. 95, 1300 (1954).

11. M. Baker and K. Johnson, Phys. Rev. D3, 2516, 2541 (1971) and references given there.