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LIE GROUPS AND SYMPLECTIC MANIFOLDS

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There are many reasons why physicists are concerned with symplectic manifolds: i) Classical mechanics is a symplectic theory containing as particular cases Lagrangian and Hamiltonian formalisms; ii) Quantization procedure is intimately connected with symplectic spaces; iii) The search for unitary representations of Lie groups are simplified by considerations of symplectic structure of orbits of Lie groups in the co-adjoint representation.

1. Definitions [1)]

a) Let M be a differentiable manifold and E_x its tangent space at point $x \in M$. Defining a tensor field φ of degree p on M consists in defining a differentiable p-linear form φ on $E_x^{\otimes p} = E_\lambda \otimes E_\lambda \otimes E_\lambda \cdots \otimes E_x$

$$\overline{d_\alpha x} \in E_x \; : \; \varphi(\overline{d_1 x}, \overline{d_2 x}, \ldots, \overline{d_p x}) = \varphi_{i_1 i_2 \cdots i_p} (d_1 x)^{i_1} (d_2 x)^{i_2} \cdots (d_p x)^{i_p} .$$

φ is called a p-exterior form if it is antisymmetric.

b) Exterior derivative of an exterior form φ: it is a form $\nabla \wedge \varphi$ the components of which are

$$(\nabla \wedge \varphi)_{i_1 i_2 \cdots i_{p+1}} = \partial_{i_1} \varphi_{i_2 \cdots i_p} - \sum_{k=2}^{p} \partial_{i_2} \varphi_{i_2 i_3 \cdots i_1 \cdots i_p} .$$

One has the properties

$$\nabla \wedge (\nabla \wedge \varphi) \equiv 0$$

$$\nabla \wedge \varphi = 0 \iff \varphi = \nabla \wedge \theta$$

2. Examples

a) M : Minkowski space (or space-time in general relativity); the electromagnetic field $F_{\mu\nu}$ is a 2-form satisfying $\nabla \wedge F = 0$. So, $F = \nabla \wedge A$ where A is a 1-form.

b) $\mathcal{E}_{\mu\nu\rho\lambda}$, the Kronecker tensor, is a 4-form on space-time.

3. Symplectic manifold[1)]

It is a differentiable manifold where a field of non-singular 2-forms $\sigma_{\mu\nu}$ is defined, such that $\nabla \wedge \sigma = 0$.

Example 1: $F_{\mu\nu}$ on space-time iff $\det F \neq 0$.

Example 2: Any surface on which the Kronecker tensor $\mathcal{E}_{ij}$ is defined (the symplectic form is the surface element).

Example 3: Let f and g be two functions on a symplectic manifold and let $\sigma^{\mu\nu}$ be the inverse form of $\sigma_{\mu\nu}$. One defines the Poisson bracket:

$$\{f, g\} = \sigma^{\mu\nu} \frac{\partial f}{\partial \lambda^{\mu}} \frac{\partial g}{\partial \lambda^{\nu}} .$$

Example 4:[2)] Let u and v be two parameters on a two-dimensional manifold M and let f, g, h be three functions on M . It is easy to prove the identity

$$dh = \frac{\{h, g\}}{\{f, g\}} df + \frac{\{h, f\}}{\{g, f\}} dg$$

where $\{f, g\} = \frac{\partial f}{\partial u} \frac{\partial g}{\partial v} - \frac{\partial f}{\partial v} \frac{\partial g}{\partial u}$. Usually $\frac{\{h, g\}}{\{f, g\}}$ is written as $\left(\frac{\partial h}{\partial f}\right)_g$.

Theorem: All symplectic manifolds are even-dimensional. All symplectic manifolds of the same dimension are locally isomorphic.

4. Consequences in classical mechanics

a) From the above theorem, if one is able to define a Lie algebra of invariance for a given problem with n degrees of freedom, the same Lie algebra can be defined for any other problem with the same number of degrees of freedom[1), 3), 4)].

b) Lagrangian and Hamiltonian theories are the only practical ways to define symplectic manifolds. A natural generalization consists in giving a priori a 2-form. One can treat a larger class of problems in this way.

5. Defining classical particles from group-theoretical considerations [5)]

Theorem: Orbits of any Lie group in the co-adjoint representation space possess a canonical symplectic structure, invariant under group transformations.

Example (rotation group): The surface element of a sphere is invariant by rotation.

Consequences

a) The orbits of the Poincaré group in the dual space of the Lie algebra can be defined as phase spaces for elementary particles (the concept of transitivity in classical mechanics playing the role of irreducibility in quantum mechanics since irreducibility = transitivity + superposition principle). Spinning particles have an eight-dimensional phase-space; this dimension is 6 for spinless and massless particles [5)].

b) With such a definition the quantization procedure consists in looking for unitary irreducible representations of the Poincaré group: the correspondence principle in its old form $\{q_i, p_j\} = \delta_{ij} \Rightarrow [\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}$ is no longer needed [5)].

c) The photon appears as the particle of a "geometrical optics" involving polarization and colour [1)]. The corresponding classical statistics has no ultraviolet divergence [1)].

d) The number of degrees of freedom of a spinning particle is 4. A wave function defined on a homogeneous space of dimension n and satisfying p equations or conditions will describe a spinning particle only if $n - p = 4$, a spinless or massless particle if $n - p = 3$, a "Regge trajectory" if $n - p = 5$. [5)]

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