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THE CONTINUOUS BAKER-CAMPBELL-HAUSDORFF FORMULA AND F-ORDERINGS

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Let $H(t)$ with $t \in \mathbb{R}$ be elements of some associative algebra $\mathcal{A}$. In many chapters of physics and mathematics it is of interest to possess an explicit algorithm for the "evolution operator" $E(t, t_0) \in \mathcal{A}$ defined by:

$$\frac{\partial}{\partial t} E(t, t_0) = H(t)\, E(t, t_0) \quad , \quad E(t, t_0)\Big|_{t=t_0} = 1 \ , \tag{1}$$

in the form:

$$E(t, t_0) = \exp \Omega(t, t_0) \ ; \ \Omega(t, t_0) = \sum_{n=1} \int_{t_0}^{t} \cdots \int_{t_0}^{t} dt_n \cdots dt_1\, H(t_n) \cdots H(t_1)\, \omega(t_n \cdots t_1) \tag{2}$$

The problem of finding the explicit form of $\omega(t_n \cdots t_1) \in \mathbb{R}$ is the continuous Baker-Hausdorff problem. Its solution is of interest, e.g., in S-matrix theory, in various applications to statistical physics and, in particular, in various applications of group theory. We obtained the solution of (2) by considering a more general problem: Let $f(z)$, $z \in \mathbb{C}$ be analytic in $z = 0$; then we seek the formal power series $F = f(E-1) = f(0) + \frac{1}{1!} f'(0)(E-1) + \frac{1}{2!} f''(0)(E-1)^2 + \cdots$ as constructed from $H(t)$:

$$f(E-1) - f(0) = \sum_{n=1} \int \cdots \int_{t_0}^{t} dt_n \cdots dt_1\, H(t_n) \cdots H(t_1)\, f(t_n \cdots t_1) \tag{3}$$

with functions $f(t_n \cdots t_1)$ to be determined. We have established the general result that:

$$f(t_n \cdots t_1) = \frac{1}{2\pi i} \oint_{z=0} \frac{dz}{zn+1}\, f(z)\, (1+z)^{\Theta_n} \tag{4}$$

where

$$\Theta_n \overset{df}{=} \theta(t_n - t_{n-1}) + \cdots\ \theta(t_2 - t_1) \ ; \Theta_1 \overset{df}{=} 0 \tag{5}$$

with $\theta(t) = 1$ for $t > 0$ and $= 0$ for $t < 0$. This specialized for $f(z) = \ln(1+z)$ yields:

$$\Omega(t,t_0) = \sum_{n=1} \int_{t_0}^{t}\cdots\int_{t_0}^{t} dt_n \cdots dt_1 \; H(t_n)\cdots H(t_1) \; (-1)^{n-1-\Theta_n} \frac{1}{n} \binom{n-1}{\Theta_n}^{-1} \qquad (6)$$

Knowing that Ω is a Lie element and applying Dynkin-Specht-Weaver identity, one can represent (6) in the form:

$$\Omega(t,t_0) = \sum_{n=1} \int_{t_0}^{t}\cdots\int_{t_0}^{t} dt_n \cdots dt_1 \left\{H(t_n)\cdots H(t_1)\right\} \frac{(-1)^{n-1-\Theta}}{n^2} \binom{n-1}{\Theta_n}^{-1} \qquad (7)$$

where $\left\{\cdots\right\}$ denotes the multiple commutator symbol. The result of course contains the "discrete" BCH formula. One obtains the algorithm on $z(x,y)$ defined by $e^z = e^x \cdot e^y$ in the form $z = \Omega(1,-1)$ with $H(t) = \theta(t)x + \theta(t)y$ which can be considered an integral representation of Dynkin's explicit form of the BCH formula. An an immediate application of (7) a closed formula for the phase shifts $\delta_\ell(K)$ in the nonrelativistic theory of scattering was obtained. The more general result (3), (4) can be interpreted in the spirit that each analytic function $f(z)$ induces $f(E-1)$ in the form of some ordering operation acting on $E = T\exp\int_{t_0}^{t} H(t_1)\, dt_1$. Indeed, let $\pi = \begin{pmatrix} n & \cdots & 1 \\ i_n & \cdots & i_1 \end{pmatrix} \in \sum_n$ be understood as an element of the natural algebra of $\sum_n$; one defines π acting on $H(t_n)\cdots H(t_1)$ as $H(t_{i_n})\cdots H(t_{i_1})$; then in the subspace spanned by $H(t_n)\cdots H(t_1)$ one defines the ordering operation $O_n(F)$ according to

$$O_n(F) \overset{df}{=} \frac{1}{2\pi i} \sum_{\pi\in\Sigma_n} \oint \frac{dz}{z^{n+1}} \; f(z)\; (1+z)^{\Theta_n(\pi)} \cdot \pi \quad . \qquad (8)$$

We hope that π in the coefficient is not confused with π understood as permutation; $\Theta_n(\pi)$ means $\theta(i_n - i_{n-1}) + \cdots \theta(i_2 - i1)$. $O_0(F)$ is to be understood as multiplication of the number by $f(0)$. With these rules, understanding the orderings as operations which have to be executed before integrations, one can rewrite (3) in the form:

$$F = f(E - 1) = O(F)E. \qquad (9)$$

The F-orderings can be composed according to the rules of the natural algebra of the group Σ_n (in the subspaces one defines the ordering $O(F)O(G)$ as $O_n(F)O_n(G)$). One can show that $L \overset{df}{=} O(\Omega)$ is idem-potent operation, $L^2 = L$. More generally, the orderings $O(\Omega^m/m!)$ represent idem-potent orthogonal operations.

Basic results presented here were published as Ref. 1 where the proofs are given based on a technique which works with some resolvent function. A more complete paper [2] where, in particular, new results concerning orderings and their composition will be presented, is prepared for publication.

REFERENCES

An excellent review of topics related to the applications of BCH formula was given by Wilcox [3]; here the pertinent references can be found. As the general references source concerning mathematical techniques related to the subject of BCH formula, the monograph, Ref. 4, can be pointed out.

1) I. Białynicki-Birula, B. Mielnik and J. Plebański, Ann. Phys. (N. Y.) 51, 187 (1969).

2) B. Mielnik and J. Plebański, in preparation.

3) R. M. Wilcox, J. Math. Phys. 8, 962 (1967).

4) W. Magnas, A. Karrass and D. Solitar, "Combinatorial Group Theory" (Interscience Publishers, New York 1967).