IC/69/54

INTERNATIONAL ATOMIC ENERGY AGENCY

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

TOPICAL CONFERENCE

ON

DYNAMICAL GROUPS AND INFINITE MULTIPLETS

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

9-14 June 1969

1969

MIRAMARE - TRIESTE

OPERATOR DISTRIBUTIONS IN GROUP REPRESENTATION THEORY AND THEIR APPLICATIONS

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Let G be a locally compact type I Lie group and let X be a homogeneous space relative to G , with a quasi-invariant measure $\mu(\cdot)$. Let $\mathcal{H} = L^2(X,\mu)$ and let $\phi(X) \subset \mathcal{H} \subset \phi'(X)$ be a Gel'fand triplet[1]. For the sake of definiteness assume that $\phi(X)$ is the Schwartz's nuclear space of functions with a compact support.

Denote by $D^{\lambda}_{m,n}(g)$ a matrix element of a unitary irreducible representation T^{λ} of G . A symbol $\lambda = \{\lambda_1 \cdots \lambda_n\}$ represents a set of eigenvalues which determine an irreducible unitary representation T^{λ} of G , whereas a symbol $n = (n_1, \cdots, n_s)$, $[m = (m_1, \cdots, m_s)]$ represents a set of eigenvalues of non-invariant operators which label eigenvectors in an irreducible carrier space of a representation T^{λ} . For simplicity of notation we assume that λ runs over a continuous set of values, whereas n(m) runs over a discrete one.

Let $g \rightarrow T_g$ be a unitary representation of G in $\mathcal{H} = L^2(X,\mu)$. Let

$$P^{\lambda}_{nm} = \rho(\lambda) \int_G dg \; \bar{D}^{\lambda}_{nm}(g) \; T_g \quad . \tag{1}$$

Here $\rho(\lambda)$ is the spectral measure associated with a set $C_1, \cdots, C_n$ of commuting invariant operators of G in the carrier space $\mathcal{H} = L^2(X,\mu)$. One verifies that P^{λ}_{nm} represents a mapping from $\phi(X)$ into $\phi'(X)$.[2] Consequently the quantity P^{λ}_{nm} cannot be considered as an operator in $\mathcal{H}$. However, it has a natural interpretation as an operator-valued distribution (cf. Ref. 2, Sec. 2).

Using the formalism of operator-valued distributions, one derives the following properties of P^{λ}_{nm} (cf. Ref. 2, Sec. 2).

1) $P^{\lambda}_{nm} \; P^{\lambda'}_{n'm'} = \delta(\lambda-\lambda')\, \delta_{mn'}\, P^{\lambda}_{nm'}$

2) $(P^{\lambda}_{nm})^{X} = P^{\lambda}_{mn}$ (2)

3) $Tg \; P^{\lambda}_{nm} = \sum_{r} D^{\lambda}_{rn}(g)\, P^{\lambda}_{rq}$

The properties 1) and 2) of eq. (2) resemble the properties of 0-1 matrices E_{ik}, which form a basis for bounded operators in a Hilbert space. One can show that operator distributions P^{λ}_{nm} form a basis for a certain class of unbounded operators. In fact we have (cf. Ref. 2, Sec. 3):

<u>Proposition</u>. Let Z be any element of an enveloping algebra E of G and let T(Z) be its image in $\mathcal{H} = L^2(X, \mu)$ induced by a representation $g \to Tg$ of G . Let $\{e_n(\lambda, x)\}$ be a complete set of generalized eigenvectors in $\mathcal{H}$. Then

$$T(Z) = \sum_{n,m} \int d\lambda \; Z_{nm}(\lambda)\; P^{\lambda}_{nm} \tag{3}$$

where

$$z_{nm}(\lambda) = \int d\rho(\lambda') \langle T(Z)'\, e_m(\lambda),\; e_n(\lambda') \rangle \; . \tag{4}$$

The operator $T(Z)'$ in eq. (4) represents an extension of the operator T(Z) given by the formula

$$\langle T(Z)'\, e_m(\lambda), \varphi \rangle = \langle e_m(\lambda), T(Z)^{*} \varphi \rangle$$

where φ is any element of the nuclear space $\phi(X)$.

One can show that an operator function

$$E(\lambda) = \sum_{n} \int_{-\infty}^{\lambda} d\lambda \; P^{\lambda}_{nn} \tag{5}$$

represents a common spectral function of a maximal set of commuting operators, which determine a complete set of generalized eigenvectors

$\{e_n(\lambda, x)\}$. In fact we have (cf. Ref. 2, Sec. 3)

$$C_i = \sum_n \int d\lambda \, C_i(\lambda) \, P^\lambda_{nn} \quad , \qquad A_i = \sum_n \int d\lambda \, A_i(n, \lambda) \, P^\lambda_{nn} \quad . \tag{6}$$

Here $C_i(\lambda)$ is an eigenvalue of an invariant operator C_i (i.e., $C_i \, e_n(\lambda, x) = C_i(\lambda) \, e_n(\lambda, x)$) and $A_i(p\lambda)$ is an eigenvalue of a non-invariant operator A_i (i.e., $A_i \, e_n(\lambda, x) = A_i(n, \lambda) \, e_n(\lambda, x)$). If Z is an operator in the enveloping algebra which does not enter into a maximal set of commuting operators then expansion (3) also contains operator distributions P^λ_{nm} with $n \neq m$.

The operator distributions P^λ_{nm} might be used for a solutio of certain operator equations. Let, for instance,

$$(\Box + m^2) \, \phi = \lambda \phi^2 \tag{7}$$

be an equation in $\mathcal{H} = L^2(M^4)$, M^4 be the Minkowski space, and let ϕ be a scalar operator field which commutes with invariant operators of Π in $\mathcal{H}$. Then the expansion (3) takes the form

$$\phi = \int dM \, d\vec{P} \, d\vec{Q} \, \varphi^M(\vec{P}, \vec{Q}) \, P^M_{\vec{P}, \vec{Q}} \tag{8}$$

where

$$P^M_{\vec{P}, \vec{Q}} = M^{-1} \int_\Pi dg \, \bar{D}^M_{\vec{P}, \vec{Q}}(g) \;\; Tg \tag{9}$$

and $D^M_{\vec{P}, \vec{Q}}(g)$ are matrix elements of the irreducible unitary representation $[M, 0]$ of the Poincaré group Π in the momentum basis. The operator $\Box + m^2$ by virtue of eq. (3) can be written in terms of $P^M_{\vec{P}\vec{Q}}$ in the form

$$\Box + m^2 = \int (M^2 + m^2) \, dM \, d\vec{P} \, d\vec{Q} \, P^M_{\vec{P}\vec{Q}} \quad . \tag{10}$$

Inserting eqs. (8) and (10) into eq. (7), utilizing orthogonality properties 1) of eq. (2) of $P^M_{\vec{P}\vec{Q}}$ and comparing c-number coefficients of $P^M_{\vec{P}\vec{Q}}$, one obtains

$$(M^2 + m^2) \, \varphi^M(\vec{P}, \vec{Q}) = \lambda \int d\vec{Q} \, \varphi^M(\vec{P}, \vec{Q}') \, \varphi^M(\vec{Q}', \vec{Q}) \, . \tag{11}$$

Thus one reduces a problem of a solution of an operator equation (7) to a problem of solution of an ordinary integral equation (11). The class of solutions of eq. (11) depends on a boundary condition imposed on the function $\varphi^{M}(\vec{P},\vec{Q})$. The simplest solution of eq. (11) has the form

$$\varphi^{M}(\vec{P},\vec{Q}) = \frac{1}{\lambda}(M^2+m^2)\,\delta(\vec{P}-\vec{Q}) \quad .$$

In general, eq. (11) might have an infinite number of solutions. The operator distributions P^{λ}_{nm} can also be used for an explicit construction of Clebsch-Gordan coefficients for a locally compact Lie group (cf. Refs. 3 and 4).

We now consider a method of classifying two-particle states of two scalar relativistic interacting particles. Let $\mathcal{H}(1,2)$ be a space of square integrable functions $\psi(\overset{1}{p},\overset{2}{p})$ relative to the measure $d_{\mu}(\overset{1}{p},\overset{2}{p}) = d_4\overset{1}{p}\,d_4\overset{2}{p}$. Since we consider an interacting system we do not assume that particles are on mass shells of free particles.

A representation $\{a,\Lambda\} \to U_{\{a,\Lambda\}}$ of the Poincaré group $\tilde{\Pi}$ in $\mathcal{H}(1,2)$ can always be written in the form

$$U_{\{a,\Lambda\}} = T_a V_{\Lambda} \quad . \tag{12}$$

The composition law in $\tilde{\Pi}$ implies

$$V_{\Lambda} T_a V_{\Lambda}^{-1} = T_{\Lambda a} \quad . \tag{13}$$

By von Neumann's theorem the operators P_{μ} of total momentum can be written in $\mathcal{H}(1,2)$ in the form

$$P_{\mu} = \int (\overset{1}{P}_{\mu} + \overset{2}{P}_{\mu})\; dE(\overset{1}{P},\overset{2}{P}) \quad . \tag{14}$$

(This is true at least in so-called direct interaction theories [5].) Using eqs. (5) and (1) for the translation subgroup T_4 of Π, one obtains

$$dE(\overset{1}{p},\overset{2}{p}) \cong \int e^{i(\overset{1}{p}+\overset{2}{p})a}\, T_a\, da \quad . \tag{15}$$

The essential information on interactions is in fact contained in the time translation operator $T_{a^0} = e^{i P_0 a^0}$ with $P_0 = H^1_{free} + H^2_{free} + H_{int}$. Using eqs. (13) and (15) one obtains

$$U_{\{a,L\}} \, dE(p^1, p^2) \, U^{-1}_{\{a,L\}} = dE(\Lambda^1_p, \Lambda^2_p) \quad . \tag{16}$$

Now Mackey's imprimitivity theorem states that if we have in $\mathcal{H}(X, \mu)$, ($X = G/G_0$ is a homogeneous space of G) a spectral measure $dE(x)$ which has the property

$$Tg \; dE(x) \; Tg^{-1} = dE(gx) \tag{17}$$

then a representation $g \to Tg$ in $\mathcal{H}(X, \mu)$ is a representation of G induced by a representation L of a stability subgroup H of the space X [6]. To use the imprimitivity theorem for a classification of admissible representations of $\tilde{\Pi}$ in $\mathcal{H}(1,2)$ we have to split a manifold $X = \{(p^1, p^2)\}$ of all momenta onto transitive submanifolds. Let (p_1, p_2) be any pair of momenta of interacting particles. By means of a Lorentz transformation one can transform a pair (p^1, p^2) to a pair $((p^1_0, 000)^0, (p^2_0 p^2_1 p^2_2 p^2_3))$. Using, further, a rotation, one obtains a "standard pair"

$((p^1_0, 000), (p^2_0, 00p^2_3))$. A stability subgroup of this pair is the subgroup $T^4 \circledS U(1)$. Thus the transitive space defined by the standard pair is a five-dimensional homogeneous space $X_5 = \tilde{\Pi}/T^4 \circledS U(1) \cong SL(2,C)/U(1)$. Consequently, by Mackey's imprimitivity theorem, a representation $U_{\{a,\Lambda\}}$ is an induced representation defined by an orbit (associated with a given "standard pair") and a representation L of U(1) . Clearly all irreducible representations of U(1) are one-dimensional, i.e., $U(1) \ni \varphi \to e^{iNU}$, $N = 0, \pm 1, \pm 2, \ldots$ One verifies that a representation $\{a, L\} \to U^L_{\{a,\Lambda\}}$ induced by an irreducible representation L of U(1) is reducible. Its decomposition into irreducible components has the form

$$U^L = \sum_{J = |N|, |N|+1, |N|+2} T^{M,J}$$

where $M^2 = (p^1 + p^2)$.

One has another standard pair of the form $(p^1, p^2) = ((p^1_0, 000), (p^2_0, 000))$. The transitive manifold defined by this standard pair is a three-dimensional homogeneous space $X_3 = \Pi / T^4 \circledS SU(2) \cong SL(2,C)/SU(2)$. An irreducible representation L of this stability subgroup leads to an irreducible representation of $\tilde{\Pi}$ as in the one-particle case.

If one admits that during the interaction a particle might arise with a space-like or light-like momenta, then one obtains yet other transitive manifolds and the classes of representations of $\tilde{\Pi}$ induced by representations of corresponding stability subgroups.

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