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Variational methods and elliptic equations in Riemannian geometry

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Notes from lectures at ICTP

by

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Abstract

These notes have their origin in a course of introductory nature given by the author at the workshop on recent trends in nonlinear variational problems held at ICTP in april 2003. Scalar curvature type problems on compact Riemannian manifolds and related problems are discussed, with a special emphasis on the H_1^2-theory for blow-up.

Contents

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1 Basic PDE material

We briefly comment here on some constructions and results that we will use in the sequel. This includes a brief discussion on Sobolev spaces, a brief discussion on the regularity and the maximum principles for elliptic type equations, and basic examples of applications of the variational method.

1.1 Sobolev spaces

Given (M, g) a smooth compact n-dimensional Riemannian manifold, one easily defines the Sobolev spaces $H_k^p(M)$, following what is done in the more traditionnal Euclidean context. For instance, when $k = 1$, and $p > 1$, one may define the Sobolev space $H_1^p(M)$ as follows: for $u \in C^\infty(M)$, we let

$$\|u\|_{H_1^p} = \|u\|_p + \|\nabla u\|_p$$

where $\|.\|_p$ is the L^p-norm with respect to the Riemannian measure dv_g. We then define $H_1^p(M)$ as the completion of $C^\infty(M)$ with respect to $\|.\|_{H_1^p}$. A similar definition holds for $H_k^p(M)$, with

$$\|u\|_{H_k^p} = \sum_{i=0}^{k} \|\nabla^i u\|_p$$

Very usefull properties of H_1^p are that Lipschitz functions on M do belong to the Sobolev spaces $H_1^p(M)$ for all p, and that if $u \in H_1^p(M)$ for some p, then $|u| \in H_1^p(M)$ and $|\nabla |u|| = |\nabla u|$ almost everywhere.

As for bounded open subsets of the Euclidean space, the Sobolev embedding theorem (continuous embeddings), and the Rellich-Kondrakov theorem (compact embeddings), do hold. Given $q \in [1, n)$, let

$$q^\star = \frac{nq}{n-q}$$

be the critical Sobolev exponent. One has in the particular case $k = 1$ that:

(1) For any $q \in [1, n)$, and any $p \in [1, q^\star]$, $H_1^q(M) \subset L^p(M)$ and this embedding is continuous, with the property that it is also compact if $p < q^\star$.

(2) For any $q > n$, and any $\alpha \in (0, 1)$ such that $(1 - \alpha)q \geq n$, $H_1^q(M) \subset C^{0,\alpha}(M)$ and this embedding is continuous, with the property that it is also compact if $(1 - \alpha)q > n$.

Similarly, one gets continuous embeddings $H_k^q(M) \subset H_m^p(M)$ if $1 \leq q < p$, $0 \leq m < k$, and $1/p = 1/q - (k - m)/n$, and continuous embeddings $H_k^q(M) \subset C^m(M)$ if $0 \leq m < k$ and $(k - m)q > n$. The same holds for the Rellich-Kondrakov theorem involving H_k^p-spaces, $k \geq 2$.

1.2 Regularity and maximum principles

Let (M, g) be a smooth compact Riemannian manifold. The Laplacian with respect to g is the second order operator whose expression in a chart is given by

$$\Delta_g u = -g^{ij}\left(\partial_{ij}u - \Gamma_{ij}^k \partial_k u\right)$$

where the Γ_{ij}^k's are the Christoffel symbols of the Levi-Civita connection in the chart. The equations we will be interested in are basically of the form

$$\Delta_g u + a(x)u = f(x)$$

where a, f are given functions on M. A function $u \in H_1^2(M)$ is said to be a weak solution of this equation if for all $\varphi \in H_1^2(M)$,

$$\int_M \langle \nabla u, \nabla \varphi \rangle_g dv_g + \int_M a(x)u\varphi dv_g = \int_M f(x)\varphi dv_g$$

Regularity results for this equation do hold. They are similar to the more traditional ones expressed in the Euclidean context (regularity is a local notion, this is not very surprising...). The regularity result we will mostly use is the following: *If a is smooth, and $f \in H_k^p(M)$ for some $k \in \mathbb{N}$ and $p > 1$, then a weak solution u to the above equation is in $H_{k+2}^p(M)$.* In particular, it follows from this result, and the Sobolev embedding theorem, that when f is smooth, u is also smooth. Needless to say, the "bible" for such topics is the exhaustive Gilbarg-Trudinger [19]. A simplier, but very nice reference, is the lecture notes [22] by Han and Lin.

In parallel with regularity, the very useful maximum principles hold for the Laplacian on Riemannian manifolds. A currently used form is as follows: *If a nonnegative $u \in C^2(M)$ is such that for any $x \in M$,*

$$\Delta_g u(x) \geq u(x)f(x, u(x))$$

for some continuous function $f : M \times \mathbb{R} \to \mathbb{R}$, then either u is everywhere positive, or u is the zero function. This easily follows from the Hopf maximum principle, as usually stated.

In order to illustrate in simple situations the variational approach we will use in the sequel, we discuss three elementary problems in what follows. For the sake of clearness, we recall the following result, that we refer to as the Lagrange multipliers theorem.

LMT: (Lagrange multipliers theorem) *Let $(E, \|\cdot\|)$ be a Banach space, Ω be an open subset of E, $f : \Omega \to \mathbb{R}$ be a differentiable function, and $\Phi : \Omega \to \mathbb{R}^n$ be of class C^1. Let also $a \in \mathbb{R}^n$ be such that $\mathcal{H} = \Phi^{-1}(a)$ is not empty. If $x_0 \in \mathcal{H}$ is such that*

$$f(x_0) = \min_{x \in \mathcal{H}} f(x)$$

and if $D\Phi(x_0)$ is surjective, then there exist $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, n$, such that

$$Df(x_0) = \sum_{i=1}^{n} \lambda_i D\Phi_i(x_0)$$

where the Φ_i's are the components of Φ.

The equation $Df(x_0) = \sum_{i=1}^{n} \lambda_i D\Phi_i(x_0)$ is referred to as the Euler-Lagrange equation of the minimization problem $f(x_0) = \min_{x \in \mathcal{H}} f(x)$. The λ_i's are referred to as the Lagrange multipliers of the equation.

1.3 The Rayleigh characterisation of the first nonzero eigenvalue

We let (M, g) be a smooth compact Riemannian manifold. As is well known, λ is an eigenvalue for Δ_g if there exists $u \in C^\infty(M)$, $u \not\equiv 0$, such that $\Delta_g u = \lambda u$. Multiplying this equation by u and integrating over M it is easily seen that if λ is an eigenvalue for Δ_g, then $\lambda \geq 0$. Similar arguments give that $\lambda_0 = 0$ is an eigenvalue for Δ_g and that u is an eigenfunction for $\lambda_0 = 0$ if and only if u is constant. More generally, the set of eigenvalues of Δ_g is a sequence

$$0 = \lambda_0 < \lambda_1 < \ldots < \lambda_n < \ldots < +\infty$$

For instance, on the unit n-sphere (S^n, g_0), $\lambda_k = k(n + k - 1)$, and on the projective space $(\mathbb{P}^n(\mathbb{R}), g_0)$, $\lambda_k = 2k(n + 2k - 1)$. We discuss here the Rayleigh characterisation of the first nonzero eigenvalue λ_1, and prove the following.

Theorem 1.1 *Let (M, g) be a smooth compact Riemannian manifold. If λ_1 is the first nonzero eigenvalue of Δ_g, then*

$$\lambda_1 = \inf_{u \in \mathcal{H}} \frac{\int_M |\nabla u|^2 dv_g}{\int_M u^2 dv_g}$$

where $\mathcal{H}$ is the set consisting of the $u \in H_1^2(M) \backslash \{0\}$ which are such that $\int_M u dv_g = 0$.

Proof: It is easily seen that a similar statement is that

$$\lambda_1 = \inf_{u \in \tilde{\mathcal{H}}} \int_M |\nabla u|^2 dv_g$$

where

$$\tilde{\mathcal{H}} = \left\{ u \in H_1^2(M) \text{ s.t. } \int_M u^2 dv_g = 1 \text{ and } \int_M u dv_g = 0 \right\}$$

We let μ be defined by

$$\mu = \inf_{u \in \tilde{\mathcal{H}}} \int_M |\nabla u|^2 dv_g$$

and let (u_i) in $\tilde{\mathcal{H}}$ be a minimizing sequence for μ. Clearly, (u_i) is bounded in $H_1^2(M)$. Since $H_1^2(M)$ is a reflexive space, and the embedding $H_1^2(M) \subset L^2(M)$ is compact by the Rellich-Kondrakov theorem (even when $n = 2$, noting that $H_1^2 \subset H_1^q$ for $q < 2$), there exists $u \in H_1^2(M)$ and a subsequence (u_i) of (u_i), such that

(1) (u_i) converges weakly to u in $H_1^2(M)$

(2) (u_i) converges to u in $L^2(M)$

By (2), $u \in \tilde{\mathcal{H}}$. Independently, it follows from (1) and a basic property of the weak limit (the norm of a weak limit is less than or equal to the infimum limit of the norms of the sequence) that

$$\|u\|_{H_1^2} \leq \liminf_{i \to +\infty} \|u_i\|_{H_1^2}$$

By (2) we then get that

$$\int_M |\nabla u|^2 dv_g \leq \mu$$

In particular, u is a minimizer for μ, and $\mu > 0$ since $\tilde{\mathcal{H}}$ does not possess constant functions. By the above mentioned Lagrange multipliers theorem, this gives the existence of two constants α and β, the Lagrange multipliers, such that for any $\varphi \in H_1^2(M)$,

$$\int_M \langle \nabla u, \nabla \varphi \rangle_g dv_g = \alpha \int_M \varphi dv_g + \beta \int_M u\varphi dv_g$$

Taking $\varphi = 1$, we get that $\alpha = 0$. Taking $\varphi = u$, we get that $\beta = \mu$. Hence, u is a weak solution of $\Delta_g u = \mu u$. By standard regularity theory, u is smooth. Hence, μ is an eigenvalue of Δ_g. It is easily seen that μ has to be the smallest nonzero eigenvalue of Δ_g, so that $\mu = \lambda_1$. This proves the theorem. □

1.4 The Laplace equation

We discuss here existence (and uniqueness) of a solution u to the Laplace equation

$$\Delta_g u = f$$

on a compact Riemannian manifold (M, g). Though not necessary, we assume for convenience that $f : M \to I\!R$ is smooth. Integrating the Laplace equation, one sees that a necessary condition for the existence of a solution is that

$$\int_M f dv_g = 0$$

The elementary result we wish to briefly discuss here is the following:

Theorem 1.2 *Let (M,g) be a smooth compact Riemannian manifold, and f a smooth function on M. The Laplace equation $\Delta_g u = f$ possesses a smooth solution if and only if $\int_M f dv_g = 0$. Moreover, the solution is unique, up to the addition of a constant.*

Proof: As already mentioned, the condition that f is of null average is a necessary condition. We prove now that it is also a sufficient condition. Let

$$\mathcal{H} = \left\{ u \in H_1^2(M) \text{ s.t. } \int_M u dv_g = 0 \text{ and } \int_M f u dv_g = 1 \right\}$$

and

$$\mu = \inf_{u\in\mathcal{H}} \int_M |\nabla u|^2 dv_g$$

Clearly, $\mathcal{H} \neq \emptyset$. Consider a minimizing sequence $(u_i) \in \mathcal{H}$ for μ: $u_i \in \mathcal{H}$ for all i, and

$$\lim_{i\to+\infty} \int_M |\nabla u_i|^2 dv_g = \mu$$

Thanks to the Rayleigh characterisation of λ_1, as discussed above, if $u \in H_1^2(M)$ is of null average, then

$$\int_M u^2 dv_g \leq \frac{1}{\lambda_1} \int_M |\nabla u|^2 dv_g$$

It easily follows that the u_i's are bounded in $H_1^2(M)$. Since $H_1^2(M)$ is a reflexive space, and the embedding $H_1^2(M) \subset L^2(M)$ is compact by the Rellich-Kondrakov theorem (even when $n = 2$, noting that $H_1^2 \subset H_1^q$ for $q < 2$), there exists $u \in H_1^2(M)$ and a subsequence (u_i) of (u_i), such that

(1) (u_i) converges weakly to u in $H_1^2(M)$

(2) (u_i) converges to u in $L^2(M)$

By (2), $u \in \mathcal{H}$. By (1), and a basic property of the weak limit (the norm of a weak limit is less than or equal to the infimum limit of the norms of the sequence), we get that

$$\int_M |\nabla u|^2 dv_g \leq \mu$$

Hence,

$$\int_M |\nabla u|^2 dv_g = \mu$$

and μ is attained. In particular, $\mu > 0$ since $\mathcal{H}$ does not possess constant functions. By the above mentioned Lagrange multipliers theorem, this gives the existence of two constants α and β, the Lagrange multipliers, such that for any $\varphi \in H_1^2(M)$,

$$\int_M \langle \nabla u, \nabla \varphi \rangle_g dv_g = \alpha \int_M \varphi dv_g + \beta \int_M f \varphi dv_g$$

Taking $\varphi = 1$, one gets that $\alpha = 0$. Taking $\varphi = u$, one gets that $\beta = \mu$. Hence, u is a weak solution of the equation

$$\Delta_g u = \mu f$$

By standard regularity results, u is smooth. The function $\mu^{-1}u$ is then the solution we were looking for.

The proof of uniqueness is also very simple. If u and v are two solutions of the Laplace equation, then $\Delta_g(v-u)=0$. Multiplying this relation by $v-u$, and integrating over M, gives that

$$\int_M |\nabla(v-u)|^2 dv_g = 0$$

Hence, $v-u$ is constant, and this ends the proof of the theorem. □

1.5 Subcritical equations

We let (M,g) be a smooth compact Riemannian manifold of dimension $n\geq 3$, and let h be a smooth function on M. Given $q\in(2,2^\star)$, where $2^\star=2n/(n-2)$ is the critical Sobolev exponent, we consider equations like

$$\begin{cases} \Delta_g u + hu = \lambda u^{q-1} \text{ in } M \\ u > 0 \text{ in } M \end{cases}$$

where $\lambda\in I\!R$. We define

$$\mu = \inf_{u\in\mathcal{H}} \int_M \left(|\nabla u|^2 + hu^2\right) dv_g$$

where

$$\mathcal{H} = \left\{u\in H_1^2(M) \text{ s.t. } \int_M |u|^q dv_g = 1\right\}$$

We prove the following theorem in what follows.

Theorem 1.3 *Let (M,g) be a smooth compact Riemannian manifold of dimension $n\geq 3$, and let h be a smooth function on M. Given $q\in(2,2^\star)$, there exists $u\in C^\infty(M)$, $u>0$ in M, such that*

$$\Delta_g u + hu = \mu u^{q-1}$$

and $\int_M u^q dv_g = 1$, where μ is as above.

Proof: (1) Existence. Here again, the idea is to prove that there exists a (positive) minimizer for μ. We let $(u_i)\in\mathcal{H}$ be a minimizing sequence for μ. Since $|\nabla|u||=|\nabla u|$ a.e., up to replacing u_i by $|u_i|$, we can assume that $u_i\geq 0$ for all i. Since $q>2$, (u_i) is bounded in L^2. In particular, μ is finite, and (u_i) is bounded in $H_1^2(M)$. Since $H_1^2(M)$ is a reflexive space, and the embedding $H_1^2(M)\subset L^q(M)$ is compact by the Rellich-Kondrakov theorem, there exists $u\in H_1^2(M)$ and a subsequence (u_i) of (u_i), such that

(1) (u_i) converges weakly to u in $H_1^2(M)$

(2) (u_i) converges to u in $L^q(M)$

(3) (u_i) converges to u a.e.

By (3), $u\geq 0$, and by (2), $u\in\mathcal{H}$. Independently, it follows from (1) and a basic property of the weak limit (the norm of a weak limit is less than or equal to the infimum limit of the norms of the sequence) that

$$\|u\|_{H_1^2} \leq \liminf_{i\to+\infty} \|u_i\|_{H_1^2}$$

By (2), and since $L^q(M) \subset L^2(M)$, we then get that

$$\mu = \int_M \left(|\nabla u|^2 + hu^2\right) dv_g$$

In particular, u is a minimizer for μ. By the above mentioned Lagrange multipliers theorem, this gives the existence of $\alpha \in I\!R$, the Lagrange multiplier, such that for any $\varphi \in H_1^2(M)$,

$$\int_M \langle \nabla u, \nabla \varphi \rangle_g dv_g + \int_M hu\varphi dv_g = \alpha \int_M u^{q-1}\varphi dv_g$$

Taking $\varphi = u$, we then get that $\alpha = \mu$. It follows that there exists $u \in \mathcal{H}$, $u \geq 0$, a weak solution of our equation.

(2) Regularity. There is still to prove that u is smooth and that $u > 0$. We use a standard bootstrap argument. Let $f = \mu u^{q-1}$, and $p_1 = 2^\star$. Since $u \in H_1^2(M)$, the Sobolev embedding theorem gives that $u \in L^{p_1}(M)$. Hence $f \in L^{p_1/(q-1)}(M)$, and it follows from standard regularity that $u \in H_2^{p_1/(q-1)}(M)$. Using once again the Sobolev embedding theorem, we then get that

$$\begin{cases} u \in L^{p_2}(M)\ , \text{ where} \\ p_2 = \frac{np_1}{n(q-1)-2p_1} \end{cases}$$

if $n(q-1) > 2p_1$, or $u \in L^s(M)$ for all s if $n(q-1) \leq 2p_1$. Going on with such a process, we get by finite induction that $u \in L^s(M)$ for all s. In order to see this, we let $p_0 = n(q-2)/2$. Then $p_1 > p_0$. We define p_i by induction letting

$$\begin{cases} p_{i+1} = \frac{np_i}{n(q-1)-2p_i} & \text{if } n(q-1) > 2p_i \\ p_{i+1} = +\infty & \text{if } n(q-1) \leq 2p_i \end{cases}$$

For any i, $p_i > p_0$. It follows that $p_{i+1} > p_i$. Moreover, $u \in L^{p_{i+1}}(M)$ if $n(q-1) > 2p_i$, and $u \in L^s(M)$ for all s if $n(q-1) \leq 2p_i$. Now, either there exists i such that $p_i > n(q-1)/2$, or $p_i \leq n(q-1)/2$ for all i. In the first case, $p_{i+1} = +\infty$ and we get that $u \in L^s(M)$ for all s. In the second case, (p_i) is an increasing sequence bounded from above. Thus (p_i) converges, and if p is the limit of the p_i's, then

$$p = \frac{np}{n(q-1) - 2p}$$

so that $p = n(q-2)/2$, which is impossible. This proves that $u \in L^s(M)$ for all s. By standard regularity results we then get that $u \in H_2^s(M)$ for all s. In particular, $u \in C^1(M)$ thanks to the Sobolev embedding theorem. Then, since $q > 2$, $u^{q-1} \in C^1(M)$, and, in particular, $u^{q-1} \in H_1^s(M)$ for all s. Thanks to standard regularity results, it follows that $u \in H_3^s(M)$ for all s, and the Sobolev embedding theorem gives that $u \in C^2(M)$. We can now apply the maximum principle. Since $u \not\equiv 0$, we get that $u > 0$ everywhere. By standard regularity results, it easily follows that $u \in C^\infty(M)$. This ends the proof of the theorem. □

1.6 Regularity for the critical equation

We let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$, and let h be a smooth function on M. We consider equations like

$$\Delta_g u + hu = \lambda u^{2^\star - 1}$$

where $\lambda \in I\!R$, $u > 0$, and $2^\star = 2n/(n-2)$ is the critical Sobolev exponent. The existence of a solution to this equation will be discussed in the following chapter. We assume here that there exists $u \in H_1^2(M)$, $u \geq 0$, a weak solution of the above equation. We prove that u is then smooth and either the zero function or everywhere positive. An important remark here is that the above bootstrap argument, as described for the subcritical equation, does not work anymore when dealing with the critical equation. The following theorem is due to Trüdinger [46]. It was then extended by Brézis-Kato [6].

Theorem 1.4 *Let (M,g) be a smooth compact Riemannian manifold of dimension $n \geq 3$, and let h be a smooth function on M. If $u \in H_1^2(M)$, $u \geq 0$, is a weak solution of an equation like*

$$\Delta_g u + hu = \lambda u^{2^\star - 1}$$

where $\lambda \in I\!R$, then u is smooth and either $u \equiv 0$, or $u > 0$ everywhere.

Proof: As already mentionned, the regularity argument discussed for the subcritical equation does not work anymore for the critical equation. However, as easily checked, the argument works if we can prove that $u \in L^s(M)$ for some $s > 2^\star$. Following Trüdinger, we prove the existence of such an $s > 2^\star$ in what follows. Given $L > 0$ we let $F_L : I\!R \to I\!R$ and $G_L : I\!R \to I\!R$ be the Lipschitz functions defined by

$$F_L(t) = |t|^{2^\star/2} \text{ if } |t| \leq L$$
$$F_L(t) = \frac{2^\star}{2} L^{(2^\star-2)/2}|t| - \frac{2^\star - 2}{2} L^{2^\star/2} \text{ if } |t| > L$$

and

$$G_L(t) = |t|^{2^\star - 1} \text{ if } |t| \leq L$$
$$G_L(t) = \frac{2^\star}{2} L^{2^\star-2}|t| - \frac{2^\star - 2}{2} L^{2^\star-1} \text{ if } |t| > L$$

It is easily checked that for $t \geq 0$,

$$F_L(t) \leq t^{2^\star/2},\ G_L(t) \leq t^{2^\star-1},\ F_L(t)^2 \geq tG_L(t)$$
$$\text{and } \left(F_L'(t)\right)^2 \leq \frac{2^\star}{2} G_L'(t) \text{ when } t \neq L$$

We let $\tilde{F}_L = F_L(u)$ and $\tilde{G}_L = G_L(u)$. Since F_L and G_L are Lipschitz functions, $\tilde{F}_L$ and $\tilde{G}_L$ are in $H_1^2(M)$. Now, since u is a weak solution of the equation $\Delta_g u + hu = \lambda u^{2^\star-1}$, we can write that

$$\int_M (\nabla u \nabla \tilde{G}_L) dv_g + \int_M hu\tilde{G}_L dv_g = \lambda \int_M u^{2^\star-1}\tilde{G}_L dv_g$$

Since $\tilde{G}_L(u) \leq u^{2^\star-1}$, and $u \in L^{2^\star}(M)$, it follows that there exist $C_1, C_2 > 0$, independent of L, such that

$$\int_M G_L'(u)|\nabla u|^2 dv_g \leq C_1 + C_2 \int_M u^{2^\star-1}\tilde{G}_L dv_g$$

and since $\left(F_L'(t)\right)^2 \leq \frac{2^\star}{2} G_L'(t)$ and $tG_L(t) \leq F_L(t)^2$, we can write that

$$\frac{2}{2^\star} \int_M |\nabla \tilde{F}_L|^2 dv_g \leq C_1 + C_2 \int_M u^{2^\star-2}\tilde{F}_L^2 dv_g$$

Given $K > 0$, let

$$K^- = \left\{ x \text{ s.t. } u(x) \le K \right\},$$
$$K^+ = \left\{ x \text{ s.t. } u(x) \ge K \right\}$$

Thanks to Hölder's inequalities, and thanks to the Sobolev inequality for the embedding $H_1^2(M) \subset L^{2^\star}(M)$,

$$\begin{aligned}
\int_M u^{2^\star-2}\tilde{F}_L^2 dv_g &= \int_{K^-} u^{2^\star-2}\tilde{F}_L^2 dv_g + \int_{K^+} u^{2^\star-2}\tilde{F}_L^2 dv_g \\
&\le \int_{K^-} u^{2^\star-2}\tilde{F}_L^2 dv_g + \left(\int_{K^+} u^{2^\star} dv_g\right)^{2/n} \left(\int_{K^+} \tilde{F}_L^{2^\star} dv_g\right)^{2/2^\star} \\
&\le \int_{K^-} u^{2^\star-2}\tilde{F}_L^2 dv_g + \varepsilon(K) \left(\int_M \tilde{F}_L^{2^\star} dv_g\right)^{2/2^\star} \\
&\le \int_{K^-} u^{2^\star-2}\tilde{F}_L^2 dv_g + C_3\varepsilon(K) \int_M \left(|\nabla \tilde{F}_L|^2 + \tilde{F}_L^2\right) dv_g
\end{aligned}$$

where $\varepsilon(K) = \left(\int_{K^+} u^{2^\star} dv_g\right)^{2/n}$, and $C_3 > 0$ does not depend on K and L. Since $u \in L^{2^\star}(M)$,

$$\lim_{K\to+\infty} \varepsilon(K) = 0$$

We fix K such that $C_2C_3\varepsilon(K) < \frac{2}{2^\star}$. When $L > K$,

$$\int_{K^-} u^{2^\star-2}\tilde{F}_L^2 dv_g \le K^{2(2^\star-1)} V_g$$

where V_g is the volume of M for g. Independently, since $u \in L^{2^\star}(M)$, and since $F_L(t) \le t^{2^\star/2}$,

$$\int_M \tilde{F}_L^2 dv_g \le C_4$$

where $C_4 > 0$ does not depend on L. It follows that there exist $C_5, C_6 > 0$, independent of L, with $C_6 < 1$, such that

$$\int_M |\nabla \tilde{F}_L|^2 dv_g \le C_5 + C_6 \int_M |\nabla \tilde{F}_L|^2 dv_g$$

Hence,

$$\int_M |\nabla \tilde{F}_L|^2 dv_g \le \frac{C_5}{1 - C_6}$$

and, thanks to the Sobolev inequality for the embedding $H_1^2(M) \subset L^{2^\star}(M)$,

$$\int_M \tilde{F}_L^{2^\star} dv_g \le C_7$$

where $C_7 > 0$ does not depend on L. Letting $L \to +\infty$, it follows that $u \in L^{(2^\star)^2/2}(M)$. Noting that $(2^\star)^2/2 > 2^\star$, we get the existence of some $s > 2^\star$ such that $u \in L^s(M)$. As already mentioned, together with the maximum principle, this proves that u is smooth and that either $u \equiv 0$ or $u > 0$. The theorem is proved.

As an important remark, we claim that there are no a priori C^0-uniform bound for solutions of equations like $\Delta_g u + hu = u^{2^\star-1}$. In order to prove this claim we consider the case of the scalar curvature equation on the unit sphere. If (S^n, g_0) is the unit n-sphere, the scalar curvature equation reads as

$$\Delta_{g_0} u + \frac{n(n-2)}{4} u = u^{2^\star-1}$$

Given $\beta > 1$ and $x_0 \in S^n$, we let $U_{x_0}^\beta$ be the function defined on S^n by

$$U_{x_0}^\beta(x) = \Big(\frac{n(n-2)}{4}(\beta^2-1)\Big)^{\frac{n-2}{4}} \Big(\beta - \cos d_{g_0}(x_0,x)\Big)^{1-\frac{n}{2}}$$

Then the $U_{x_0}^\beta$'s are solutions of the above equation. This follows from conformal invariance, relating the scalar curvature equation on the sphere to the critical equation $\Delta u = u^{2^\star-1}$ on the Euclidean space. Noting that

$$U_{x_0}^\beta(x_0) = \Big(\frac{n(n-2)(\beta+1)}{4(\beta-1)}\Big)^{\frac{n-2}{4}}$$

so that $U_{x_0}^\beta(x_0) \to +\infty$ as $\beta \to 1$, this proves the above claim. On the other hand, thanks to the De Giorgi-Nash-Moser iterative scheme, a C^0-uniform bound follows from L^p-uniform bounds, $p > 2^\star$. The De Giorgi-Nash-Moser iterative scheme is often referred to in the literature as the Moser iterative scheme. The De Giorgi-Nash-Moser iterative scheme gives the following.

MIS: (De Giorgi-Nash-Moser iterative scheme) *Let (M,g) be a smooth compact Riemannian n-manifold, $n \geq 3$, h be a smooth function on M, and $u \in H_1^2(M)$, $u \geq 0$, be such that for any nonnegative $\varphi \in H_1^2(M)$,*

$$\int_M \langle \nabla u, \nabla \varphi \rangle dv_g + \int_M hu\varphi dv_g \leq \int_M u^{2^\star-1}\varphi dv_g \,.$$

Then $u \in L^\infty(M)$. Moreover, for any x in M, any $\Lambda > 0$, any $p > 0$, any $q > 2^\star$, and any $\delta > 0$, if u is some nonnegative function of $H_1^2(M)$ satisfying the above inequation and

$$\int_{B_x(2\delta)} u^q dv_g \leq \Lambda$$

then

$$\sup_{y \in B_x(\delta)} u(y) \leq C\Big(\int_{B_x(2\delta)} u^p dv_g\Big)^{1/p}$$

where $C > 0$ does not depend on u.

A proof of this result can be found in the very nice reference Han-Lin [22]. As stated here, in its first part, the above theorem makes also use of the Trüdinger arguments developed in the proof of Theorem 1.4. As a remark, one passes from small δ's (given by Han-Lin [22]) to arbitrary δ's by writing that

$$\overline{B_x(\delta)} \subset \bigcup_{y \in B_x(\delta)} B_y(\delta_y)$$

where $\delta_y < \delta$. Hence $\overline{B_x(\delta)} \subset \bigcup_{i=1,\dots,N} B_{y_i}(\delta_{y_i})$ for some $y_i \in B_x(\delta)$, $i = 1,\dots,N$. The equations

$$\sup_{B_{y_i}(\delta_{y_i})} u \le C_i \|u\|_{L^p(B_x(2\delta_{y_i}))}$$

then give that

$$\sup_{B_x(\delta)} u \le C\|u\|_{L^p(B_x(2\delta))}$$

since $B_{y_i}(2\delta_{y_i}) \subset B_x(2\delta)$.

2 Existence theory for critical equations

The existence of solutions in the examples we discussed in the preceding section comes from the compactness of the embeddings of H_1^2 in L^p, $p < 2^\star$. We discuss here existence results for a critical equation for which compactness does not hold anymore. A particular case of this equation is the Yamabe equation. The Yamabe equation that we discuss below, is perhaps the most important historical example where the Sobolev embedding we have to consider is critical: continuous, but not anymore compact.

We let (M, g) be a smooth compact Riemannian manifold of dimension $n \ge 3$, and let h be a smooth function on M. We consider equations like

$$\begin{cases} \Delta_g u + hu = \lambda u^{2^\star - 1} \text{ in } M \\ u > 0 \text{ in } M \end{cases}$$

where $\lambda \in I\!R$, and $2^\star = 2n/(n-2)$ is the critical Sobolev exponent. We define

$$\mu = \inf_{u\in\mathcal{H}} \int_M \left(|\nabla u|^2 + hu^2\right) dv_g$$

where

$$\mathcal{H} = \left\{u \in H_1^2(M) \text{ s.t. } \int_M |u|^{2^\star} dv_g = 1\right\}$$

Since $L^{2^\star}(M) \subset L^2(M)$, μ is finite. We distinguish here three cases. The negative case where $\mu < 0$, the null case where $\mu = 0$, and the positive case where $\mu > 0$.

2.1 The negative case

Given $q \in (2, 2^\star)$, we let

$$\mu_q = \inf_{u\in\mathcal{H}_q} \int_M \left(|\nabla u|^2 + hu^2\right) dv_g$$

where

$$\mathcal{H}_q = \left\{u \in H_1^2(M) \text{ s.t. } \int_M |u|^q dv_g = 1\right\}$$

Thanks to the preceding section, Theorem 1.3, there exists $u_q \in C^\infty(M)$, $u_q > 0$, such that

$$\Delta_g u_q + hu_q = \mu_q u_q^{q-1}$$

and $\int_M u_q^q dv_g = 1$. We assume from now on that $\mu < 0$. Hence there exists $u \in \mathcal{H}$ such that $I(u) < 0$, where

$$I(u) = \int_M \left(|\nabla u|^2 + hu^2\right) dv_g$$

Noting that

$$\mu_q \leq I\left(\frac{u}{\left(\int |u|^q\right)^{1/q}}\right)$$

and that $\int_M |u|^q dv_g \leq V_g^{1-\frac{q}{2^\star}}$, where V_g is the volume of M with respect to g, we easily get that there exists $\varepsilon_0 > 0$ such that $\mu_q \leq -\varepsilon_0$ for all $q \in (2, 2^\star)$. In a similar way, we easily get that there exists $K > 0$ such that $\mu_q \geq -K$ for all $q \in (2, 2^\star)$. Hence, there exists $\varepsilon_0 > 0$ such that

$$-\frac{1}{\varepsilon_0} \leq \mu_q \leq -\varepsilon_0$$

for all $q \in (2, 2^\star)$. We let x_q be a point where u_q is maximum. Then $\Delta_g u_q(x_q) \geq 0$. It follows from the equation satisfied by u_q that

$$h(x_q)u_q(x_q) \leq \mu_q u_q^{q-1}(x_q)$$

In particular, $h(x_q) < 0$, and

$$u_q^{q-2}(x_q) \leq \frac{1}{\varepsilon_0} \max_{x \in M} |h(x)|$$

so that the u_q's are uniformly bounded. By standard elliptic theory, the u_q's are then bounded in $H_2^p(M)$ for all p. In particular, a subsequence of the u_q's converge to some u in $C^1(M)$ as $q \to 2^\star$. Assuming that the μ_q's converge to some λ as $q \to 2^\star$, we get that u is a weak solution of

$$\Delta_g u + hu = \lambda u^{2^\star - 1}$$

Moreover, u is nonzero since $\int_M u_q^q dv_g = 1$ and $u_q \to u$ uniformly as $q \to 2^\star$, so that

$$\int_M u^{2^\star} dv_g = 1$$

Thanks to standard regularity theory, and the maximum principle, we then get that u is smooth and everywhere positive. In particular, u is a strong solution of the above equation.

With similar arguments to the ones we discussed above, see also subsection 2.4, we easily get that $\limsup_{q\to 2^\star} \mu_q \leq \mu$. Independently, it is straightforward that $\mu \leq I\left(\left(\int u_q^{2^\star}\right)^{-1/2^\star} u_q\right)$ so that

$$\left(\int_M u_q^{2^\star} dv_g\right)^{2/2^\star} \mu \leq \mu_q$$

for all q. Since $u_q \to u$ uniformly, we have that $\int u_q^{2^\star} \to \int u^{2^\star} = 1$. Hence, we also have that $\liminf_{q\to 2^\star} \mu_q \geq \mu$. It follows that $\mu_q \to \mu$ as $q \to 2^\star$ so that $\lambda = \mu$.

Summarizing, we proved that if $\mu < 0$, then there exists $u \in C^\infty(M)$, $u > 0$, such that

$$\Delta_g u + hu = \mu u^{2^\star - 1}$$

and $\int_M u^{2^\star} dv_g = 1$. In particular, u is a minimizing solution of the equation. Moreover, u is obtained as the uniform limit of a subsequence of the u_q's.

2.2 The null case

Everything here comes from Theorem 1.3. We assume that $\mu = 0$ where

$$\mu = \inf_{u \in \mathcal{H}} \int_M \left(|\nabla u|^2 + hu^2 \right) dv_g$$

and

$$\mathcal{H} = \left\{ u \in H_1^2(M) \text{ s.t. } \int_M |u|^{2^\star} dv_g = 1 \right\}$$

Given $q \in (2, 2^\star)$, we let

$$\mu_q = \inf_{u \in \mathcal{H}_q} \int_M \left(|\nabla u|^2 + hu^2 \right) dv_g$$

where

$$\mathcal{H}_q = \left\{ u \in H_1^2(M) \text{ s.t. } \int_M |u|^q dv_g = 1 \right\}$$

Also, we let $u_q \in C^\infty(M)$, $u_q > 0$, given by Theorem 1.3, be such that

$$\Delta_g u_q + hu_q = \mu_q u_q^{q-1}$$

and $\int_M u_q^q dv_g = 1$. First we claim that if $\mu = 0$, then $\mu_q = 0$ for all q. Given $\varepsilon > 0$, we let $u_\varepsilon \in \mathcal{H}$ be such that $I(u_\varepsilon) \leq \varepsilon$. Thanks to the Sobolev inequality, there exists $A > 0$ such that for any $u \in H_1^2(M)$,

$$\|u\|_{2^\star}^2 \leq A \left(\|\nabla u\|_2^2 + \|u\|_2^2 \right)$$

Taking $u = u_\varepsilon$ in this equation, we get that for any $\varepsilon > 0$,

$$1 \leq A \left(\varepsilon + B \|u_\varepsilon\|_2^2 \right)$$

where $B = 1 + \max_{x \in M} |h(x)|$. Hence, there exists $C > 0$ such that $\|u_\varepsilon\|_2 \geq C$ for all $\varepsilon > 0$ sufficiently small. In particular, for $q > 2$, there exists $C_q > 0$ such that

$$\int_M |u_\varepsilon|^q dv_g \geq C_q$$

Independently, it is clear that $\mu_q \leq I \left(\|u_\varepsilon\|_q^{-1} u_\varepsilon \right)$, so that

$$\|u_\varepsilon\|_q^2 \mu_q \leq \varepsilon$$

Fixing $q > 2$, and letting $\varepsilon \to 0$, it follows that $\mu_q \leq 0$. On the other hand,

$$\mu_q = I(u_q) = \|u_q\|_{2^\star}^2 I \left(\|u_q\|_{2^\star}^{-1} u_q \right) \geq \|u_q\|_{2^\star}^2 \mu$$

so that $\mu_q \geq 0$. This proves the above claim that if $\mu = 0$, then $\mu_q = 0$ for all q. Letting $u = \|u_q\|_{2^\star}^{-1} u_q$ for some q, we get that u is a smooth positive solution of

$$\Delta_g u + hu = \mu u^{2^\star - 1}$$

such that $\int_M u^{2^\star} dv_g = 1$. In particular, u is a minimizing solution of the equation.

2.3 Sharp constants for the Sobolev inequality

We start with a short discussion on the Euclidean space $I\!R^n$, $n \geq 3$. Thanks to Sobolev [42], see also Gagliardo [18] and Nirenberg [33], there exists a positive constant K such that for any $u \in C_0^\infty(I\!R^n)$,

$$\left(\int_{I\!R^n} |u|^{2^\star} dx\right)^{1/2^\star} \leq K \sqrt{\int_{I\!R^n} |\nabla u|^2 dx}$$

where $C_0^\infty(I\!R^n)$ is the space of smooth functions with compact support in $I\!R^n$. The value of the sharp constant K in this inequality is known, thanks for instance to Aubin [2] and Talenti [45]. If K_n stands for the sharp constant,

$$K_n = \sqrt{\frac{4}{n(n-2)\omega_n^{2/n}}}$$

where ω_n is the volume of the unit n-sphere. Taking $K = K_n$ in the above inequality, we get what is referred to as the sharp Euclidean Sobolev inequality.

The extremal functions for the sharp Euclidean Sobolev inequality are known. We let $u : I\!R^n \to I\!R$ be the function defined by

$$u(x) = \left(\frac{1}{1 + \frac{|x|^2}{n(n-2)}}\right)^{\frac{n-2}{2}}$$

Then, u is an extremal function for the sharp Euclidean Sobolev inequality. Moreover, we refer to Caffarelli-Gidas-Spruck [9], u is the unique positive solution of the critical Euclidean equation

$$\Delta u = u^{2^\star - 1}$$

such that $u(0) = 1$ and such that $u(0) = \max_{x \in I\!R^n} u(x)$. Then, we refer once more to Caffarelli-Gidas-Spruck [9], any positive solution $\tilde{u}$ of the critical Euclidean equation $\Delta u = u^{2^\star-1}$ is of the form

$$\tilde{u}(x) = \lambda^{\frac{n-2}{2}} u\big(\lambda(x-a)\big)$$

where $\lambda > 0$ and $a \in I\!R^n$. In particular, the positive solutions of $\Delta u = u^{2^\star-1}$ are extremal functions for the sharp Euclidean Sobolev inequality. For such functions, $\int_{I\!R^n} |\nabla u|^2 dx = K_n^{-n}$ and $\int_{I\!R^n} u^{2^\star} dx = K_n^{-n}$. As a remark, similar arguments to the ones developed in the proof of Theorem 1.4, see for instance Struwe [44], give that if $u \in H_{1,loc}^2(I\!R^n)$ is a weak nonnegative solution of $\Delta u = u^{2^\star-1}$, then u is smooth and either $u \equiv 0$ or $u > 0$.

We let now (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$. We know that $H_1^2(M) \subset L^{2^\star}(M)$. Hence, there exist positive constants A and B such that for any $u \in H_1^2(M)$,

$$\left(\int_M |u|^{2^\star} dv_g\right)^{2/2^\star} \leq A \int_M |\nabla u|^2 dv_g + B \int_M u^2 dv_g$$

Then, easy claims are as follows:

(i) any constant A in this inequality has to be such that $A \geq K_n^2$, and

(ii) for any $\varepsilon > 0$, there exists $B_\varepsilon > 0$ such that the inequality holds with $A = K_n^2 + \varepsilon$ and $B = B_\varepsilon$.

It follows that the sharp constant A in the inequality is K_n^2. A long standing conjecture of Aubin [2] was whether or not the above inequality holds with $A = K_n^2$. The conjecture was proved in Hebey-Vaugon [27, 28]. It follows that for any smooth compact Riemannian manifold (M, g) of dimension $n \geq 3$, there exists $B > 0$ such that for any $u \in H_1^2(M)$,

$$\left(\int_M |u|^{2^\star} dv_g\right)^{2/2^\star} \leq K_n^2 \int_M |\nabla u|^2 dv_g + B \int_M u^2 dv_g$$

and the inequality is sharp. We will use such a sharp inequality many times in the sequel. More material on sharp Sobolev inequalities can be found in the monographs Druet-Hebey [13] and Hebey [23]. We refer also to the notes Druet [11].

2.4 The positive case

We assume in what follows that $\mu > 0$. Recall that

$$\mu = \inf_{u \in \mathcal{H}} \int_M \left(|\nabla u|^2 + hu^2\right) dv_g$$

where

$$\mathcal{H} = \left\{u \in H_1^2(M) \text{ s.t. } \int_M |u|^{2^\star} dv_g = 1\right\}$$

Then, the operator $\Delta_g + h$ is coercive in the sense that its energy controls the H_1^2-norm. More precisely, if $\mu > 0$, then there exists $\lambda > 0$ such that for any $u \in H_1^2(M)$,

$$\int_M \left(|\nabla u|^2 + hu^2\right) dv_g \geq \lambda \|u\|_{H_1^2}^2$$

In order to see this, we note that by Hölder's inequalities, if $\mu > 0$ then there exists $\tilde{\mu} > 0$ such that for any $u \in H_1^2(M)$,

$$\int_M \left(|\nabla u|^2 + hu^2\right) dv_g \geq \tilde{\mu} \int_M u^2 dv_g$$

We let $0 < \varepsilon < \tilde{\mu}/2$ be such that $(1-\varepsilon)\tilde{\mu} + \varepsilon h \geq \tilde{\mu}/2$. Then

$$\begin{aligned}
\int_M \left(|\nabla u|^2 + hu^2\right) dv_g &\geq \varepsilon \int_M \left(|\nabla u|^2 + hu^2\right) dv_g + (1-\varepsilon)\tilde{\mu} \int_M u^2 dv_g \\
&\geq \varepsilon \int_M |\nabla u|^2 dv_g + \frac{\tilde{\mu}}{2} \int_M u^2 dv_g \\
&\geq \varepsilon \int_M \left(|\nabla u|^2 + u^2\right) dv_g
\end{aligned}$$

In particular, if $\mu > 0$, then the operator $\Delta_g + h$ is coercive. We prove the following theorem in this subsection.

Theorem 2.1 *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$, and h be a smooth function on M. If*

$$\inf_{u \in \mathcal{H}} \int_M \left(|\nabla u|^2 + hu^2\right) dv_g < \frac{1}{K_n^2}$$

where K_n is as in subsection 2.3, and $\mathcal{H}$ is as above, then there exists $u \in C^\infty(M)$, $u > 0$, such that

$$\Delta_g u + hu = \mu u^{2^\star - 1}$$

and $\int_M u^{2^\star} dv_g = 1$. In particular, u is a minimizing solution of the critical equation.

There are several historical elementary proofs of this theorem. We prove here the theorem using two distinct elementary approaches.

Proof 1: This is the historical approach, as initiated by Yamabe [49], and then developed by Aubin [3] and Trüdinger [46]. We prove that the solution u of Theorem 2.1 can be obtained as the limit of the subcritical solutions given by Theorem 1.3. Given $q \in (2, 2^\star)$, we let

$$\mu_q = \inf_{u \in \mathcal{H}_q} \int_M \left(|\nabla u|^2 + hu^2\right) dv_g$$

where

$$\mathcal{H}_q = \left\{u \in H_1^2(M) \text{ s.t. } \int_M |u|^q dv_g = 1\right\}$$

Also, we let $u_q \in C^\infty(M)$, $u_q > 0$, given by Theorem 1.3, be such that

$$\Delta_g u_q + hu_q = \mu_q u_q^{q-1}$$

and $\int_M u_q^q dv_g = 1$. An easy claim is that $\mu_q \to \mu$ as $q \to 2^\star$. In order to prove this claim, we proceed as follows. First, given $\varepsilon > 0$, we let $u \in \mathcal{H}$ be such that $I(u) \leq \mu + \varepsilon$. It is clear that $\|u\|_q \to \|u\|_{2^\star}$ as $q \to 2^\star$. Since $u \in \mathcal{H}$, so that $\|u\|_{2^\star} = 1$, it follows that $I\left(\|u\|_q^{-1} u\right) \to I(u)$ as $q \to 2^\star$. Noting that $\|u\|_q^{-1} u \in \mathcal{H}_q$, so that $\mu_q \leq I\left(\|u\|_q^{-1} u\right)$, we then get that $\limsup \mu_q \leq \mu + \varepsilon$ as $q \to 2^\star$. This holds for all $\varepsilon > 0$. Hence,

$$\limsup_{q \to 2^\star} \mu_q \leq \mu$$

Conversely, it follows from Hölder's inequality that

$$1 = \|u_q\|_q^q \leq \|u_q\|_{2^\star}^q V_g^{1 - \frac{q}{2^\star}}$$

where V_g is the volume of M with respect to g. Hence, $\liminf \|u_q\|_{2^\star} \geq 1$ as $q \to 2^\star$. Noting that

$$\mu \|u_q\|_{2^\star} \leq \mu_q = I(u_q)$$

we then get that $\mu_q \geq 0$ for all q, and that

$$\mu \leq \liminf_{q \to 2^\star} \mu_q$$

It follows that

$$\lim_{q \to 2^\star} \mu_q = \mu$$

and the above claim is proved. Since $\Delta_g + h$ is coercive, the u_q's are bounded in $H_1^2(M)$. Therefore, there exists $u \in H_1^2(M)$ such that, up to a subsequence,

(1) $u_q \rightharpoonup u$ in $H_1^2(M)$,

(2) $u_q \to u$ in $L^2(M)$, and

(3) $u_q \to u$ a.e.

Thanks to (3), u is nonnegative. Thanks to (2), $u_q \to u$ in $L^2(M)$. By standard integration theory, if (f_q) is a bounded sequence in $L^p(M)$ for some $p > 1$, and if (f_q) converges a.e. to f, then $f \in L^p(M)$ and $f_q \rightharpoonup f$ in $L^p(M)$. It is clear that the f_q's given by $f_q = u_q^{q-1}$ are bounded in $L^{2^\star/(2^\star-1)}(M)$. Since $f_q \to u^{2^\star-1}$ a.e., it follows that for any $\varphi \in H_1^2(M)$,

$$\int_M u_q^{q-1}\varphi dv_g \to \int_M u^{2^\star-1}\varphi dv_g$$

as $q \to 2^\star$. Independently, we get with (1) and (2) that for any $\varphi \in H_1^2(M)$,

$$\int_M (\Delta_g u_q + hu_q)\varphi dv_g \to \int_M (\langle \nabla u, \nabla\varphi\rangle_g + hu\varphi) dv_g$$

as $q \to 2^\star$. Multiplying by φ the equation satisfied by u_q, and integrating over M, we then get that u is a weak solution of the equation

$$\Delta_g u + hu = \mu u^{2^\star-1}$$

By standard regularity results, as discussed above, and the maximum principle, we get that u is smooth and that either $u \equiv 0$ or $u > 0$ everywhere. In order to prove that $u \not\equiv 0$, we use the energy assumption of the theorem and the sharp Sobolev inequality. It follows from the sharp Sobolev inequality that there exists $B > 0$ such that

$$\|u_q\|_{2^\star}^2 \le K_n^2 \|\nabla u_q\|_2^2 + B\|u_q\|_2^2$$

for all q. Thanks to Hölder's inequality and the equation satisfied by the u_q's we can write that

$$\begin{aligned} 1 = \|u_q\|_q^2 &\le V_g^{2(\frac{1}{q}-\frac{1}{2^\star})}\|u_q\|_{2^\star}^2 \\ &\le V_g^{2(\frac{1}{q}-\frac{1}{2^\star})} K_n^2 \left(I(u_q) + C\|u_q\|_2^2\right) \end{aligned}$$

where V_g is as above, and where

$$C = \frac{B}{K_n^2} + \max_{x\in M} |h(x)|$$

Letting $q \to 2^\star$, it follows that

$$1 \le K_n^2 \left(\mu + C\|u\|_2^2\right)$$

By assumption, $1 > \mu K_n^2$. Hence, $\|u\|_2 > 0$, and $u \not\equiv 0$. In particular, as already mentioned, u is smooth and everywhere positive. Now, in order to end the proof of Theorem 2.1, there is still to prove that $\|u\|_{2^\star} = 1$. Since $u_q \rightharpoonup u$ in $H_1^2(M)$, we also have that $u_q \rightharpoonup u$ in $L^{2^\star}(M)$. Hence, $\|u\|_{2^\star} \le \liminf \|u_q\|_{2^\star}$ as $q \to 2^\star$. It easily follows that $\|u\|_{2^\star} \le 1$ since $\|u_q\|_q = 1$ and $q < 2^\star$. Conversely, multiplying by u the equation satisfied by u, and integrating over M, we get that $I(u) = \mu\|u\|_{2^\star}^{2^\star}$. Hence,

$$I\left(\|u\|_{2^\star}^{-1} u\right) = \mu \|u\|_{2^\star}^{2^\star-2}$$

and since $I\left(\|u\|_{2^\star}^{-1}u\right) \geq \mu$, we also have that $\|u\|_{2^\star} \geq 1$. It follows that $\|u\|_{2^\star} = 1$, and Theorem 2.1 is proved.

Proof 2: We present a more direct and more elegant proof based on convergence arguments as developed in Brézis-Lieb [7]. We let $(u_i) \in \mathcal{H}$ be a minimizing sequence for μ. Up to replacing u_i by $|u_i|$, we can assume that the u_i's are nonnegative. Clearly, (u_i) is bounded in $H_1^2(M)$. After passing to a subsequence, we may thus assume that there exists $u \in H_1^2(M)$ such that $u_i \rightharpoonup u$ weakly in $H_1^2(M)$, $u_i \to u$ strongly in $L^2(M)$, and $u_i \to u$ almost everywhere as $i \to +\infty$. In particular, u is nonnegative. It easily follows from the weak convergence that

$$\|\nabla u_i\|_2^2 = \|\nabla(u_i - u)\|_2^2 + \|\nabla u\|_2^2 + o(1)$$

for all i, where $o(1) \to 0$ as $i \to +\infty$. We also have, see for instance Brézis-Lieb [7], that

$$\|u_i\|_{2^\star}^{2^\star} = \|u_i - u\|_{2^\star}^{2^\star} + \|u\|_{2^\star}^{2^\star} + o(1)$$

for all i, where, as above, $o(1) \to 0$ as $i \to +\infty$. Thanks to the sharp Sobolev inequality, there exists $B > 0$ such that for any i,

$$\|u_i - u\|_{2^\star}^2 \leq K_n^2 \|\nabla(u_i - u)\|_2^2 + B\|u_i - u\|_2^2 \, .$$

Since $u_i \in \mathcal{H}$, it follows that

$$\left(1 - \|u\|_{2^\star}^{2^\star}\right)^{2/2^\star} \leq K_n^2 \left(\|\nabla u_i\|_2^2 - \|\nabla u\|_2^2\right) + o(1) \, .$$

Since $I(u_i) \to \mu$, and since $u_i \to u$ in $L^2(M)$, we also have that

$$\begin{aligned} K_n^2\left(\|\nabla u_i\|_2^2 - \|\nabla u\|_2^2\right) &= K_n^2\mu - K_n^2\left(\int_M |\nabla u|^2 dv_g + \int_M hu^2 dv_g\right) + o(1) \\ &\leq K_n^2\mu - K_n^2\mu\|u\|_{2^\star}^2 + o(1) \, . \end{aligned}$$

Hence,

$$\left(1 - \|u\|_{2^\star}^{2^\star}\right)^{2/2^\star} \leq K_n^2\mu\left(1 - \|u\|_{2^\star}^2\right) \, .$$

We assumed that $\mu K_n^2 < 1$. Noting that

$$1 - \|u\|_{2^\star}^2 \leq \left(1 - \|u\|_{2^\star}^{2^\star}\right)^{2/2^\star} \, ,$$

this implies that $\|u\|_{2^\star} = 1$. Then, $\|\nabla u_i\|_2 \to \|\nabla u\|_2$ as $i \to +\infty$, and since

$$\|\nabla u_i\|_2^2 = \|\nabla(u_i - u)\|_2^2 + \|\nabla u\|_2^2 + o(1) \, ,$$

we get that $u_i \to u$ strongly in $H_1^2(M)$ as $i \to +\infty$. In particular, u is a minimizer for μ, and u is a weak nonnegative solution of the equation

$$\Delta_g u + hu = \mu u^{2^\star - 1} \, .$$

By standard regularity results, and thanks to the maximum principle, u is smooth and positive. This proves Theorem 2.1.

2.5 Sharpness of the condition in Theorem 2.1

An important remark about the condition in Theorem 2.1 is the following.

Proposition 2.1 *For any smooth compact Riemannian manifold (M, g) of dimension $n \geq 3$, and any smooth function h on M,*

$$\inf_{u\in\mathcal{H}} \int_M \left(|\nabla u|^2 + hu^2\right) dv_g \leq \frac{1}{K_n^2}$$

where $\mathcal{H}$ is the set consisting of functions in $H_1^2(M)$ such that $\int_M |u|^{2^\star} dv_g = 1$.

The proposition is a reformulation of the fact that for any (M, g), any $B > 0$, and any constant A, if

$$\|u\|_{2^\star}^2 \leq A\|\nabla u\|_2^2 + B\|u\|_2^2$$

for all $u \in H_1^2(M)$, then A has to be such that $A \geq K_n^2$. A possible proof of this proposition, there are others, is as follows. We let x_0 be some point in M and let $\delta > 0$ small. For $\varepsilon > 0$, we define u_ε by

$$u_\varepsilon(x) = \left(\varepsilon + r^2\right)^{1-\frac{n}{2}} - \left(\varepsilon + \delta^2\right)^{1-\frac{n}{2}} \quad \text{if } r \leq \delta$$
$$u_\varepsilon(x) = 0 \quad \text{if } r \geq \delta$$

where r stands for the distance to x_0. Thus we recognize the extremal functions of the sharp Euclidean Sobolev inequality in the definition of the u_ε's. The u_ε's were introduced in Aubin [3]. Noting that

$$\lim_{\varepsilon\to 0} \frac{\int_M \left(|\nabla u_\varepsilon|^2 + hu_\varepsilon^2\right) dv_g}{\left(\int_M u_\varepsilon^{\frac{2n}{n-2}} dv_g\right)^{\frac{n-2}{n}}} = \frac{1}{K_n^2}$$

we then get the proposition. In particular, the condition in Theorem 2.1 is sharp. The case of equality in Proposition 2.1, and the notions of weakly critical and critical functions are discussed in Hebey-Vaugon [29].

3 The Yamabe problem

We discuss here the Yamabe problem, a problem that is often referred to in the literature on elliptic type equations with critical Sobolev growth. The goal with the Yamabe problem is to prove that, up to conformal changes of the metric, there always exists a metric of constant scalar curvature. This was announced to be true by Yamabe [49] in 1960. Roughly eight years later, Trudinger [46] discovered a serious difficulty in Yamabe's proof. He repaired the proof when the conformal class of the reference metric is nonpositive. Eight years later after Trudinger [46], Aubin [3] improved Yamabe's approach and reduced the problem to the proof of some strict inequality on the Yamabe invariant of the manifold. Such an inequality was proved to be true by Aubin [3] in some cases, and then by Schoen [35] in the remaining more difficult cases. In particular, in his remarkable work, Schoen [35] discovered the unexpected relevance of the positive mass theorem. The Yamabe problem, whose origin goes back to the beginning of the 1960's, was solved something like twenty five years later.

3.1 Introduction

Let (M, g) be a smooth compact n-dimensional Riemannian manifold. We assume in what follows that $n \geq 3$. We denote by $[g]$ the conformal class of the reference metric g. By definition,

$$[g] = \{e^u g \ / \ u \in C^\infty(M)\}$$

A metric $\tilde{g}$ of the form $\tilde{g} = e^u g$ is referred to as a conformal metric to g. We let Rm_g be the Riemann curvature tensor of g, and $Rm_{\tilde{g}}$ be the Riemann curvature of $\tilde{g}$, where $\tilde{g} = e^{2u}g$. By standard computations, one gets that

$$e^{-2u}Rm_{\tilde{g}} = Rm_g - g \odot \left(\nabla^2 u - \nabla u \otimes \nabla u + \frac{1}{2}|\nabla u|_g^2 g\right)$$

In this expression, ∇u and $\nabla^2 u$ stand for the covariant derivatives of u with respect to g. In local coordinates, $(\nabla u)_i = \partial_i u$, while

$$\left(\nabla^2 u\right)_{ij} = \partial_{ij}u - \Gamma_{ij}^k \partial_k u$$

where the Γ_{ij}^k's are the Christoffel symbols of the Levi-Civita connection in the chart. The symbol $\otimes$ stands for the tensorial product, and the symbol $\odot$ stands for the Kulkarni-Nomizu product. It is defined on a vector space E, and for two symmetric bilinear forms h, k on E by

$$\begin{aligned}(h \odot k)(X, Y, Z, T) &= h(X, Z)k(Y, T) + h(Y, T)k(X, Z) \\ &\quad -h(X, T)k(Y, Z) - h(Y, Z)k(X, T)\end{aligned}$$

Contracting twicely the transformation law that relates the Riemann curvatures of g and $\tilde{g}$, and if S_g and $S_{\tilde{g}}$ are the scalar curvatures of g and $\tilde{g}$, one easily gets that

$$e^{2u}S_{\tilde{g}} = S_g + 2(n-1)\Delta_g u - (n-1)(n-2)|\nabla u|_g^2$$

As above, $\Delta_g u$ stands for the Laplacian of u with respect to g, that is minus the trace with respect to g of $\nabla^2 u$:

$$\Delta_g u = -g^{ij}\left(\nabla^2 u\right)_{ij} = -g^{ij}\left(\partial_{ij}u - \Gamma_{ij}^k \partial_k u\right)$$

in local coordinates.

Let us now write $\tilde{g}$ under the form $\tilde{g} = u^{\frac{4}{n-2}}g$, for $u : M \to I\!R$ some smooth positive function. The above relation becomes

$$\Delta_g u + \frac{n-2}{4(n-1)}S_g u = \frac{n-2}{4(n-1)}S_{\tilde{g}}u^{\frac{n+2}{n-2}}$$

This is easily checked.

As already mentioned in the very beginning of this section, the Yamabe problem consists of proving that, up to conformal changes of the metric, there always exists a metric of constant scalar curvature. According to what we just said, this problem also receives a PDE formulation.

The Yamabe problem: (1) Geometric formulation. *For any smooth compact Riemannian manifold (M, g) of dimension n, $n \geq 3$, there exists $\tilde{g} \in [g]$ of constant scalar curvature.*

(2) PDE formulation. *For any smooth compact Riemannian manifold (M, g) of dimension n, $n \geq 3$, there exists $u \in C^\infty(M)$, $u > 0$, and there exists $\lambda \in \mathbb{R}$ such that*

$$\Delta_g u + \frac{n-2}{4(n-1)} S_g u = \lambda u^{\frac{n+2}{n-2}} \tag{E}$$

where Δ_g is the Laplacian with respect to g, and S_g is the scalar curvature of g.

If u and λ satisfy equation (E), and if $\tilde{g} = u^{\frac{4}{n-2}} g$, one gets by the transformation law that relates the scalar curvatures of g and $\tilde{g}$ that

$$S_{\tilde{g}} = \frac{4(n-1)}{n-2} \lambda$$

In particular, this gives a conformal metric to g of constant scalar curvature.

The left hand side in this equation is referred to as the conformal Laplacian. More precisely, the conformal Laplacian, denoted by L_g, is the operator defined by

$$L_g u = \Delta_g u + \frac{n-2}{4(n-1)} S_g u$$

It is important to note that L_g is conformally invariant in the following sense: If $\tilde{g} = \varphi^{\frac{4}{n-2}} g$ is a conformal metric to g, then, for all $u \in C^\infty(M)$,

$$L_{\tilde{g}}(u) = \varphi^{-\frac{n+2}{n-2}} L_g(u\varphi)$$

This follows by computing the difference between the Laplacian with respect to $\tilde{g}$ and the Laplacian with respect to g.

3.2 The Yamabe invariant

We let $\mathcal{H}$ be defined by

$$\mathcal{H} = \left\{ u \in H_1^2(M) \ / \ \int_M |u|^{\frac{2n}{n-2}} dv_g = 1 \right\}$$

and μ_g by

$$\mu_g = \inf_{u \in \mathcal{H}} I(u)$$

where the functional I is as in the statement of theorem 2.1. Our first claim is that μ_g is a conformal invariant. This is the subject of the following lemma:

Lemma 3.1 *If g and $\tilde{g}$ are two conformal metrics, then $\mu_{\tilde{g}} = \mu_g$.*

Proof: The proof of this claim goes in a very simple way, by direct computations. Write that $\tilde{g} = \varphi^{\frac{4}{n-2}} g$. Then, $dv_{\tilde{g}} = \varphi^{2n/(n-2)} dv_g$. Hence, for any u,

$$\int_M |u|^{\frac{2n}{n-2}} dv_{\tilde{g}} = \int_M |u\varphi|^{\frac{2n}{n-2}} dv_g$$

On the other hand, by conformal invariance of the conformal Laplacian,

$$I_{\tilde{g}}(u) = I_g(u\varphi)$$

where the indices g and $\tilde{g}$ mean that the functional has to be considered with respect to $\tilde{g}$ on the left hand side of the relation, and with respect to g on the right hand side of the relation. Clearly, this proves the lemma.

Regarding terminology, μ_g is referred to as the Yamabe invariant of (M,g). Its conformal invariance can be seen from a more interesting point of view. What is referred to as the Hilbert functional in Riemannian geometry, is just the integral of the scalar curvature:

$$\mathcal{H}(g) = \int_M S_g dv_g$$

An Einstein metric g on M is characterized by the property that it is a critical point of this functional when restricted to the set of metrics having the same volume than g. Restricting instead the Hilbert functional to a conformal class, and fixing once more the volume, one gets conformal metrics of constant scalar curvarture. The relation between the Hilbert functional and the Yamabe invariant is expressed in the following lemma.

Lemma 3.2 *Given (M,g) smooth compact of dimension $n \geq 3$,*

$$\mu_g = \frac{n-2}{4(n-1)} \inf_{\tilde{g}\in[g]} V_{\tilde{g}}^{-\frac{n-2}{n}} \int_M S_{\tilde{g}} dv_{\tilde{g}}$$

where μ_g is the Yamabe invariant of (M,g), and $V_{\tilde{g}}$ stands for the volume of M with respect to $\tilde{g}$.

Proof: We just sketch the proof of this result. Let $C_+^\infty(M)$ be the set of smooth positive functions on M. What we claim here, without any proof, is that

$$\mu_g = \inf_{\left\{u\in C_+^\infty(M)\ ,\ \int_M u^{\frac{2n}{n-2}} dv_g=1\right\}} I(u)$$

In other words, one can replace $H_1^2(M)$ by $C_+^\infty(M)$ in the definition of μ_g. This is not very difficult to prove. Let now $\tilde{g}$ be a conformal metric to g. Write $\tilde{g}$ under the form $\tilde{g} = u^{\frac{4}{n-2}} g$. On the one hand,

$$V_{\tilde{g}} = \int_M u^{\frac{2n}{n-2}} dv_g$$

This is very easy to check. On the other hand, multiplying by u the equation that relates the scalar curvature of $\tilde{g}$ to the scalar curvature of g, and then, integrating over M, give that

$$I(u) = \frac{n-2}{4(n-1)} \int_M S_{\tilde{g}} dv_{\tilde{g}}$$

Clearly, this proves the lemma.

Before we go on with the study of the Yamabe problem, let us point out that the sign of μ_g is basically the sign of the scalar curvature in a conformal class. This is the subject of the following theorem:

Theorem 3.1 *Let (M,g) be a smooth compact Riemannian manifold of dimension $n \geq 3$. Then the following holds:*

$$\begin{aligned}
\mu_g > 0 &\Leftrightarrow \exists \tilde{g} \in [g] \ , \quad S_{\tilde{g}} > 0 \\
\mu_g = 0 &\Leftrightarrow \exists \tilde{g} \in [g] \ , \quad S_{\tilde{g}} = 0 \\
\mu_g < 0 &\Leftrightarrow \exists \tilde{g} \in [g] \ , \quad S_{\tilde{g}} < 0
\end{aligned}$$

where by $S_{\tilde{g}} > 0$, respectively $= 0$ and < 0, we mean that the relation holds at any point in M. In particular, one can not get two conformal metrics with scalar curvatures of distinct signs.

Proof: Let us just prove one of these implications, for instance that if $\mu_g > 0$, then there exists $\tilde{g}$ conformal to g with the property that $S_{\tilde{g}}$ is positive. An easy claim here is that if μ_g is positive, then, μ_q in Theorem 1.3, $q \in (2, 2^\star)$, is also positive. Fix q, and set

$$\tilde{g} = u_q^{\frac{4}{n-2}} g$$

where u_q is given by Theorem 1.3. Then, according to the equation satisfied by u_q, here $h = \frac{n-2}{4(n-1)} S_g$, and to the transformation law that relates the scalar curvature of g and $\tilde{g}$, we get that

$$S_{\tilde{g}} = \frac{4(n-1)}{n-2} \mu_q u_q^{q - \frac{n+2}{n-2}}$$

In particular, $S_{\tilde{g}}$ is positive.

Note that according to this theorem, the sign of the scalar curvature is a conformal invariant. In dimension $n = 2$, the sign of the scalar curvature is even a topological invariant according to the Gauss-Bonnet theorem. One has indeed when $n = 2$ that the Euler characteristic $\chi(M)$ of M is related to the scalar curvature by the relation

$$\chi(M) = \frac{1}{4\pi} \int_M S_g dv_g$$

For $n \geq 3$, the sign of the scalar curvature is not anymore a global invariant. One may prove that any compact Riemannian manifold of dimension $n \geq 3$ possesses a metric of negative scalar curvature. This result extends to the Ricci curvature. On the contrary, according to works by Gromov-Lawson [20, 21], Lichnerowicz [31], and also Schoen and Yau [38], there are manifolds which do not possess metrics of positive scalar curvature. On the one hand, negative scalar curvature, and even negative Ricci curvature, are given for free. On the other hand, the question of the existence of a metric of positive scalar curvature on a given manifold carries obstructions. An interesting case to keep in mind is that torii do not possess metrics of positive scalar curvature.

3.3 Resolution of the problem

Given (M,g) compact, we assume that $\mu_g > 0$. By proposition 2.1, $\mu_g \leq K_n^{-2}$. Moreover, we can prove that $\mu_g = K_n^{-2}$ when (M,g) is conformally diffeomorphic to the unit sphere. Thanks to this remark, and thanks to Theorem 2.1, the Yamabe problem was historically reduced to the following.

RP: (Reduced problem) *Prove that if (M,g) is not conformally diffeomorphic to the unit sphere (S^n,h) of same dimension, then $\mu_g < K_n^{-2}$.*

In other words, the Yamabe problem was reduced to the proof that if (M,g) is not conformally diffeomorphic to the unit sphere of same dimension, then there exists $u \in H_1^2(M)$, $u \not\equiv 0$, such that

$$J(u) < \frac{1}{K_n^2}$$

where

$$J(u) = \frac{\int_M |\nabla u|_g^2 dv_g + \frac{n-2}{4(n-1)} \int_M S_g u^2 dv_g}{\left(\int_M |u|^{\frac{2n}{n-2}} dv_g\right)^{\frac{n-2}{n}}}$$

This approach leads to the following question.

Question: *Given (M,g) compact with $\mu_g > 0$, find condition(s) that ensure that (M,g) is not conformally diffeomorphic to the unit sphere.*

Since conformal diffeomorphisms preserve conformal flatness, and (S^n,h) is conformally flat, an easy condition is as follows:

C1 (Aubin, [3]): (M,g) is not conformally flat.

Under this condition, Aubin [3] was able to solve the problem when $n \geq 6$. We outline his argument here. As test functions, we consider the extremals for the sharp Euclidean Sobolev inequality

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{n}} \leq K_n^2 \int_{\mathbb{R}^n} |\nabla u|^2 dx$$

that we cut off to make them zero at the boundary of a small ball. In other words, given $x \in M$, $\delta > 0$ small, and $\varepsilon > 0$, we let $u_{x,\varepsilon}$ be given by

$$u_{x,\varepsilon} = \left(\varepsilon + r^2\right)^{1-\frac{n}{2}} - \left(\varepsilon + \delta^2\right)^{1-\frac{n}{2}} \quad \text{if } r \leq \delta$$
$$u_{x,\varepsilon} = 0 \ \text{ if not}$$

where r is the distance to x. For $n \geq 4$, the conformal flatness of (M,g) is characterized by the nullity of the Weyl tensor W_g. Assuming $C1$, there exists x in M such that $|W_g(x)| > 0$. We fix such an x. Up to a conformal change of the metric, we may assume that $Rc_g(x) = 0$, where Rc_g is the Ricci curvature of g. This is an easy claim. By conformal invariance of the Weyl tensor, the relation $|W_g(x)| > 0$ still holds. Standard computations, see [3], then give the following:

$$J\left(u_{x,\varepsilon}\right) = \frac{1}{K_6^2}\left(1 + C_1|W_g(x)|^2\varepsilon^2 \ln\varepsilon + o\left(\varepsilon^2 \ln\varepsilon\right)\right) \quad \text{if } n = 6$$
$$J\left(u_{x,\varepsilon}\right) = \frac{1}{K_n^2}\left(1 - C_2|W_g(x)|^2\varepsilon^2 + o\left(\varepsilon^2\right)\right) \quad \text{if } n > 6$$

where $C_1 > 0$ and $C_2 > 0$ depend only on n. Hence, for $\varepsilon > 0$ small,

$$J\left(u_{x,\varepsilon}\right) < \frac{1}{K_n^2}$$

Summarizing, $\mu_g < K_n^{-2}$ if (M,g) is not conformally flat and $n \geq 6$. This solves the Yamabe problem for such manifolds.

When (M,g) is conformally flat, the global geometry has to be taken into consideration and we need to be more subtle. This case, together with the case of manifolds of dimensions 3, 4, or 5, was solved by Schoen [35] in a remarkable paper. We outline here his approach when the manifold is conformally flat. We therefore assume in what follows that (M,g) is conformally flat with $\mu_g > 0$. Given $x \in M$, we let G_x be the Green function at x of the conformal Laplacian. If the metric g is chosen such that it is flat around x, then, for y close to x,

$$G_x(y) = \frac{1}{(n-2)\omega_{n-1}r^{n-2}} + \alpha_x(y)$$

where r is the distance from x to y, and α_x is a smooth function. This is easy to check. Much more subtle is the observation, due to Schoen [35], that $\alpha_x(x)$ is, up to a positive scale factor, the mass of the asymptotically flat manifold

$$(M_\infty, g_\infty) = \left(M\backslash\{x\}, G_x^{\frac{4}{n-2}}g\right)$$

A long standing conjecture in the Physics litterature, going back to works by Arnowitt, Deser, Gibbons, Hawking, Misner, and Perry, is that, roughly speaking, the mass of an asymptotically flat manifold is nonnegative, and null if and only if the manifold is isometric to the Euclidean space of same dimension. This conjecture was solved, at least when n is not too large, by Schoen and Yau, see [39, 40], but also Schoen [36]. We refer also to Witten [48] for the proof of this conjecture in the spinorial case. Coming back to our initial situation, one then gets the following statement:

PMT: (Positive mass theorem - Weak form) *The quantity $\alpha_x(x)$, usually referred to as the energy of g at x, is nonnegative, and null if and only if (M,g) is conformally diffeomorphic to the unit sphere.*

A very nice direct proof of this weak form of the positive mass theorem is in Schoen and Yau [41]. With respect to the Yamabe problem, one gets here the desired condition that ensures that a conformally flat manifold is not conformally diffeomorphic to the unit sphere:

C2 (Schoen, [35]): (M,g) is conformally flat, but has positive energy.

In such a case, the argument of Schoen [35] goes as follows: Given $x \in M$, we choose g such that it is flat around x, and, for $\varepsilon > 0$, we let the $u_{x,\varepsilon}$'s be defined by

$$\begin{aligned} &u_{x,\varepsilon} = \left(\varepsilon + r^2\right)^{1-\frac{n}{2}} \text{ if } r \leq \delta \\ &u_{x,\varepsilon} = A\left(G_x - \eta\tilde{\alpha}_x\right) \text{ if } \delta \leq r \leq 2\delta \\ &u_{x,\varepsilon} = AG_x \text{ if } r \geq 2\delta \end{aligned}$$

Here, r is the distance to x, $\tilde{\alpha}_x = \alpha_x - \alpha_x(x)$, η is a cut-off function between δ, 2δ with suitable properties, and A is chosen such that the $u_{x,\varepsilon}$ are continuous across $\partial B_x(\delta)$. Following Schoen [35], one then find that for $\delta > 0$ small enough,

$$I\left(u_{x,\varepsilon}\right) \leq \frac{1}{K_n^2}\left(\int_M u_{x,\varepsilon}^{\frac{2n}{n-2}}dv_g\right)^{\frac{n-2}{n}} - C_1\alpha_x(x)\varepsilon^{\frac{n}{2}-1} + o\left(\varepsilon^{\frac{n}{2}-1}\right)$$

where $C_1 > 0$ does not depend on ε. Here again, this gives that for ε small, $J(u_{x,\varepsilon}) < K_n^{-2}$. Hence, $\mu_g < K_n^{-2}$ if (M, g) is conformally flat. This solves the Yamabe problem for such manifolds. When $n = 3$, 4, or 5, though more tricky, the argument basically goes in the same way. We refer to Schoen [35] for details.

Depending on whether the manifold is not conformally flat of dimension $n \geq 6$, conformally flat, or of dimension $n = 3$, 4, or 5, distinct arguments have to be used. This is clearly not a satisfactory situation, and we are left with the problem of finding an argument that unifies those of Aubin and Schoen. This was first done by Lee and Parker in their very nice survey paper [30]. The unified argument that we present here is due to Hebey and Vaugon [26].

Theorem 3.2 *Let (M, g) be a smooth compact n-dimensional Riemannian manifold, $n \geq 3$, that we assume not to be conformally diffeomorphic to the unit sphere. Given $x \in M$, and $\delta > 0$ small, the functions $u_{x,\varepsilon}$, $\varepsilon > 0$, defined by*

$$u_{x,\varepsilon} = \left(\varepsilon + r^2\right)^{1-\frac{n}{2}} \quad if\, r \leq \delta$$
$$u_{x,\varepsilon} = \left(\varepsilon + \delta^2\right)^{1-\frac{n}{2}} \quad if\, r \geq \delta$$

give the inequality $\mu_g < K_n^{-2}$. In particular, the Yamabe problem is affirmatively solved.

Proof: We discuss the proof of this theorem when the manifold is either not conformally flat and of dimension $n \geq 6$, or conformally flat. When $n = 3$, 4, or 5, the argument basically goes in the same way.

We suppose first that (M, g) is not conformally flat and of dimension $n \geq 6$. We fix as above x such that $|W_g(x)| > 0$, and choose the metric g such that $Rc_g(x) = 0$. The same computations as those made in Aubin [3] then lead to similar expansions. In particular, for $\varepsilon > 0$ small,

$$J(u_{x,\varepsilon}) < \frac{1}{K_n^2}$$

and the theorem is proved when (M, g) is not conformally flat of dimension $n \geq 6$.

Suppose now that g is conformally flat. Given $x \in M$, choose g such that it is flat around x. Easy computations then give that

$$J(u_{x,\varepsilon}) \leq \frac{1}{K_n^2} + C\varepsilon^{\frac{n}{2}-1}\left(\frac{n-2}{4(n-1)}\int_M S_g dv_g - \frac{(n-2)\omega_{n-1}\delta^n}{\varepsilon + \delta^2}\right) + o\left(\varepsilon^{\frac{n}{2}-1}\right)$$

The argument then follows from the equation

$$\frac{n-2}{4(n-1)}\alpha_x(x) = \sup\left(\frac{1}{\int_M S_{\tilde{g}} dv_{\tilde{g}}} - \frac{1}{4(n-1)\omega_{n-1}\rho^{n-2}}\right)$$

where, for δ such that g is flat on $B_x(\delta)$, the supremum is taken over $\rho \in (0, \delta)$ and $\tilde{g} \in [g]$ such that $\tilde{g} = g$ in $B_x(\rho)$. (See [26] for the proof of this equation). Since $\alpha_x(x) > 0$ if (M, g) is not conformally diffeomorphic to the unit sphere, we may choose g and δ such that

$$\frac{n-2}{4(n-1)}\int_M S_g dv_g - \frac{(n-2)\omega_{n-1}\delta^n}{\varepsilon + \delta^2} < 0$$

Hence, for $\varepsilon > 0$ small,

$$J\left(u_{x,\varepsilon}\right) < \frac{1}{K_n^2}$$

and the theorem is proved when (M, g) is conformally flat, but not conformally diffeomorphic to the unit sphere. As already mentioned, the argument basically goes in the same way for manifolds of dimension $n = 3$, 4, or 5.

Other proofs of the Yamabe problem have appeared recently. This includes a "topological" proof by Bahri [4] when the manifold is conformally flat. This includes also in the same spirit a proof by Bahri and Brezis [5] for manifolds of dimension $n = 3$, 4, and 5. A remarkable approach was presented by Schoen [36, 37]. In particular, it follows from Schoen [36, 37] that solutions of the Yamabe equation are compact in the C^2-topology. This result was first proved for conformally flat manifolds in [36] under the additional assumption that the energy of the solutions is bounded. The bound on the energy was removed, and a more direct and elegant proof was presented in [37], however still in the case of conformally flat manifolds. A proof of this result of Schoen for arbitrary manifolds of dimensions 3, 4, and 5 was recently obtained by Druet [12]. The argument in Druet [12] is based on the C^0-theory for blow-up developed by Druet-Hebey-Robert [15, 16].

4 Palais-Smale sequences

As above, we let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$. We consider equations like

$$\Delta_g u + hu = u^{2^\star - 1}$$

and $u > 0$, where h is a smooth function on M. We assume in what follows that the operator $\Delta_g + h$ is coercive so that there exists $\lambda > 0$ such that for any $u \in H_1^2(M)$,

$$\int_M \left(|\nabla u|^2 + hu^2\right) dv_g \geq \lambda \|u\|_{H_1^2}^2$$

This is necessarily the case if h is a positive function.

4.1 Definition and the mountain pass lemma

We let J be the free functional defined on $H_1^2(M)$ by

$$J(u) = \frac{1}{2} \int_M \left(|\nabla u|^2 + hu^2\right) dv_g - \frac{1}{2^\star} \int_M |u|^{2^\star} dv_g$$

By definition, a sequence (u_i) of functions in $H_1^2(M)$ is said to be a Palais-Smale sequence for J if:

(i) $J(u_i)$ is bounded with respect to i, and

(ii) $DJ(u_i) \to 0$ in $H_1^2(M)'$

as $i \to +\infty$. A basic tool to construct Palais-Smale sequences is the mountain pass lemma of Ambrosetti-Rabinowitz [1]. We use the mountain pass lemma under the following form:

MPL: (Ambrosetti-Rabinowitz) *Let Φ be a C^1 function on a Banach space E. Suppose that there exist a neighbourhood U of 0 in E, $u_0 \in E \backslash U$, and a constant ρ such that*

$$\Phi(0) < \rho \ , \ \ \Phi(u_0) < \rho \ , \ \ and \, \Phi(u) \geq \rho$$

for all $u \in \partial U$. Let

$$c = \inf_{\gamma \in \Gamma} \max_{u \in \gamma} \Phi(u)$$

where Γ stands for the class of continuous paths joining 0 to u_0. Then there exists a sequence (u_i) in E such that $\Phi(u_i) \to c$ and $D\Phi(u_i) \to 0$ in E' as $i \to +\infty$.

In our case, $E = H_1^2(M)$, and $\Phi = J$. We let $U = B_0(r)$ be the ball of center 0 and radius r in $H_1^2(M)$. Since $\Delta_g + h$ is coercive, and thanks to the Sobolev inequality corresponding to the embedding of H_1^2 in $L^{2^\star}$, there exists positive constants $C_1, C_2 > 0$ such that for any $u \in H_1^2(M)$,

$$J(u) \geq C_1 \|u\|_{H_1^2}^2 - C_2 \|u\|_{H_1^2}^{2^\star}$$

Taking $r > 0$ sufficiently small, it follows that there exists $\rho > 0$ such that for any $u \in \partial B_0(r)$, $J(u) \geq \rho$. Independently, $J(0) = 0$, while, for $v_0 \in H_1^2(M)$, $v_0 \not\equiv 0$,

$$\lim_{t \to +\infty} J(tv_0) = -\infty$$

It follows that there exists $r > 0$, $\rho > 0$, and $u_0 = tv_0$ such that $J(0) < \rho$, $J(u_0) < \rho$, $u_0 \in H_1^2(M) \backslash B_0(r)$, and $J(u) \geq \rho$ for all $u \in \partial B_0(r)$. The mountain pass lemma then gives the existence of a Palais-Smale sequence for J. A slightly more subtle argument allows one to construct Palais-Smale sequences of positive functions.

4.2 The existence part of Theorem 2.1

As a remark, we prove here that this notion of a Palais-Smale sequence allows one to recover the existence part in Theorem 2.1, namely that there exists $u > 0$ smooth, a solution of

$$\Delta_g u + hu = u^{2^\star - 1}$$

This was first noticed by Brézis and Nirenberg [8]. We follow their approach. In order to get positive solutions, we slightly change J into J^+ defined by

$$J^+(u) = \frac{1}{2} \int_M \left(|\nabla u|^2 + hu^2 \right) dv_g - \frac{1}{2^\star} \int_M (u^+)^{2^\star} dv_g$$

where $u^+ = \max(0, u)$. As in Theorem 2.1, we assume that

$$\inf_{u \in \mathcal{H}} I(u) < \frac{1}{K_n^2}$$

where $\mathcal{H}$ is the set of functions in $H_1^2(M)$ which are such that $\|u\|_{2^\star} = 1$, and where I is the functional defined by

$$I(u) = \int_M (|\nabla u|^2 + hu^2) dv_g$$

Then there exists $v_0 \in \mathcal{H}$, $v_0 \geq 0$, such that $I(v_0) < K_n^{-2}$. Noting that

$$J^+(tv_0) = \frac{I(u_0)}{2}t^2 - \frac{1}{2^\star}t^{2^\star}$$

we then get that

$$\begin{aligned}\max_{t\geq 0} J^+(tv_0) &= J^+(t_0 v_0) \\ &= \frac{1}{n} I(v_0)^{n/2}\end{aligned}$$

where $t_0 = I(v_0)^{4/(n-2)}$. Hence,

$$\max_{t\geq 0} J^+(tv_0) < \frac{1}{nK_n^n}$$

Let $r > 0$ be small, and $t > 0$ be large. Using the MPL as above, with $U = B_0(r)$ and $u_0 = tv_0$, we then get the existence of a Palais-Smale sequence (u_i) for J^+ such that $J^+(u_i) \to c$ as $i \to +\infty$, where

$$\begin{aligned} c &= \inf_{\gamma\in\Gamma} \max_{u\in\gamma} J^+(u) \\ &\leq \max_{t\geq 0} J^+(tu_0) \\ &< \frac{1}{nK_n^n}\end{aligned}$$

where Γ stands for the class of continuous paths joining 0 to u_0. In particular, we can write that

$$\frac{1}{2}\int_M \left(|\nabla u_i|^2 + hu_i^2\right) dv_g = \frac{1}{2^\star}\int_M (u_i^+)^{2^\star} dv_g + c + o(1) \tag{4.1}$$

and writing that $DJ^+(u_i).\tilde{u}_i \to 0$ as $i \to +\infty$, where $\tilde{u}_i = \|u_i\|_{H_1^2}^{-1} u_i$, we can write that

$$\int_M \left(|\nabla u_i|^2 + hu_i^2\right) dv_g = \int_M (u_i^+)^{2^\star} dv_g + o(\|u_i\|_{H_1^2}) \tag{4.2}$$

Considering (4.1) $- \frac{1}{2^\star}$(4.2), and since $\Delta_g + h$ is coercive, we get that the u_i's are bounded in $H_1^2(M)$. It follows that there exists $u \in H_1^2(M)$ such that, up to a subsequence,

(i) $u_i \rightharpoonup u$ in $H_1^2(M)$,

(ii) $u_i \to u$ and $u_i^+ \to u^+$ in $L^2(M)$, and

(iii) $u_i \to u$ a.e.

as $i \to +\infty$. Since $DJ^+(u_i) \to 0$ as $i \to +\infty$, we have that for any $\varphi \in H_1^2(M)$,

$$\int_M (\nabla u_i \nabla \varphi) dv_g + \int_M hu_i\varphi dv_g = \int_M (u_i^+)^{2^\star - 1}\varphi dv_g$$

Passing to the limit as $i \to +\infty$, it follows that u is a weak solution of

$$\Delta_g u + hu = (u^+)^{2^\star - 1}$$

In particular,

$$\int_M (\nabla u \nabla u^-) dv_g + \int_M u u^- dv_g = 0$$

so that

$$\int_M \left(|\nabla u^-|^2 + h(u^-)^2 \right) dv_g = 0$$

and since $\Delta_g + h$ is coercive, it follows that $u^- \equiv 0$. Hence, $u \geq 0$, and u is a weak solution of

$$\Delta_g u + hu = u^{2^\star - 1}$$

Now there is still to prove that $u \not\equiv 0$. Let us assume that $u \equiv 0$. Without loss of generality, we can also assume that

$$\int_M |\nabla u_i|^2 dv_g \to S$$

as $i \to +\infty$ for some $S \geq 0$. Passing to the limit in (4.1) and (4.2), noting that

$$\int_M u_i^2 dv_g \to 0$$

as $i \to +\infty$, we then get that

$$\int_M (u_i^+)^{2^\star} dv_g \to S$$

as $i \to +\infty$, and that $S = nc$, where c is as above. By the sharp Sobolev inequality,

$$\begin{aligned} \|u_i^+\|_{2^\star}^2 &\leq \|u_i\|_{2^\star}^2 \\ &\leq K_n^2 \|\nabla u_i\|_2^2 + B\|u_i\|_2^2 \end{aligned}$$

for some $B > 0$ and all i. Passing to the limit as $i \to +\infty$, it follows that

$$S^{2/2^\star} \leq K_n^2 S$$

so that

$$K_n S^{1/n} \geq 1$$

Noting that this is in contradiction with the equations

$$S = nc \text{ and } c < K_n^{-n}/n$$

we get that $u \not\equiv 0$. Thus, there exists $u > 0$ smooth, a solution of the equation

$$\Delta_g u + hu = u^{2^\star - 1}$$

When dealing with Palais-Smale sequences we loose a priori the minimality of the solution. On the other hand, approaches based on Palais-Smale sequences avoid the use of Euler-Lagrange multipliers. Thus we can deal with more general equations. As in Brézis-Nirenberg [8], this includes the case of equations like $\Delta_g u + hu = u^{2^\star - 1} + fu^q$ where f is a smooth function and $1 < q < 2^\star - 1$.

5 Blow-up theory in the H_1^2-Sobolev space

The very first arguments we developed were based on the compactness of the embeddings of the Sobolev space H_1^2 into Lebesgue spaces L^p. Then we discussed arguments where the critical exponent arises, but the energy is low. There is still to explain what happens when we do face a critical exponent problem and the energy is arbitrary. An important notion here is the notion of blow-up points, sometimes referred to as concentration points. A setting where this notion appears naturally is when discussing Palais-Smale sequences associated to equations like

$$\Delta_g u + hu = u^{2^\star - 1} \tag{E}$$

where h is a smooth function on M, not necessarily such that $\Delta_g + h$ is coercive. As in the preceding section, we let J be the functional defined on $H_1^2(M)$ by

$$J(u) = \frac{1}{2}\int_M \left(|\nabla u|^2 + hu^2\right) dv_g - \frac{1}{2^\star}\int_M |u|^{2^\star} dv_g$$

By definition, as above, a sequence (u_i) of functions in $H_1^2(M)$ is said to be a Palais-Smale sequence for J if:

(i) $J(u_i)$ is bounded with respect to i, and

(ii) $DJ(u_i) \to 0$ in $H_1^2(M)'$

as $i \to +\infty$. The very general question we are concerned with in this section is to characterize the asymptotic behaviour of Palais-Smale sequences as $i \to +\infty$. For the sake of clearness we restrict ourselves to Palais-Smale sequences of nonnegative functions. A similar result exists when no sign assumption is made on the u_i's.

The answer to the question we are concerned with in this section involves several contributions. Among others, we quote Lions [32], Sacks-Uhlenbeck [34], Wente [47], and various works by Schoen. However, the final result, as we state it below, is due to Struwe [43]. Struwe was concerned with the Euclidean equation $\Delta u = u^{2^\star - 1}$ on bounded domains with a zero Dirichlet condition on the boundary. As noticed by several authors, his result extends with basically no changes in the proof to the Riemannian setting and equations like the ones we consider. The Struwe result is then what we refer to as the H_1^2-theory for blow-up. The stronger C^0-theory was recently developed by Druet-Hebey-Robert [15, 16].

In order to state the Struwe result, we need the important notion of a bubble. We define the notion of a bubble as follows.

Definition 5.1 *Given a sequence (x_i) of points in M, and a sequence (μ_i) of positive real numbers, such that $\mu_i \to 0$ as $i \to +\infty$, we define a bubble as a sequence (B_i) of functions given by the following equations:*

$$B_i(x) = \left(\frac{\mu_i}{\mu_i^2 + \frac{d_g(x_i,x)^2}{n(n-2)}}\right)^{\frac{n-2}{2}}$$

where d_g is the distance with respect to g. We refer to the x_i's as the centers of the bubble, and to the μ_i's as the weights of the bubble.

It is easily seen that a bubble converges strongly to 0 outside the x_i's, while $B_i(x_i) \to +\infty$ as $i \to +\infty$. Note that in the above definition we recover the expression of the extremal functions of the sharp Euclidean Sobolev inequality

$$\left(\int_{\mathbb{R}^n} |u|^{2^\star} dx\right)^{\frac{2}{2^\star}} \le K_n^2 \int_{\mathbb{R}^n} |\nabla u|^2 dx$$

Bubbles are, in some sense, rescalings of fundamental solutions of the Euclidean equation $\Delta u = u^{2^\star - 1}$. The Struwe result can then be stated as follows.

Theorem 5.1 *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \ge 3$, and h be a smooth function on M. Let also (u_i) be a Palais-Smale sequence of nonnegative functions for J. Then there exists $k \in \mathbb{N}$, there exists $u^0 \ge 0$ a solution of equation (E), and there exist k bubbles (B_i^m), $m = 1, \dots, k$, such that, up to a subsequence,*

$$u_i = u^0 + \sum_{m=1}^{k} B_i^m + R_i$$

where (R_i) is a sequence in $H_1^2(M)$ such that $R_i \to 0$ in $H_1^2(M)$ as $i \to +\infty$.

When $k = 0$ in this theorem, the u_i's converge in H_1^2 to u^0. When $k \ge 1$, we face a blow-up situation. Up to a subsequence, we may assume that the centers x_i^m of the bubbles converge as $i \to +\infty$. Let $\mathcal{S}$ be the set of these limits. Then $\mathcal{S}$ is finite, possibly reduced to one point, and

$$\mathcal{S} = \left\{x \in M \text{ s.t. } \exists m\ ,\ x = \lim_{i \to +\infty} x_i^m\right\}$$

The points in $\mathcal{S}$ are referred to as the geometric blow-up points of (u_i). Then, it follows from the decomposition in the theorem that $u_i \to u^0$ in $H_{1,loc}^2(M \backslash \mathcal{S})$.

A very important remark on this theorem is that the energy respects the decomposition. We then have that

$$\|u_i\|_{H_1^2}^2 = \|u^0\|_{H_1^2}^2 + \sum_{m=1}^{k} \|B_i^m\|_{H_1^2}^2 + o(1)$$

where $o(1) \to 0$ as $i \to +\infty$. It is easily checked that if (B_i) is a bubble as defined above, then

$$\|B_i\|_{H_1^2}^2 = K_n^{-n} + o(1)$$

Hence,

$$\|u_i\|_{H_1^2}^2 = \|u^0\|_{H_1^2}^2 + kK_n^{-n} + o(1)$$

for all i, where $o(1) \to 0$ as $i \to +\infty$. We then get a classification of the energy levels for which compactness does not hold. In particular, non-compact energies are quantified, and K_n^{-n} is the minimal energy under which compactness holds. This provides another interpretation of results like Theorem 2.1. The assumption in this theorem guarantees that the sequences under consideration have an energy which is below the minimal energy for which blow-up may occur. Thus compactness holds.

5.1 Proof of Theorem 5.1

We sketch the proof of Theorem 5.1, following the original reference Struwe [43]. For more details we refer to Struwe [43, 44]. Among other possible references, we refer also to Druet-Hebey-Robert [15] and Hebey-Robert [25] for proofs in the Riemannian case and extensions to other types of operators. We divide the proof into several steps. Step 1 is as follows.

Step 1: We claim that Palais-Smale sequences for J are bounded in $H_1^2(M)$. We let (u_i) be a Palais-Smale sequence for J. Since $DJ(u_i).u_i = o(\|u_i\|_{H_1^2})$,

$$J(u_i) = \frac{1}{n}\int_M |u_i|^{2^\star} dv_g + o\left(\|u_i\|_{H_1^2}\right)$$

and since $(J(u_i))$ is a bounded sequence, we also have that

$$\int_M |u_i|^{2^\star} dv_g \leq C + o\left(\|u_i\|_{H_1^2}\right)$$

for some $C > 0$. By Hölder's inequalities, this implies in turn that

$$\int_M u_i^2 dv_g \leq C + o\left(\|u_i\|_{H_1^2}^{2/2^\star}\right)$$

Similarly, we can write that

$$\int_M \left(|\nabla u_i|^2 + hu_i^2\right) dv_g = 2J(u_i) + \frac{2}{2^\star}\int_M |u_i|^{2^\star} dv_g$$

so that

$$\int_M \left(|\nabla u_i|^2 + hu_i^2\right) dv_g \leq C + o\left(\|u_i\|_{H_1^2}\right)$$

Noting that

$$\|u_i\|_{H_1^2}^2 \leq \int_M \left(|\nabla u_i|^2 + hu_i^2\right) dv_g + C\|u_i\|_2^2$$

it follows that

$$\|u_i\|_{H_1^2}^2 \leq C + o\left(\|u_i\|_{H_1^2}\right) + o\left(\|u_i\|_{H_1^2}^{2/2^\star}\right)$$

In particular, $\|u_i\|_{H_1^2} \leq C$ for some $C > 0$. This proves the claim in step 1.

From now on, we let $\hat{J}$ be the functional defined on $H_1^2(M)$ by

$$\hat{J}(u) = \frac{1}{2}\int_M |\nabla u|^2 dv_g - \frac{1}{2^\star}\int_M |u|^{2^\star} dv_g$$

Then, $\hat{J} = J$ when $h = 0$. A Palais-Smale sequence for $\hat{J}$ is a sequence $(\hat{u}_i)$ in $H_1^2(M)$ such that:

(i) $\hat{J}(\hat{u}_i)$ is bounded with respect to i, and

(ii) $D\hat{J}(\hat{u}_i) \to 0$ in $H_1^2(M)'$

as $i \to +\infty$. Step 2 is as follows.

Step 2: Let (u_i) be a Palais-Smale sequence of nonnegative functions for J. Thanks to step 1 we may assume that, up to a subsequence, $u_i \rightharpoonup u^0$ in $H_1^2(M)$ as $i \to +\infty$. We may also assume that $u_i \to u^0$ in $L^2(M)$ and that $u_i \to u^0$ a.e. In particular, $u^0 \geq 0$. We let $\hat{u}_i = u_i - u^0$. Then we claim that $(\hat{u}_i)$ is a Palais-Smale sequence for $\hat{J}$ with the property that

$$\hat{J}(\hat{u}_i) = J(u_i) - J(u^0) + o(1)$$

where $o(1) \to 0$ as $i \to +\infty$. Moreover, we claim that u^0 is a solution of equation (E). In order to prove these two claims, we first observe that for any $\varphi \in H_1^2(M)$,

$$\begin{aligned} DJ(u_i).\varphi &= \int_M (\nabla u_i \nabla \varphi) dv_g + \int_M h u_i \varphi dv_g - \int_M u_i^{2^\star - 1} \varphi dv_g \\ &= o(1) \end{aligned}$$

where $o(1) \to 0$ as $i \to +\infty$. Passing to the limit as $i \to +\infty$, it easily follows that

$$\int_M (\nabla u^0 \nabla \varphi) dv_g + \int_M h u^0 \varphi dv_g = \int_M (u^0)^{2^\star - 1} \varphi dv_g$$

In particular, u^0 is a solution of equation (E). This proves the second claim in step 2. Now we compute the energy $\hat{J}(\hat{u}_i)$. Clearly,

$$\int_M h u_i^2 dv_g = \int_M h (u^0)^2 dv_g + o(1)$$

Then, since $u_i = u^0 + \hat{u}_i$, we can write that

$$J(u_i) = J(u^0) + \hat{J}(\hat{u}_i) - \int_M \Phi_i dv_g + o(1)$$

where

$$\Phi_i = \frac{1}{2^\star} \left(|u^0 + \hat{u}_i|^{2^\star} - |\hat{u}_i|^{2^\star} - |u^0|^{2^\star} \right)$$

Thanks to Brézis-Lieb [7], noting that $\hat{u}_i \to 0$ a.e. and that the $\hat{u}_i$'s are bounded in $L^{2^\star}$ thanks to step 1, we can write that

$$\int_M \Phi_i dv_g = o(1)$$

It follows that

$$\hat{J}(\hat{u}_i) = J(u_i) - J(u^0) + o(1)$$

where $o(1) \to 0$ as $i \to +\infty$. Let $\varphi \in H_1^2(M)$. Thanks to Hölder's inequalities, and to the Sobolev inequality,

$$\int_M h u_i \varphi dv_g = \int_M h u^0 \varphi dv_g + o\left(\|\varphi\|_{H_1^2} \right)$$

Since u^0 is a solution of (E), and thanks to arguments like the ones used in Brézis-Lieb [7], we can write that

$$DJ(u_i).\varphi = D\hat{J}(\hat{u}_i).\varphi - \int_M \Psi_i \varphi dv_g + o\left(\|\varphi\|_{H_1^2} \right)$$

where $\Psi_i \in L^{2^\star/(2^\star - 1)}(M)$ is such that $\Psi_i \to 0$ in $L^{2^\star/(2^\star-1)}(M)$ as $i \to +\infty$. By Hölder's inequality and the Sobolev embedding theorem,

$$\begin{aligned} \int_M |\Psi_i \varphi| dv_g &\leq \|\Psi_i\|_{2^\star/(2^\star-1)} \|\varphi\|_{2^\star} \\ &\leq C \|\Psi_i\|_{2^\star/(2^\star-1)} \|\varphi\|_{H_1^2} \end{aligned}$$

It follows that

$$\int_M \Psi_i \varphi dv_g = o\left(\|\varphi\|_{H_1^2}\right)$$

and we thus get that

$$D\hat{J}(\hat{u}_i).\varphi = DJ(u_i).\varphi + o\left(\|\varphi\|_{H_1^2}\right)$$

In particular, $(\hat{u}_i)$ is a Palais-Smale sequence of $\hat{J}$, and this proves the first claim in step 2. Step 2 is proved.

We let $\beta^\star = \frac{1}{n}K_n^{-n}$. If u is a fundamental positive solution of the critical Euclidean equation $\Delta u = u^{2^\star-1}$, as discussed in subsection 2.3, then

$$\frac{1}{2}\int_{I\!R^n} |\nabla u|^2 dx - \frac{1}{2^\star}\int_{I\!R^n} u^{2^\star} dx = \frac{1}{n}K_n^{-n}$$

A third step in the proof of Theorem 5.1 is as follows.

Step 3: Let $(\hat{u}_i)$ be a Palais-Smale sequence for $\hat{J}$ such that $\hat{u}_i \rightharpoonup 0$ weakly in $H_1^2(M)$ as $i \to +\infty$, and such that $\hat{J}(\hat{u}_i) \to \beta < \beta^\star$ as $i \to +\infty$. Then we claim that $\hat{u}_i \to 0$ strongly in $H_1^2(M)$ as $i \to +\infty$. In order to prove this claim we note that, thanks to step 1 with $h \equiv 0$, the $\hat{u}_i$'s are bounded in $H_1^2(M)$. Hence,

$$\begin{aligned} D\hat{J}(\hat{u}_i).\hat{u}_i &= \int_M |\nabla \hat{u}_i|^2 dv_g - \int_M |\hat{u}_i|^{2^\star} dv_g \\ &= o(1) \end{aligned}$$

and it follows that

$$\begin{aligned} \hat{J}(\hat{u}_i) &= \frac{1}{n}\int_M |\hat{u}_i|^{2^\star} dv_g + o(1) \\ &= \frac{1}{n}\int_M |\nabla \hat{u}_i|^2 dv_g + o(1) \\ &= \beta + o(1) \end{aligned}$$

where $o(1) \to 0$ as $i \to +\infty$. In particular, $\beta \geq 0$. Independently, thanks to the sharp Sobolev inequality, there exists $B > 0$ such that

$$\|\hat{u}_i\|_{2^\star}^2 \leq K_n^2 \|\nabla \hat{u}_i\|_2^2 + B\|\hat{u}_i\|_2^2$$

for all i. Moreover, the embedding $H_1^2 \subset L^2$ being compact, we necessarily have that $\hat{u}_i \to 0$ in $L^2(M)$. It follows that

$$(n\beta)^{2/2^\star} \leq K_n^2 n\beta$$

Since $\beta < \beta^\star$, this implies that $\beta = 0$. Then $\hat{u}_i \to 0$ in $H_1^2(M)$ as $i \to +\infty$. Step 3 is proved.

Palais-Smale sequences for $\hat{J}$, typically $\hat{u}_i = u_i - u^0$, do not necessarily have a sign. We thus need to modify slightly the notion of a bubble in order to get bubbles which may change sign. Given $\delta > 0$, we let η_δ be a smooth cut-off function in $I\!R^n$ such that $\eta_\delta = 1$ in $B_0(\delta)$ and $\eta_\delta = 0$ in $I\!R^n \backslash B_0(2\delta)$. For $x_0 \in M$, and $\delta < i_g/2$, where i_g is the injectivity radius, we let η_{δ,x_0} be the smooth cut-off function in M given by

$$\eta_{\delta,x_0}(x) = \eta_\delta\left(\exp_{x_0}^{-1}(x)\right)$$

where $\exp_{x_0}$ is the exponential map at x_0. We let $D_1^2(\mathbb{R}^n)$ be the homogeneous Sobolev space defined as the completion of $C_0^\infty(\mathbb{R}^n)$ for the norm $\|u\| = \|\nabla u\|_2$. If $u \in D_1^2(\mathbb{R}^n)$ is a nontrivial solution of the equation

$$\Delta u = |u|^{2^\star-2}u$$

in $\mathbb{R}^n$, we define a bubble (B_i) by

$$B_i(x) = \eta_{\delta,x_i}(x)\left(\frac{1}{\mu_i}\right)^{\frac{n-2}{2}} u\Big(\frac{1}{\mu_i}\exp_{x_i}^{-1}(x)\Big)$$

where (x_i) is a converging sequence of points in M, and (μ_i) is a sequence of positive real numbers such that $\mu_i \to 0$ as $i \to +\infty$. If (B_i) is such a bubble, we then define the energy $E(B_i)$ of (B_i) by

$$E(B_i) = \frac{1}{2}\int_{\mathbb{R}^n}|\nabla u|^2 dx - \frac{1}{2^\star}\int_{\mathbb{R}^n}|u|^{2^\star}dx$$

As an important remark, if u in the above definition is nonnegative, then there exist sequences $(\tilde{x}_i)$ in M and $(\tilde{\mu}_i)$ of positive real numbers, with $\tilde{\mu}_i \to 0$ as $i \to +\infty$, such that

$$B_i(x) = \left(\frac{\tilde{\mu}_i}{\tilde{\mu}_i^2 + \frac{d_g(\tilde{x}_i,x)^2}{n(n-2)}}\right)^{\frac{n-2}{2}} + R_i(x)$$

for almost all x, where the R_i's are functions in $H_1^2(M)$ such that $R_i \to 0$ in $H_1^2(M)$ as $i \to +\infty$. We thus recover our original definition of a bubble. In order to prove this, we proceed as follows. If u is nonnegative, then, see subsection 2.3,

$$u(x) = \left(\frac{\lambda}{\lambda^2 + \frac{|x-a|^2}{n(n-2)}}\right)^{\frac{n-2}{2}}$$

for some $\lambda > 0$ and $a \in \mathbb{R}^n$. We define $\tilde{x}_i$ and $\tilde{\mu}_i$ by

$$\tilde{x}_i = \exp_{x_i}(\mu_i a)$$
$$\tilde{\mu}_i = \lambda\mu_i$$

Let $(\tilde{B}_i)$ be the bubble as in Definition 5.1, defined with respect to the $\tilde{x}_i$'s and $\tilde{\mu}_i$'s. Easy computations give that

$$\int_M(\nabla B_i\nabla\tilde{B}_i)dv_g = \int_{\mathbb{R}^n}|\nabla u|^2 dx + o(1)$$

where $o(1) \to 0$ as $i \to +\infty$. It follows that

$$\int_M\left|\nabla(B_i - \tilde{B}_i)\right|^2 dv_g = o(1)$$

and since we also have that $\int_M B_i^2 dv_g = o(1)$ and $\int_M \tilde{B}_i^2 dv_g = o(1)$, we get that

$$B_i - \tilde{B}_i \to 0 \text{ in } H_1^2(M)$$

as $i \to +\infty$. In other words, $B_i = \tilde{B}_i + R_i$ where the R_i's are functions in $H_1^2(M)$ such that $R_i \to 0$ in $H_1^2(M)$ as $i \to +\infty$.

The following step is the main argument in the proof of Theorem 5.1. We refer to Struwe [43, 44] for its proof. See also Druet-Hebey-Robert [15] for the proof in the Riemannian context, and Hebey-Robert [25] for the proof in the case of other types of operators.

Step 4: Let $(\hat{u}_i)$ be a Palais-Smale sequence for $\hat{J}$ such that $\hat{u}_i \rightharpoonup 0$ weakly in $H_1^2(M)$ but not strongly as $i \to +\infty$. Then we claim that there exists a bubble (B_i) such that, up to a subsequence, the sequence consisting of the $\tilde{u}_i$'s given by

$$\tilde{u}_i = \hat{u}_i - B_i$$

is such that $(\tilde{u}_i)$ is a Palais-Smale sequence for $\hat{J}$, $\tilde{u}_i \rightharpoonup 0$ in $H_1^2(M)$ as $i \to +\infty$, and

$$\hat{J}(\tilde{u}_i) = \hat{J}(\hat{u}_i) - E(B_i) + o(1)$$

where $o(1) \to 0$ as $i \to +\infty$.

The following step concerns the energy of a bubble. For nonnegative bubbles, the energy is precisely $\beta^\star$. See subsection 2.3.

Step 5: Let (B_i) be a bubble as above. Then we claim that $E(B_i) \geq \beta^\star$. Indeed, if u is a nontrivial solution of the equation $\Delta u = |u|^{2^\star-2}u$, then

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx = \int_{\mathbb{R}^n} |u|^{2^\star} dx$$

while, thanks to the sharp Euclidean Sobolev inequality,

$$\left(\int_{\mathbb{R}^n} |u|^{2^\star} dx\right)^{2/2^\star} \leq K_n^2 \int_{\mathbb{R}^n} |\nabla u|^2 dx$$

In particular,

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \frac{1}{K_n^n}$$

and

$$\frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2 dx - \frac{1}{2^\star}\int_{\mathbb{R}^n} |u|^{2^\star} dx = \frac{1}{n}\int_{\mathbb{R}^n} |\nabla u|^2 dx$$

so that $E(B_i) \geq \beta^\star$. This proves step 5.

Now we are in position to prove Theorem 5.1. The proof proceeds as follows, using steps 1 to 5. To some extent, k in the theorem is the number of bubbles we have to substract in order to get a Palais-Smale sequence of small energy for which we do have compactness.

Proof of Theorem 5.1: Let (u_i) be a Palais-Smale sequence of nonnegative functions for J. According to step 1, (u_i) is bounded in $H_1^2(M)$. Up to a subsequence, we may therefore assume that for some $u^0 \in H_1^2(M)$, $u_i \rightharpoonup u^0$ weakly in $H_1^2(M)$, $u_i \to u^0$ strongly in $L^2(M)$, and $u_i \to u^0$ almost everywhere as $i \to +\infty$. We may also assume that $J(u_i) \to c$ as $i \to +\infty$. By step 2, u^0 is a nonnegative solution of equation (E) and $\hat{u}_i = u_i - u^0$ is a Palais-Smale sequence for $\hat{J}$ such that

$$\hat{J}(\hat{u}_i) = J(u_i) - J(u^0) + o(1)$$

If $\hat{u}_i \to 0$ strongly in $H_1^2(M)$, then $u_i = u^0 + o(1)$, and the theorem is proved. If not, we apply step 4, and thanks to step 5 we get a new Palais-Smale sequence $(\hat{u}_i^1)$ such that

$$\hat{J}(\hat{u}_i^1) \leq \hat{J}(\hat{u}_i) - \beta^\star + o(1)$$

Here again, either $\hat{u}_i^1 \to 0$ strongly in $H_1^2(M)$, in which case the theorem is proved, or $\hat{u}_i^1 \rightharpoonup 0$ weakly but not strongly in $H_1^2(M)$, in which case we apply again step 4. By induction, at some point, either we do have compactness, or the Palais-Smale sequence $(\hat{u}_i^k)$ we get with this process has an energy which converges to some $\beta < \beta^\star$. Then, by step 3, $\hat{u}_i^k \to 0$ strongly in $H_1^2(M)$. It follows that

$$u_i = u^0 + \sum_{m=1}^{k} B_i^m + R_i$$

where the (B_i)'s are bubbles as in step 4, and $R_i \to 0$ in $H_1^2(M)$ as $i \to +\infty$. Up to the positivity of the bubbles, see below, this proves Theorem 5.1.

In order to get the complete Theorem 5.1, we need to prove that the the bubbles we got in the above process come from positive fundamental solutions of the critical Euclidean equation $\Delta u = u^{2^\star-1}$. A complete proof of this fact can be found in Druet-Hebey-Robert [15]. We refer also to Hebey-Robert [25]. If the bubbles (B_i^m) in Theorem 5.1 are defined with respect to sequences (x_i^m) and (μ_i^m), representing their centers and weights, we get in the process that for any $m_1 \neq m_2$,

$$\frac{\mu_i^{m_2}}{\mu_i^{m_1}} + \frac{\mu_i^{m_1}}{\mu_i^{m_2}} + \frac{d_g(x_i^{m_1}, x_i^{m_2})^2}{\mu_i^{m_1}\mu_i^{m_2}} \to +\infty$$

as $i \to +\infty$. This equation is discussed in the following subsection.

Concerning the energy, the proof of Theorem 5.1 gives that

$$J(u_i) = J(u^0) + \sum_{m=1}^{k} E(B_i^m) + o(1)$$

where the energy $E(B_i)$ of a bubble is as defined above, and where $o(1) \to 0$ as $i \to +\infty$. Since (u_i) is a Palais-Smale sequence for J, and (u_i) is bounded in $H_1^2(M)$,

$$\int_M |\nabla u_i|^2 dv_g = \int_M |u_i|^{2^\star} dv_g + o(1)$$

It follows that

$$J(u_i) = \frac{1}{n}\int_M |\nabla u_i|^2 dv_g + o(1)$$

Noting that for a bubble (B_i) as in Definition 5.1,

$$E(B_i) = \frac{1}{n}\|\nabla B_i\|_2^2 + o(1)$$

that $B_i \to 0$ in $L^2(M)$ as $i \to +\infty$, and that $u_i \to u^0$ in $L^2(M)$ as $i \to +\infty$, we get that

$$\|u_i\|_{H_1^2}^2 = \|u^0\|_{H_1^2}^2 + \sum_{m=1}^{k} \|B_i^m\|_{H_1^2}^2 + o(1)$$

where $o(1) \to 0$ as $i \to +\infty$. This proves the remark after Theorem 5.1.

5.2 Uniqueness of the Struwe decomposition

We discuss uniqueness of the Struwe decomposition. As far as we know, the material in this subsection, as well as in the following subsection, has never appeared before in the literature. First we need to define what we mean when we speak of a Struwe decomposition. Given a Palais-Smale sequence (u_i) of nonnegative functions for J, we define a Struwe decomposition of (u_i) by the equations

$$u_i = u^0 + \sum_{m=1}^{k} B_i^m + R_i \ , \text{ and}$$

$$\|u_i\|_{H_1^2}^2 = \|u^0\|_{H_1^2}^2 + \sum_{m=1}^{k} \|B_i^m\|_{H_1^2}^2 + o(1)$$

where the (B_i^m)'s are bubbles as in Definition 5.1, so that

$$B_i^m(x) = \left(\frac{\mu_i^m}{(\mu_i^m)^2 + \frac{d_g(x_i^m,x)^2}{n(n-2)}}\right)^{\frac{n-2}{2}}$$

where the R_i's are functions in $H_1^2(M)$ such that $R_i \to 0$ in $H_1^2(M)$ as $i \to +\infty$, and where $o(1) \to 0$ as $i \to +\infty$. Then, of course, $u_i \to u^0$ a.e., and both u^0 and k are invariants of such decompositions.

Now we discuss the equation

$$\frac{\mu_i^{m_2}}{\mu_i^{m_1}} + \frac{\mu_i^{m_1}}{\mu_i^{m_2}} + \frac{d_g(x_i^{m_1}, x_i^{m_2})^2}{\mu_i^{m_1}\mu_i^{m_2}} \to +\infty$$

of the preceding section. If (B_i) and $(\tilde{B}_i)$ are two bubbles, of respective centers x_i and $\tilde{x}_i$, and respective weights μ_i and $\tilde{\mu}_i$, we let $F(B_i, \tilde{B}_i)$ be given by

$$F(B_i, \tilde{B}_i) = \frac{\tilde{\mu}_i}{\mu_i} + \frac{\mu_i}{\tilde{\mu}_i} + \frac{d_g(x_i, \tilde{x}_i)^2}{\mu_i\tilde{\mu}_i}$$

We claim here that $F(B_i, \tilde{B}_i) \to +\infty$ as $i \to +\infty$ if and only if (B_i) and $(\tilde{B}_i)$ do not interact one with the other at the H_1^2-level. In other words, we claim that $F(B_i, \tilde{B}_i) \to +\infty$ as $i \to +\infty$ if and only if

$$\int_M \left((\nabla B_i \nabla \tilde{B}_i) + B_i\tilde{B}_i\right) dv_g \to 0$$

as $i \to +\infty$. Given $\lambda > 0$ and $a \in I\!R^n$ we let $U_{a,\lambda} : I\!R^n \to I\!R$ be the function defined by

$$U_{a,\lambda}(x) = \left(\frac{\lambda}{\lambda^2 + \frac{|x-a|^2}{n(n-2)}}\right)^{\frac{n-2}{2}}$$

Then, see subsection 2.3, $U_{a,\lambda}$ is a positive fondamental solution of the critical Euclidean equation $\Delta u = u^{2^\star-1}$. If the $F(B_i, \tilde{B}_i)$'s are bounded, we let $\lambda_0 > 0$ and $x_0 \in I\!R^n$ be such that, up to a subsequence,

$$\frac{\tilde{\mu}_i}{\mu_i} \to \lambda_0 \text{ , and}$$
$$\frac{1}{\mu_i} \exp_{x_i}^{-1}(\tilde{x}_i) \to x_0$$

as $i \to +\infty$, where $\exp_{x_i}$ is the exponential map at x_i. Then, easy claims are as follows:

(i) for any $R > 0$,

$$\int_{B_{x_i}(R\mu_i)} (\nabla B_i \nabla \tilde{B}_i) dv_g \to 0$$

as $i \to +\infty$ when $F(B_i, \tilde{B}_i) \to +\infty$ as $i \to +\infty$,

(ii) for any $R > 0$,

$$\limsup_{i \to +\infty} \int_{M \backslash B_{x_i}(R\mu_i)} |\nabla B_i|^2 dv_g = \varepsilon_R$$

where $\varepsilon_R \to 0$ as $R \to +\infty$,

(iii) for any $R > 0$,

$$\int_{B_{x_i}(R\mu_i)} (\nabla B_i \nabla \tilde{B}_i) dv_g \to \int_{B_0(R)} (\nabla U_{0,1} \nabla U_{x_0,\lambda_0}) dx$$

as $i \to +\infty$ when the $F(B_i, \tilde{B}_i)$'s are bounded.

Integrating by parts we also have that

$$\int_{B_0(R)} (\nabla U_{0,1} \nabla U_{x_0,\lambda_0}) dx = \int_{B_0(R)} U_{0,1}^{2^\star-1} U_{x_0,\lambda_0} dx + \varepsilon_R$$

where $\varepsilon_R \to 0$ as $R \to +\infty$. Independently, it is easily checked that

$$\int_M B_i^2 dv_g \to 0 \text{ and } \int_M \tilde{B}_i^2 dv_g \to 0$$

as $i \to +\infty$. In particular,

$$\int_M B_i \tilde{B}_i dv_g \to 0$$

as $i \to +\infty$. We know that bubbles are bounded in H_1^2. We can then write that

$$\int_M (\nabla B_i \nabla \tilde{B}_i) dv_g = \int_{B_{x_i}(R\mu_i)} (\nabla B_i \nabla \tilde{B}_i) dv_g + \int_{M \backslash B_{x_i}(R\mu_i)} (\nabla B_i \nabla \tilde{B}_i) dv_g$$

and that

$$\int_{M \backslash B_{x_i}(R\mu_i)} \left| (\nabla B_i \nabla \tilde{B}_i) \right| dv_g \le C \sqrt{\int_{M \backslash B_{x_i}(R\mu_i)} |\nabla B_i|^2 dv_g}$$

for some $C > 0$. By (i) and (ii), letting $i \to +\infty$, and then $R \to +\infty$, it follows that

$$\int_M (\nabla B_i \nabla \tilde{B}_i) dv_g \to 0$$

as $i \to +\infty$ if $F(B_i, \tilde{B}_i) \to +\infty$ as $i \to +\infty$. Conversely, if the $F(B_i, \tilde{B}_i)$'s are bounded, we can write with (i) and (iii) that

$$\begin{aligned}\int_M (\nabla B_i \nabla \tilde{B}_i) dv_g &= \int_{B_{x_i}(R\mu_i)} (\nabla B_i \nabla \tilde{B}_i) dv_g + \int_{M \backslash B_{x_i}(R\mu_i)} (\nabla B_i \nabla \tilde{B}_i) dv_g \\ &= \int_{B_0(R)} U_{0,1}^{2^\star - 1} U_{x_0, \lambda_0} dx + \varepsilon_{i,R}\end{aligned}$$

where

$$\lim_{R \to +\infty} \lim_{i \to +\infty} \varepsilon_{i,R} = 0$$

In particular,

$$\begin{aligned}\lim_{i \to +\infty} \int_M (\nabla B_i \nabla \tilde{B}_i) dv_g &= \int_{\mathbb{R}^n} U_{0,1}^{2^\star - 1} U_{x_0, \lambda_0} dx \\ &> 0\end{aligned}$$

if the $F(B_i, \tilde{B}_i)$'s are bounded. It follows that if

$$\int_M (\nabla B_i \nabla \tilde{B}_i) dv_g \to 0$$

as $i \to +\infty$, then $F(B_i, \tilde{B}_i) \to +\infty$ as $i \to +\infty$. The above claim is proved.

We know from subsection 5.1 that, given a Palais-Smale sequence (u_i) of nonnegative functions for J, there exists a Struwe decomposition of (u_i) with the property that the centers and weights of the bubbles in this decomposition satisfy that

$$\frac{\mu_i^{m_2}}{\mu_i^{m_1}} + \frac{\mu_i^{m_1}}{\mu_i^{m_2}} + \frac{d_g(x_i^{m_1}, x_i^{m_2})^2}{\mu_i^{m_1} \mu_i^{m_2}} \to +\infty$$

as $i \to +\infty$ for all $m_1 \neq m_2$. Since

$$\lim_{i \to +\infty} \int_M (\nabla B_i \nabla \tilde{B}_i) dv_g > 0$$

if the $F(B_i, \tilde{B}_i)$'s are bounded, a consequence of what we just said is that the above equation holds for all the Struwe decompositions of (u_i). In other words, the condition

$$\|u_i\|_{H_1^2}^2 = \|u^0\|_{H_1^2}^2 + \sum_{m=1}^{k} \|B_i^m\|_{H_1^2}^2 + o(1)$$

is equivalent to the condition

$$\frac{\mu_i^{m_2}}{\mu_i^{m_1}} + \frac{\mu_i^{m_1}}{\mu_i^{m_2}} + \frac{d_g(x_i^{m_1}, x_i^{m_2})^2}{\mu_i^{m_1} \mu_i^{m_2}} \to +\infty$$

as $i \to +\infty$ for all $m_1 \neq m_2$. Note here that for $u^0 \in H_1^2(M)$,

$$\int_M \Big((\nabla u^0 \nabla B_i) + u^0 B_i \Big) dv_g = o(1)$$

if (B_i) is a bubble as in Definition 5.1.

Another consequence of what we just said is as follows. We let (B_i^m), $m = 1, 2, 3$, be three bubbles. We assume that $F(B_i^1, B_i^2) \to +\infty$ as $i \to +\infty$, and that the $F(B_i^1, B_i^3)$'s are bounded. Then, as easily checked from the definition of F, $F(B_i^2, B_i^3) \to +\infty$ as $i \to +\infty$. In other words

$$\begin{aligned} & F(B_i^1, B_i^2) \to +\infty \text{ and } F(B_i^1, B_i^3) \leq C \\ & \quad \Rightarrow \quad F(B_i^2, B_i^3) \to +\infty \end{aligned}$$

and it follows that if (B_i^1) and (B_i^2) do not interact at the H_1^2-level, while (B_i^1) and (B_i^3) interact at the H_1^2-level, then (B_i^2) and (B_i^3) do not interact at the H_1^2-level.

Now we discuss uniqueness in a Struwe decomposition. Let us assume that we have two sets (B_i^m) and $(\tilde{B}_i^m)$ of bubbles, $m = 1, \dots, k$, corresponding to two Struwe's decompositions of (u_i). We denote by x_i^m and $\tilde{x}_i^m$ the respective centers of these bubbles, and by μ_i^m and $\tilde{\mu}_i^m$ their respective weights. Since bubbles in a given decomposition do not interact one with each other, we can write that for any $\tilde{m}$,

$$K_n^{-n} = \sum_{m=1}^{k} \langle B_i^m, \tilde{B}_i^{\tilde{m}} \rangle_{H_1^2} + o(1)$$

It follows, thanks to the above remark, that for any $\tilde{m}$, there exists m such that

$$K_n^{-n} = \langle B_i^m, \tilde{B}_i^{\tilde{m}} \rangle_{H_1^2} + o(1)$$

or equivalently such that

$$\lim_{i \to +\infty} \| B_i^m - \tilde{B}_i^{\tilde{m}} \|_{H_1^2} = 0$$

Thanks to (i)-(iii) above, and since $\langle x, y \rangle = \|x\|.\|y\|$ if and only if x and y are colinear, we easily get that

$$K_n^{-n} = \langle B_i^m, \tilde{B}_i^{\tilde{m}} \rangle_{H_1^2} + o(1)$$

if and only if

$$\begin{aligned} & \lim_{i \to +\infty} \frac{\tilde{\mu}_i^{\tilde{m}}}{\mu_i^m} = 1 \text{ , and} \\ & \lim_{i \to +\infty} \frac{d_g(x_i^m, \tilde{x}_i^{\tilde{m}})}{\mu_i^m} = 0 \end{aligned}$$

Hence, we proved that if we have two sets (B_i^m) and $(\tilde{B}_i^m)$ of bubbles, $m = 1, \dots, k$, corresponding to two Struwe's decompositions of (u_i), then, up to renumbering,

$$\begin{aligned} & \lim_{i \to +\infty} \frac{\tilde{\mu}_i^m}{\mu_i^m} = 1 \text{ , and} \\ & \lim_{i \to +\infty} \frac{d_g(x_i^m, \tilde{x}_i^m)}{\mu_i^m} = 0 \end{aligned}$$

where the x_i^m's and $\tilde{x}_i^m$'s are respectively the centers of (B_i^m) and $(\tilde{B}_i^m)$, and the μ_i^m and $\tilde{\mu}_i^m$ are respectively the weights of (B_i^m) and $(\tilde{B}_i^m)$. In other words, a Struwe decomposition is unique up to the above equations.

5.3 A remark on the definition of a Struwe's decomposition

Let (u_i) be a Palais-Smale sequence of nonnegative functions for J. We defined a Struwe's decomposition of (u_i) by the equations

$$u_i = u^0 + \sum_{m=1}^{k} B_i^m + R_i$$

and

$$\|u_i\|_{H_1^2}^2 = \|u^0\|_{H_1^2}^2 + \sum_{m=1}^{k} \|B_i^m\|_{H_1^2}^2 + o(1)$$

where the (B_i^m)'s are bubbles as in Definition 5.1, the R_i's are functions in $H_1^2(M)$ such that $R_i \to 0$ in $H_1^2(M)$ as $i \to +\infty$, and $o(1) \to 0$ as $i \to +\infty$. We claim here that the second equation is actually a consequence of the first one, so that we can restrict the definition of a Struwe's decomposition of (u_i) to the equation

$$u_i = u^0 + \sum_{m=1}^{k} B_i^m + R_i$$

where the (B_i^m)'s are bubbles as in Definition 5.1, and the R_i's are functions in $H_1^2(M)$ such that $R_i \to 0$ in $H_1^2(M)$ as $i \to +\infty$.

In order to prove the claim, we let (u_i) be a Palais-Smale sequence for J, and assume that

$$u_i = u^0 + \sum_{m=1}^{k} B_i^m + R_i$$

for some $k \in \mathbb{N}\backslash\{0\}$, where the (B_i^m)'s are bubbles as in Definition 5.1, and the R_i's are functions in $H_1^2(M)$ such that $R_i \to 0$ in $H_1^2(M)$ as $i \to +\infty$. We let the x_i^m's be the centers of (B_i^m), and the μ_i^m's be the weights of (B_i^m). Given $\varphi \in C_0^\infty(\mathbb{R}^n)$, and $m \in \{1, \dots, k\}$, we let φ_i^m be defined on M by

$$\varphi_i^m(x) = (\mu_i^m)^{-\frac{n-2}{2}} \varphi\left((\mu_i^m)^{-1} \exp_{x_i^m}^{-1}(x)\right)$$

We let also I_m be defined by

$$I_m = \left\{\tilde{m} \text{ s.t. the } F(B_i^m, B_i^{\tilde{m}})\text{'s are bounded}\right\}$$

where F is as in the preceding section. In particular, $m \in I_m$. If $\tilde{m} \in I_m$, we let $\lambda_{\tilde{m}} > 0$ and $x_{\tilde{m}} \in \mathbb{R}^n$ be such that, up to a subsequence,

$$\frac{\mu_i^{\tilde{m}}}{\mu_i^m} \to \lambda_{\tilde{m}}\ , \text{ and}$$
$$\frac{1}{\mu_i^m} \exp_{x_i^m}^{-1}(x_i^{\tilde{m}}) \to x_{\tilde{m}}$$

as $i \to +\infty$. We let also $U_{\tilde{m}} : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$U_{\tilde{m}}(x) = \left(\frac{\lambda_{\tilde{m}}}{\lambda_{\tilde{m}}^2 + \frac{|x-x_{\tilde{m}}|^2}{n(n-2)}}\right)^{\frac{n-2}{2}}$$

Then, easy claims, similar to the ones we used in the preceding section, are as follows:

(i) $\int_M |\nabla\varphi_i^m|^2 dv_g = \int_{\mathbb{R}^n} |\nabla\varphi|^2 dx + o(1)$,

(ii) $\int_M |\varphi_i^m|^{2^\star} dv_g = \int_{\mathbb{R}^n} |\varphi|^{2^\star} dx + o(1)$,

(iii) $\int_M (\nabla\varphi_i^m \nabla B_i^{\tilde{m}}) dv_g = \int_{\mathbb{R}^n} (\Delta U_{\tilde{m}})\varphi dx + o(1)$ for all $\tilde{m} \in I_m$,

(iv) $\int_M (\nabla\varphi_i^m \nabla B_i^{\tilde{m}}) dv_g = o(1)$ for all $\tilde{m} \notin I_m$,

(v) $\int_M (\nabla u^0 \nabla\varphi_i^m) dv_g = o(1)$ and $\int_M (\nabla R_i \nabla\varphi_i^m) dv_g = o(1)$,

(vi) $\int_M (B_i^{\tilde{m}})^{2^\star-1} \varphi_i^m dv_g = o(1)$ for all $\tilde{m} \notin I_m$,

(vii) $\int_M (u^0)^{2^\star-1} \varphi_i^m dv_g = o(1)$ and $\int_M |R_i|^{2^\star-1} \varphi_i^m dv_g = o(1)$,

where $o(1) \to 0$ as $i \to +\infty$. Let

$$\Phi_i = \sum_{\tilde{m}\in I_m} B_i^{\tilde{m}}\ , \text{ and}$$
$$\Psi_i = u^0 + \sum_{\tilde{m}\notin I_m} B_i^{\tilde{m}} + R_i$$

so that $u_i = \Phi_i + \Psi_i$. Thanks to Hölder's inequalities, and thanks to the above equations,

$$\begin{aligned}
&\int_M \Phi_i^{2^\star-2} |\Psi_i||\varphi_i^m| dv_g \\
&= \int_M \Phi_i^{2^\star-2} |\varphi_i^m|^{(2^\star-2)/(2^\star-1)} |\Psi_i||\varphi_i^m|^{1/(2^\star-1)} dv_g \\
&\le \left(\int_M \Phi_i^{2^\star-1} |\varphi_i^m| dv_g\right)^{(2^\star-2)/(2^\star-1)} \left(\int_M |\Psi_i|^{2^\star-1} |\varphi_i^m| dv_g\right)^{1/(2^\star-1)}
\end{aligned}$$

while

$$\begin{aligned}
\int_M \Phi_i^{2^\star-1} |\varphi_i^m| dv_g &\le \left(\int_M \Phi_i^{2^\star} dv_g\right)^{(2^\star-1)/2^\star} \left(\int_M |\varphi_i^m|^{2^\star} dv_g\right)^{1/2^\star} \\
&\le C
\end{aligned}$$

and

$$\int_M |\Psi_i|^{2^\star-1} |\varphi_i^m| dv_g = o(1)$$

It follows that

$$\int_M \Phi_i^{2^\star-2} |\Psi_i||\varphi_i^m| dv_g = o(1)$$

Similarly, we get that

$$\int_M \Phi_i |\Psi_i|^{2^\star-2} \varphi_i^m dv_g = o(1)$$

Writing that

$$\left||\Phi_i + \Psi_i|^{2^\star-1} - \Phi_i^{2^\star-1}\right| \le |\Psi_i|^{2^\star-1} + C\left(\Phi_i^{2^\star-2}|\Psi_i| + \Phi_i|\Psi_i|^{2^\star-2}\right)$$

we get that

$$\int_M u_i^{2^\star-1}\varphi_i^m dv_g = \int_M \Phi_i^{2^\star-1}\varphi_i^m dv_g + o(1)$$

Independently we can prove that

$$\int_M \Phi_i^{2^\star-1}\varphi_i^m dv_g = \int_{I\!R^n} \Big(\sum_{\tilde m\in I_m} U_{\tilde m}\Big)^{2^\star-1}\varphi dx + o(1)$$

Since (u_i) is a Palais-Smale sequence for J, and thanks to (i)-(ii),

$$\begin{aligned} DJ(u_i).\varphi_i^m &= \int_M (\nabla u_i \nabla\varphi_i^m) dv_g - \int_M u_i^{2^\star-1}\varphi_i^m dv_g + o(1) \\ &= o(1) \end{aligned}$$

Plugging

$$u_i = u^0 + \sum_{\tilde m=1}^k B_i^{\tilde m} + R_i$$

into this equation, letting $i \to +\infty$, and thanks to the above equations, we get that

$$\int_{I\!R^n}\left(\sum_{\tilde m\in I_m} U_{\tilde m}^{2^\star-1} - \Big(\sum_{\tilde m\in I_m} U_{\tilde m}\Big)^{2^\star-1}\right)\varphi dx = 0$$

Note here that $\Delta U_{\tilde m} = U_{\tilde m}^{2^\star-1}$. Since φ is any function in $C_0^\infty(I\!R^n)$, it follows that

$$\sum_{\tilde m\in I_m} U_{\tilde m}^{2^\star-1} = \Big(\sum_{\tilde m\in I_m} U_{\tilde m}\Big)^{2^\star-1}$$

and thus that $I_m = \{m\}$. In other words, we proved that if (u_i) is a Palais-Smale sequence of nonnegative functions for J, and if

$$u_i = u^0 + \sum_{m=1}^k B_i^m + R_i$$

where the (B_i^m)'s are bubbles as in Definition 5.1, of centers x_i^m and weights μ_i^m, and the R_i's are functions in $H_1^2(M)$ such that $R_i \to 0$ in $H_1^2(M)$ as $i \to +\infty$, then, for any $m_1 \neq m_2$ in $\{1,\dots,k\}$,

$$\frac{\mu_i^{m_2}}{\mu_i^{m_1}} + \frac{\mu_i^{m_1}}{\mu_i^{m_2}} + \frac{d_g(x_i^{m_1},x_i^{m_2})^2}{\mu_i^{m_1}\mu_i^{m_2}} \to +\infty$$

as $i \to +\infty$ According to what was said in the preceding section, this proves that for any $m_1 \neq m_2$ in $\{1,\dots,k\}$,

$$\int_M \Big((\nabla B_i^{m_1}\nabla B_i^{m_2}) + B_i^{m_1}B_i^{m_2}\Big) dv_g = o(1)$$

and thus that

$$\|u_i\|_{H_1^2}^2 = \|u^0\|_{H_1^2}^2 + \sum_{m=1}^k \|B_i^m\|_{H_1^2}^2 + o(1)$$

This proves the claim we made at the beginning of this subsection.

6 The dynamical viewpoint

We let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$, and let $\alpha > 0$. We consider the equation

$$\Delta_g u + \alpha u = u^{2^\star - 1} \tag{E_α}$$

where u is required to be positive. As a very basic remark, it is easily seen that the constant function $\overline{u}_\alpha = \alpha^{(n-2)/4}$ satisfies (E_α).

The type of questions we considered in the preceding sections were whether or not such nonlinear elliptic equations of critical Sobolev growth possess one (or several) solution. Such studies are concerned with equation (E_α) when the parameter α is fixed. Almost no studies are concerned with the equation as part of a whole family or, equivalently, with these equations when α is considered as a free parameter. Emphasizing the opposition between what we could refer to as a statical approach, and what we could refer to as a dynamical approach (with respect to the parameter α), the very general question we ask, following Hebey [24], is to understand the structure of (E_α) when α varies. This question requires a deep understanding of the blow-up behaviour of sequences of solutions of equations like (E_α).

6.1 The energy function

Given a function $u \in H_1^2$, we define the energy $E(u)$ of u as the $L^{2^\star}$-norm of u. In other words,

$$E(u) = \left(\int_M |u|^{2^\star} dv_g \right)^{\frac{1}{2^\star}} .$$

If u is a solution of (E_α) we recover the classical energy associated to such types of equations. The minimum energy Λ_{min} is then given by $\Lambda_{min} = K_n^{-(n-2)/2}$ where K_n is the sharp constant K in the Euclidean Sobolev inequality $\|u\|_{2^\star} \leq K \|\nabla u\|_2$. The minimum energy is characterized by the property that blowing up sequences of solutions of (E_α) have an energy which is greater than or equal to Λ_{min}, as described in Struwe's decomposition. Another energy we define is the following:

Definition 6.1 *Let $\alpha > 0$. We define the energy function E_m associated to equation (E_α) by*

$$E_m(\alpha) = \inf_{u \in S_\alpha} E(u)$$

where S_α consists of the solutions of (E_α).

Basic arguments, as developed in the preceding sections, show that for any $\alpha > 0$, $E_m(\alpha) > 0$. As already mentioned, the question of understanding the structure of equation (E_α) can be interpreted in two ways. On the one hand, we may want to understand the structure for a given linear term. We refer to this approach as the statical viewpoint. On the other hand, we may want to understand the structure as α varies. We refer to this approach as the dynamical viewpoint, where, needless to say, the dynamic has to be understood with respect to the parameter α. Several questions can be asked when studying the dynamical viewpoint. They constitute a program of research on such equations. We refer to Hebey [24] for the precise statement of this program, and concentrate in these notes on one of the questions of this program:

Question: is E_m a continuous function ?

This question is still open. However, we can prove that E_m is lower semi-continuous. We prove such a result here when the manifold we consider is conformally flat. The round sphere and real projective space are examples of conformally flat manifolds. The assumption that the manifold is conformally flat makes the argument simple. The non conformally flat case is much more difficult and requires the C^0-theory developed in Druet-Hebey-Robert [15, 16]. In this case, the local geometry acts as a barrier in the Pohozaev identity. We refer to Druet [10, 12], and Druet-Hebey-Robert [15, 16], for the deep analysis involved in the non conformally flat case. See also Druet-Hebey [14] for nontrivial examples of blowing-up sequences of solutions of equations like (E_α). If S_g stands for the scalar curvature of g, we let

$$\alpha_0 = \frac{n-2}{4(n-1)} \max_{x\in M} S_g(x)$$

and prove the following theorem of Druet-Hebey-Vaugon [17].

Theorem 6.1 *Let (M, g) be a smooth compact conformally flat manifold of dimension $n \geq 4$. The energy function E_m is lower semi-continuous on $(\alpha_0, +\infty)$.*

In such a result, the energy is minimal in some sense, but not in the sense of Theorem 2.1. For instance, see Druet-Hebey-Vaugon [17], we can prove that $E_m(\alpha) \to +\infty$ as $\alpha \to +\infty$. In particular, multi-bubbles are involved in this result. Examples of manifolds for which we have that $E_m(\alpha) < E(\alpha^{(n-2)/4})$ are in Druet-Hebey-Vaugon [17].

6.2 Proof of Theorem 6.1

We prove Theorem 6.1 using the Struwe decomposition. The case $\alpha \to +\infty$ is also treated in Druet-Hebey-Vaugon [17]. Different technics are required when $\alpha \to +\infty$.

Let $\alpha > \alpha_0$ and let (α_i) be a sequence of real numbers such that $\alpha_i \to \alpha$ as $i \to +\infty$. We want to prove that

$$\liminf_{i\to+\infty} E_m(\alpha_i) \geq E_m(\alpha)$$

Let (ε_i) be a sequence of positive real numbers such that $\varepsilon_i \to 0$ as $i \to +\infty$. By the definition of E_m, for any i, there exists $u_i \in S_{\alpha_i}$ such that

$$E_m(\alpha_i) \leq E(u_i) \leq E_m(\alpha_i) + \varepsilon_i$$

In particular, (u_i) is a bounded sequence in $H_1^2(M)$. Moreover, since $u_i \in S_{\alpha_i}$, the sequence (u_i) is a Palais-Smale sequence for the functional

$$J_i(u) = \frac{1}{2}\int_M \left(|\nabla u|^2 + \alpha_i u^2\right) dv_g - \frac{1}{2^\star}\int_M |u|^{2^\star} dv_g$$

Note that $E_m(\alpha_i) \leq E(\alpha_i^{(n-2)/4})$. With respect to section 5, h is replaced by the sequence (α_i). However, since $\alpha_i \to \alpha$, and $\alpha \in I\!R$, it is easily checked that the Struwe result is still valid.

Almost no changes in the proof we presented are required. In particular, there exist $k \in \mathbb{N}$, $u^0 \in S_\alpha \cup \{0\}$, and k bubbles (B_i^m), $m = 1, \dots, k$, such that

$$u_i = u^0 + \sum_{m=1}^{k} B_i^m + R_i$$

where $R_i \to 0$ in $H_1^2(M)$ as $i \to +\infty$, and such that

$$E(u_i)^{2^\star} = E(u^0)^{2^\star} + kK_n^{-n} + o(1)$$

where $o(1) \to 0$ as $i \to +\infty$. Therefore, $E(u_i) \geq E(u^0) + o(1)$, and if $u^0 \not\equiv 0$, so that $u^0 \in S_\alpha$, then

$$\begin{aligned} \liminf_{i \to +\infty} E_m(\alpha_i) &\geq E(u^0) \\ &\geq E_m(\alpha) \end{aligned}$$

and E_m is lower semi-continuous at α. The proof of the theorem then reduces to the proof that if (α_i) is a sequence of real numbers such that $\alpha_i \to \alpha$ as $i \to +\infty$, and if $u_i \in S_{\alpha_i}$ is such that $E(u_i) = E_m(\alpha_i) + o(1)$, then $\int_M u_i^2 dv_g \geq C$ for all i and some $C > 0$ independent of i.

We proceed by contradiction and assume that, up to a subsequence, $u_i \to 0$ in $L^2(M)$ as $i \to +\infty$. Then

$$u_i = \sum_{m=1}^{k} B_i^m + R_i$$

where the (B_i^m)'s are bubbles as in Definition 5.1, and $R_i \to 0$ in $H_1^2(M)$ as $i \to +\infty$. We let the x_i^m's and μ_i^m's be the respective centers and weights of the (B_i^m)'s. We let also S be the set of geometrical blow-up points defined, up to a subsequence, by

$$S = \left\{ \lim_{i \to +\infty} x_i^m \ , \ m = 1, \dots, k \right\}$$

It is easily seen that $k \geq 1$, so that S is not empty. Indeed, since $u_i \in S_{\alpha_i}$,

$$\int_M |\nabla u_i|^2 dv_g + \alpha_i \int_M u_i^2 dv_g = \int_M u_i^{2^\star} dv_g$$

and thanks to the Sobolev inequality we can write that

$$\begin{aligned} \left(\int_M u_i^{2^\star} dv_g \right)^{2/2^\star} &\leq A \int_M \left(|\nabla u_i|^2 + u_i^2 \right) dv_g \\ &\leq A \int_M u_i^{2^\star} dv_g + A \left(\frac{1}{\alpha_i} + 1 \right) \alpha_i \int_M u_i^2 dv_g \\ &\leq A \left(\frac{1}{\alpha_i} + 2 \right) \int_M u_i^{2^\star} dv_g \end{aligned}$$

where $A > 0$ is independent of i. It follows that there exists $c > 0$, independent of i, such that $\|u_i\|_{2^\star} \geq c$ for all i. This implies $k \geq 1$. We let then $S = \{x_0, \dots, x_p\}$, $p + 1 \leq k$, and divide the proof of Theorem 6.1 in three steps.

Step 1: We claim that

$$u_i \to 0 \text{ in } C^0_{loc}(M\backslash S)$$

as $i \to +\infty$. In order to prove the claim, we let $x \in M\backslash S$, and let $\delta_x > 0$ be such that $B_x(2\delta_x) \cap S = \emptyset$. We let also η be a smooth nonnegative cut-off function such that $\eta = 1$ in $B_x(\delta_x/2)$, and $\eta = 0$ in $M\backslash B_x(\delta_x)$. Since $u_i \in S_{\alpha_i}$,

$$\Delta_g u_i + \alpha_i u_i = u_i^{2^\star-1} \tag{E_i}$$

We multiply (E_i) by $\eta^2 u_i^{2^\star-1}$, and integrate over M. Easy computations give that

$$\int_M \left|\nabla(\eta\tilde{u}_i^{2^\star/2})\right|^2 dv_g \le C_1 \left(\int_{B_x(\delta_x)} \tilde{u}_i^{2^\star} dv_g\right)^{(2^\star-2)/2^\star} \int_M \left|\nabla(\eta\tilde{u}_i^{2^\star/2})\right|^2 dv_g + C_2$$

where $C_1, C_2 > 0$ do not depend on i. Since $B_x(2\delta_x) \cap S = \emptyset$, it follows from the Struwe decomposition of (u_i) that

$$\lim_{i\to+\infty} \int_{B_x(\delta_x)} u_i^{2^\star} dv_g = 0$$

Therefore,

$$\int_M \left|\nabla(\eta\tilde{u}_i^{2^\star/2})\right|^2 dv_g \le C_3$$

where $C_3 > 0$ does not depend on i. In particular, thanks to the Sobolev embedding theorem, $(\tilde{u}_i)$ is bounded in $L^{(2^\star)^2/2}(B_x(\delta_x/2))$. Since $(2^\star)^2/2 > 2^\star$, noting that

$$\Delta_g u_i \le u_i^{2^\star-1}$$

we can apply the De Giorgi-Nash-Moser iterative scheme as stated in subsection 1.6. We then get that $u_i \to 0$ in $C^0(B_x(\delta_x/4))$. Since x is arbitrary in $M\backslash S$, this proves step 1.

Going on with the proof of Theorem 6.1, we claim now that global L^2-concentration holds for the u_i's. We let $\delta > 0$ be such that $B_{x_i}(\delta) \cap B_{x_j}(\delta) = \emptyset$ for all $i \neq j$ in $\{0, \ldots, p\}$, where $S = \{x_0, \ldots, x_p\}$, and set

$$R_\delta(i) = \frac{\int_{M\backslash B_\delta} \tilde{u}_i^2 dv_g}{\int_M \tilde{u}_i^2 dv_g}$$

where B_δ is the union of the $B_{x_i}(\delta)$'s, $i = 0, \ldots, p$. Then we say that global L^2-concentration holds for the u_i's if $R_\delta(i) \to 0$ as $i \to +\infty$ for all δ as above. Another formulation is that the L^2-mass of the $\tilde{u}_i$'s concentrate around the points in S. As noticed by Druet and Robert, see [13], such a concentration does not hold when $n = 3$. We prove here that the concentration holds when $n \ge 4$. The case $n \ge 5$ is very easy. The case $n = 4$ is more tricky.

Step 2: We claim that when $n \ge 4$, for any $\delta > 0$ such that $B_{x_i}(\delta) \cap B_{x_j}(\delta) = \emptyset$ for all $i \neq j$,

$$\lim_{i\to+\infty} R_\delta(i) = 0$$

Thanks to the De Giorgi-Nash-Moser iterative scheme, see subsection 1.6, we can write that

$$\begin{aligned} \int_{M\backslash B_\delta} u_i^2 dv_g &\le \left(\max_{x\in M\backslash B_\delta} u_i\right) \int_M u_i dv_g \\ &\le C\sqrt{\int_M u_i^2 dv_g} \int_M u_i dv_g \end{aligned}$$

where $C > 0$ does not depend on i. Thanks to (E_i) we then get that

$$\int_{M\backslash B_\delta} u_i^2 dv_g \le C\|u_i\|_2 \int_M u_i^{2^\star-1} dv_g$$

It follows that

$$R_\delta(i) \le C\frac{\int_M u_i^{2^\star-1}dv_g}{\sqrt{\int_M u_i^2 dv_g}}$$

where $C > 0$ does not depend on i. If $n \ge 6$, $2^\star - 1 \le 2$, and, thanks to Hölder's inequalities,

$$\int_M u_i^{2^\star-1} dv_g \le V_g^{\frac{3-2^\star}{2}} \left(\int_M u_i^2 dv_g\right)^{\frac{2^\star-1}{2}}$$

where V_g is the volume of M with respect to g. Since $2^\star > 2$, and $u_i \to 0$ in L^2, it follows that $R_\delta(i) \to 0$ as $i \to +\infty$ when $n \ge 6$. If $n = 5$, $2 \le 2^\star - 1 \le 2^\star$, and we get by Hölder's inequalities that

$$\left(\int_M u_i^{2^\star-1} dv_g\right)^{\frac{1}{2^\star-1}} \le \left(\int_M u_i^2 dv_g\right)^{\frac{\alpha}{2}} \left(\int_M u_i^{2^\star} dv_g\right)^{\frac{1-\alpha}{2^\star}}$$

where $\alpha = \frac{3}{2(2^\star-1)}$. Since $\|u_i\|_{2^\star} \le C$, for some $C > 0$ independent of i, we get that

$$\int_M u_i^{2^\star-1} dv_g \le C\left(\int_M u_i^2 dv_g\right)^{\frac{3}{4}}$$

Noting that $\frac{3}{4} > \frac{1}{2}$, and since $u_i \to 0$ in L^2, it follows here again that $R_\delta(i) \to 0$ as $i \to +\infty$. Now we assume that $n = 4$. We write that

$$\begin{aligned}\int_M u_i^{2^\star-1} dv_g &= \int_{M\backslash B_\delta} u_i^{2^\star-1} dv_g + \int_{B_\delta} u_i^{2^\star-1} dv_g\\ &\le \left(\max_{M\backslash B_\delta} u_i\right) \int_M u_i^{2^\star-2} dv_g + \int_{B_\delta} u_i^{2^\star-1} dv_g\end{aligned}$$

so that

$$\frac{\int_M u_i^{2^\star-1}dv_g}{\sqrt{\int_M u_i^2 dv_g}} \le \left(\max_{M\backslash B_\delta} u_i\right)\sqrt{\int_M u_i^2 dv_g} + \frac{\int_{B_\delta} u_i^{2^\star-1}dv_g}{\sqrt{\int_M u_i^2 dv_g}}$$

since $2^\star = 4$. By step 1, $u_i \to 0$ in $C^0(M\backslash B_\delta)$, and we also have that $u_i \to 0$ in $L^2(M)$. It follows that

$$\lim_{i\to+\infty}\left(\max_{M\backslash B_\delta} \tilde{u}_i\right)\sqrt{\int_M \tilde{u}_i^2 dv_g} = 0$$

Given $R > 0$, we let

$$\Omega_i(R) = \bigcup_{m=1}^{k} B_{x_i^m}(R\mu_i^m)$$

where k, the x_i^m's and the μ_i^m's are given by the Struwe decomposition of (u_i). Since $2^\star = 4$, we write, thanks to Hölder's inequalities, that

$$\int_{B_\delta} u_i^{2^\star-1} dv_g \le \int_{\Omega_i(R)} u_i^{2^\star-1} dv_g + \sqrt{\int_{B_\delta\backslash\Omega_i(R)} u_i^{2^\star} dv_g}\sqrt{\int_M u_i^2 dv_g}$$

Then,

$$\frac{\int_{B_\delta} u_i^{2^\star-1}dv_g}{\sqrt{\int_M u_i^2 dv_g}} \le \sqrt{\int_{B_\delta\backslash\Omega_i(R)} u_i^{2^\star}dv_g} + \frac{\int_{\Omega_i(R)} u_i^{2^\star-1}dv_g}{\sqrt{\int_M u_i^2 dv_g}}$$

If $\varphi \in C_0^\infty(I\!R^n)$, where $C_0^\infty(I\!R^n)$ is the set of smooth functions with compact support in $I\!R^n$, we let φ_i^m be, as in subsection 5.3, the function on M defined by the equation

$$\varphi_i^m(x) = (\mu_i^m)^{-\frac{n-2}{2}}\varphi\left((\mu_i^m)^{-1}\exp_{x_i^m}^{-1}(x)\right)$$

Similar computations to the ones developed in section 5 give that for any $\tilde{m} \neq m$,

(i) $\displaystyle\int_M (B_i^{\tilde{m}})^{2^\star-1}\varphi_i^m dv_g = o(1)$,

where $o(1) \to 0$ as $i \to +\infty$. Similarly, for any $R > 0$,

(ii) $\displaystyle\int_{M\backslash B_{x_i^m}(R\mu_i^m)} (B_i^m)^{2^\star}dv_g = \varepsilon_R(i)$,

(iii) $\displaystyle\int_{B_{x_i^m}(R\mu_i^m)} (B_i^m)^{2^\star-1}\varphi_i^m dv_g = \int_{B_0(R)} u^{2^\star-1}\varphi dx + o(1)$,

(iv) $\displaystyle\int_{B_{x_i^m}(R\mu_i^m)} (B_i^m)^2(\varphi_i^m)^{2^\star-2}dv_g = \int_{B_0(R)} u^2\varphi^{2^\star-2}dx + o(1)$

where u is the fundamental positive solution

$$u(x) = \left(\frac{1}{1+\frac{|x|^2}{n(n-2)}}\right)^{\frac{n-2}{2}}$$

of the Euclidean equation $\Delta u = u^{2^\star-1}$, where $\varepsilon_R(i)$ is such that

$$\lim_{R\to+\infty}\limsup_{i\to+\infty}\varepsilon_R(i) = 0$$

and where $o(1) \to 0$ as $i \to +\infty$. It easily follows from (ii) that

$$\int_{B_\delta\backslash\Omega_i(R)} u_i^{2^\star}dv_g = \varepsilon_R(i)$$

where

$$\lim_{R\to+\infty}\limsup_{i\to+\infty}\varepsilon_R(i) = 0$$

Now, we choose $\varphi \in C_0^\infty(I\!R^n)$ such that $\varphi = 1$ in $B_0(R)$. Then,

$$\int_{\Omega_i(R)} u_i^{2^\star-1}dv_g \le \sum_{m=1}^{k}(\mu_i^m)^{\frac{n-2}{2}}\int_{B_{x_i^m}(R\mu_i^m)} u_i^{2^\star-1}\varphi_i^m dv_g$$

while, thanks to (i) and (iii),

$$\begin{aligned}\int_{B_{x_i^m}(R\mu_i^m)} u_i^{2^\star-1}\varphi_i^m dv_g &\le C\int_{B_{x_i^m}(R\mu_i^m)} (B_i^m)^{2^\star-1}\varphi_i^m dv_g + o(1)\\ &\le C\int_{B_0(R)} u^{2^\star-1}dx + o(1)\end{aligned}$$

Independently, for any m,

$$\begin{aligned}\int_M u_i^2 dv_g &\geq \int_{B_{x_i^m}(R\mu_i^m)} u_i^2 dv_g \\ &\geq (\mu_i^m)^{n-2} \int_{B_{x_i^m}(R\mu_i^m)} u_i^2 (\varphi_i^m)^{2^\star-2} dv_g\end{aligned}$$

Here, $2^\star - 2 = 2$. As easily checked, we can write that

$$\begin{aligned}\int_{B_{x_i^m}(R\mu_i^m)} u_i^2 (\varphi_i^m)^{2^\star-2} dv_g &= \int_{B_{x_i^m}(R\mu_i^m)} \left(\sum_{\tilde{m}=1}^{k} B_i^{\tilde{m}}\right)^2 (\varphi_i^m)^{2^\star-2} dv_g + o(1) \\ &\geq \int_{B_{x_i^m}(R\mu_i^m)} (B_i^m)^2 (\varphi_i^m)^{2^\star-2} dv_g + o(1)\end{aligned}$$

and thanks to (iv) we get that

$$\int_{B_{x_i^m}(R\mu_i^m)} u_i^2 (\varphi_i^m)^{2^\star-2} dv_g \geq \int_{B_0(R)} u^2 dx + o(1)$$

Hence, for any m,

$$\int_M u_i^2 dv_g \geq (\mu_i^m)^{n-2} \left(\int_{B_0(R)} u^2 dx + o(1)\right)$$

and we can write that

$$\int_M u_i^2 dv_g \geq \left(\max_{m=1,\dots,k} \mu_i^m\right)^{n-2} \left(\int_{B_0(R)} u^2 dx + o(1)\right)$$

Then, thanks to the above equations, we get that for any $R > 0$,

$$\limsup_{i\to+\infty} R_\delta(i) \leq \varepsilon_R + C\frac{\int_{B_0(R)} u^{2^\star-1} dx}{\sqrt{\int_{B_0(R)} u^2 dx}}$$

where $\varepsilon_R \to 0$ as $R \to +\infty$, and $C > 0$ does not depend on R. It is easily seen that

$$\begin{aligned}\lim_{R\to+\infty} \int_{B_0(R)} u^{2^\star-1} dx &= \int_{\mathbb{R}^n} u^{2^\star-1} dx \\ &< +\infty\end{aligned}$$

On the other hand, when $n = 4$,

$$\lim_{R\to+\infty} \int_{B_0(R)} u^2 dx = +\infty$$

Hence, $R_\delta(i) \to 0$ as $i \to +\infty$, and global L^2-concentration holds also when $n = 4$. This proves step 2.

With steps 1 and 2 we are now in position to prove Theorem 6.1. The final argument consists in plugging the u_i's into the Euclidean Pohozaev identity.

Step 3: We let $x_0 \in S$ be a geometrical blow-up point of (u_i). Since g is assumed to be conformally flat, there exists $\delta_0 > 0$ and $\varphi \in C^\infty(M)$, $\varphi > 0$, such that $\tilde{g} = \varphi^{4/(n-2)} g$ is flat in $\Omega = B_{x_0}(\delta_0)$. Let ξ be the Euclidean metric. By conformal invariance of the conformal Laplacian,

$$\Delta_\xi \frac{u_i}{\varphi} = \frac{1}{\varphi^{2^\star-1}} \left(\Delta_g u_i + \frac{n-2}{4(n-1)} S_g u_i \right)$$

where S_g is the scalar curvature of g. We let $\hat{u}_i = \varphi^{-1} u_i$. Then,

$$\Delta_\xi \hat{u}_i + h_i \hat{u}_i = \hat{u}_i^{2^\star-1} \tag{$\hat{E}_i$}$$

in Ω, where

$$h_i = \frac{\alpha_i - \frac{n-2}{4(n-1)} S_g}{\varphi^{2^\star-2}}$$

Without loss of generality, we may assume that $x_0 = 0$. We choose $\delta > 0$ such that $B_0(4\delta) \subset \Omega$ and 0 is the only geometrical blow-up point of $(\tilde{u}_i)$ in $B_0(4\delta)$, the Euclidean ball of center 0 and radius 4δ. We let also $0 \leq \eta \leq 1$ be a smooth radially symmetrical nonincreasing function such that $\eta = 1$ in $B_0(\delta)$ and $\eta = 0$ in $I\!R^n \backslash B_0(2\delta)$. For convenience, we let $\Delta = \Delta_\xi$ be the Euclidean Laplacian, and $B = B_0(2\delta)$ be the Euclidean ball of center 0 and radius 2δ. By the Pohozaev identity,

$$2 \int_B \left(x^k \partial_k (\eta \hat{u}_i) \right) \Delta(\eta \hat{u}_i) dx + (n-2) \int_B \eta \hat{u}_i \Delta(\eta \hat{u}_i) dx \leq 0$$

As easily checked,

$$\begin{aligned}
\int_B \left(x^k \partial_k (\eta \hat{u}_i) \right) \Delta(\eta \hat{u}_i) dx &= \int_B \left(x^k \partial_k \eta \right) \hat{u}_i \Delta(\eta \hat{u}_i) dx \\
&+ \int_B \eta \left(x^k \partial_k \hat{u}_i \right) \hat{u}_i \Delta \eta dx - 2 \int_B \left(x^k \partial_k \hat{u}_i \right) \eta \left(\nabla \eta \nabla \hat{u}_i \right) dx \\
&+ \int_B \eta^2 \left(x^k \partial_k \hat{u}_i \right) \Delta \hat{u}_i dx
\end{aligned}$$

where $(\nabla \eta \nabla \hat{u}_i) = \sum_i \partial_i \hat{u}_i \partial_i \eta$. Similarly,

$$\int_B \eta \hat{u}_i \Delta(\eta \hat{u}_i) dx = \int_B \eta \hat{u}_i^2 \Delta \eta dx - 2 \int_B \eta \hat{u}_i \left(\nabla \hat{u}_i \nabla \eta \right) dx + \int_B \eta^2 \hat{u}_i \Delta \hat{u}_i dx$$

Set

$$\begin{aligned}
\mathcal{R}_1(\eta, \hat{u}_i) &= \int_B \left(x^k \partial_k \eta \right) \hat{u}_i \Delta(\eta \hat{u}_i) dx + \int_B \eta \left(x^k \partial_k \hat{u}_i \right) \hat{u}_i \Delta \eta dx \\
&\quad -2 \int_B \left(x^k \partial_k \hat{u}_i \right) \eta \left(\nabla \eta \nabla \hat{u}_i \right) dx
\end{aligned}$$

and

$$\mathcal{R}_2(\eta, \hat{u}_i) = \int_B \eta \hat{u}_i^2 \Delta \eta dx - 2 \int_B \eta \hat{u}_i \left(\nabla \hat{u}_i \nabla \eta \right) dx$$

Then,

$$2 \int_B \eta^2 \left(x^k \partial_k \hat{u}_i \right) \Delta \hat{u}_i dx + (n-2) \int_B \eta^2 \hat{u}_i \Delta \hat{u}_i dx + \mathcal{R}(\eta, \hat{u}_i) \leq 0$$

where

$$\mathcal{R}(\eta,\hat{u}_i)=2\mathcal{R}_1(\eta,\hat{u}_i)+(n-2)\mathcal{R}_2(\eta,\hat{u}_i)$$

In particular, we have that

$$\begin{aligned}&2\int_B\eta^2\left(x^k\partial_k\hat{u}_i\right)\hat{u}_i^{2^\star-1}dx+(n-2)\int_B\eta^2\hat{u}_i^{2^\star}dx+\mathcal{R}(\eta,\hat{u}_i)\\&\leq 2\int_B\eta^2\left(x^k\partial_k\hat{u}_i\right)h_i\hat{u}_idx+(n-2)\int_B\eta^2h_i\hat{u}_i^2dx\end{aligned}$$

Integrating by parts,

$$\begin{aligned}\int_B\eta^2h_i\left(x^k\partial_k\hat{u}_i\right)\hat{u}_idx &= -\int_B\eta^2h_i\left(x^k\partial_k\hat{u}_i\right)\hat{u}_idx\\&\quad -n\int_B\eta^2h_i\hat{u}_i^2dx-\int_B\left(x^k\partial_k(\eta^2h_i)\right)\hat{u}_i^2dx\end{aligned}$$

so that

$$\begin{aligned}&2\int_B\eta^2\left(x^k\partial_k\hat{u}_i\right)h_i\hat{u}_idx+(n-2)\int_B\eta^2h_i\hat{u}_i^2dx\\&=-n\int_B\eta^2h_i\hat{u}_i^2dx-\int_B\left(x^k\partial_k(\eta^2h_i)\right)\hat{u}_i^2dx+(n-2)\int_B\eta^2h_i\hat{u}_i^2dx\\&=-2\int_B\eta^2h_i\hat{u}_i^2dx-\int_B\left(x^k\partial_k(\eta^2h_i)\right)\hat{u}_i^2dx\\&=-2\int_B\eta^2h_i\hat{u}_i^2dx-2\int_B\eta\left(x^k\partial_k\eta\right)h_i\hat{u}_i^2dx-\int_B\eta^2\left(x^k\partial_kh_i\right)\hat{u}_i^2dx\end{aligned}$$

Similarly,

$$\begin{aligned}\int_B\eta^2\left(x^k\partial_k\hat{u}_i\right)\hat{u}_i^{2^\star-1}dx &= -(2^\star-1)\int_B\eta^2\left(x^k\partial_k\hat{u}_i\right)\hat{u}_i^{2^\star-1}dx\\&\quad -n\int_B\eta^2\hat{u}_i^{2^\star}dx-\int_B\left(x^k\partial_k\eta^2\right)\hat{u}_i^2dx\end{aligned}$$

so that

$$\int_B\eta^2\left(x^k\partial_k\hat{u}_i\right)\hat{u}_i^{2^\star-1}dx=-\frac{n-2}{2}\int_B\eta^2\hat{u}_i^{2^\star}dx-\frac{n-2}{2n}\int_B\left(x^k\partial_k\eta^2\right)\hat{u}_i^{2^\star}dx$$

and

$$2\int_B\eta^2\left(x^k\partial_k\hat{u}_i\right)\hat{u}_i^{2^\star-1}dx+(n-2)\int_B\eta^2\hat{u}_i^{2^\star}dx=-\frac{n-2}{n}\int_B\left(x^k\partial_k\eta^2\right)\hat{u}_i^{2^\star}dx$$

It follows that

$$\begin{aligned}&-\frac{n-2}{n}\int_B\left(x^k\partial_k\eta^2\right)\hat{u}_i^{2^\star}dx+\mathcal{R}(\eta,\hat{u}_i)+2\int_B\eta^2h_i\hat{u}_i^2dx\\&+2\int_B\eta\left(x^k\partial_k\eta\right)h_i\hat{u}_i^2dx+\int_B\eta^2\left(x^k\partial_kh_i\right)\hat{u}_i^2dx\leq 0\end{aligned}$$

Regarding $\mathcal{R}_2(\eta,\hat{u}_i)$, an integration by parts gives that

$$\int_B\eta\hat{u}_i\left(\nabla\eta\nabla\hat{u}_i\right)dx=\frac{1}{2}\int_B\eta(\Delta\eta)\hat{u}_i^2dx-\frac{1}{2}\int_B|\nabla\eta|^2\hat{u}_i^2dx$$

and we therefore get that

$$\mathcal{R}_2(\eta,\hat{u}_i) = \int_B |\nabla\eta|^2\hat{u}_i^2dx$$

Regarding $\mathcal{R}_1(\eta,\hat{u}_i)$, we have

$$\begin{aligned}\int_B \eta\left(x^k\partial_k\hat{u}_i\right)\hat{u}_i\Delta\eta dx &= -n\int_B \eta(\Delta\eta)\hat{u}_i^2dx - \int_B\left(x^k\partial_k\eta\right)(\Delta\eta)\hat{u}_i^2dx\\ &\quad -\int_B \eta\left(x^k\partial_k\hat{u}_i\right)\hat{u}_i\Delta\eta dx - \int_B\left(x^k\partial_k\Delta\eta\right)\eta\hat{u}_i^2dx\end{aligned}$$

so that

$$\begin{aligned}&\int_B \eta\left(x^k\partial_k\hat{u}_i\right)\hat{u}_i\Delta\eta dx = -\frac{n}{2}\int_B \eta(\Delta\eta)\hat{u}_i^2dx\\ &-\frac{1}{2}\int_B\left(x^k\partial_k\eta\right)(\Delta\eta)\hat{u}_i^2dx - \frac{1}{2}\int_B\left(x^k\partial_k\Delta\eta\right)\eta\hat{u}_i^2dx\end{aligned}$$

Independently,

$$\begin{aligned}&\int_B\left(x^k\partial_k\eta\right)\hat{u}_i\Delta(\eta\hat{u}_i)dx = \int_B\left(x^k\partial_k\eta\right)(\Delta\eta)\hat{u}_i^2dx\\ &+\int_B\left(x^k\partial_k\eta\right)\eta\hat{u}_i\Delta\hat{u}_idx - 2\int_B\left(x^k\partial_k\eta\right)\hat{u}_i\left(\nabla\eta\nabla\hat{u}_i\right)dx\end{aligned}$$

By $(\hat{E}_i)$,

$$\int_B\left(x^k\partial_k\eta\right)\eta\hat{u}_i\Delta\hat{u}_idx = \int_B\left(x^k\partial_k\eta\right)\eta\hat{u}_i^{2^\star}dx - \int_B\left(x^k\partial_k\eta\right)\eta h_i\hat{u}_i^2dx$$

while, integrating by parts,

$$\begin{aligned}&\int_B\left(x^k\partial_k\eta\right)\hat{u}_i\left(\nabla\eta\nabla\hat{u}_i\right)dx = \int_B\left(x^k\partial_k\eta\right)(\Delta\eta)\hat{u}_i^2dx\\ &-\int_B\left(x^k\partial_k\eta\right)\hat{u}_i\left(\nabla\eta\nabla\hat{u}_i\right)dx - \int_B\left(\nabla(x^k\partial_k\eta)\nabla\eta\right)\hat{u}_i^2dx\end{aligned}$$

so that

$$\begin{aligned}\int_B\left(x^k\partial_k\eta\right)\hat{u}_i\left(\nabla\eta\nabla\hat{u}_i\right)dx &= \frac{1}{2}\int_B\left(x^k\partial_k\eta\right)(\Delta\eta)\hat{u}_i^2dx\\ &\quad -\frac{1}{2}\int_B\left(\nabla(x^k\partial_k\eta)\nabla\eta\right)\hat{u}_i^2dx\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_B\left(x^k\partial_k\eta\right)\hat{u}_i\Delta(\eta\hat{u}_i)dx = \int_B\left(x^k\partial_k\eta\right)\eta\hat{u}_i^{2^\star}dx\\ &-\int_B\left(x^k\partial_k\eta\right)\eta h_i\hat{u}_i^2dx + \int_B\left(\nabla(x^k\partial_k\eta)\nabla\eta\right)\hat{u}_i^2dx\end{aligned}$$

and we get that

$$\begin{aligned}&\mathcal{R}_1(\eta,\hat{u}_i) = \int_B\left(x^k\partial_k\eta\right)\eta\hat{u}_i^{2^\star}dx - \int_B\left(x^k\partial_k\eta\right)\eta h_i\hat{u}_i^2dx\\ &+\int_B\left(\nabla(x^k\partial_k\eta)\nabla\eta\right)\hat{u}_i^2dx - \frac{n}{2}\int_B\eta(\Delta\eta)\hat{u}_i^2dx\\ &-\frac{1}{2}\int_B\left(x^k\partial_k\eta\right)(\Delta\eta)\hat{u}_i^2dx - \frac{1}{2}\int_B\left(x^k\partial_k\Delta\eta\right)\eta\hat{u}_i^2dx\\ &-2\int_B\left(x^k\partial_k\hat{u}_i\right)\eta\left(\nabla\eta\nabla\hat{u}_i\right)dx\end{aligned}$$

Therefore,

$$
\begin{aligned}
&2\int_B \eta^2 h_i \hat{u}_i^2 dx + \int_B \eta^2 \left(x^k\partial_k h_i\right)\hat{u}_i^2 dx \\
&-4\int_B \left(x^k\partial_k \hat{u}_i\right)\eta\left(\nabla\eta\nabla\hat{u}_i\right)dx \\
&+\int_B f(\eta)\hat{u}_i^2 dx + \int_B g(\eta)\hat{u}_i^{2^\star}dx \le 0
\end{aligned}
$$

where

$$
\begin{aligned}
f(\eta) &= 2\left(\nabla(x^k\partial_k\eta)\nabla\eta\right) - n\eta\Delta\eta - \left(x^k\partial_k\eta\right)\Delta\eta \\
&\quad - \left(x^k\partial_k\Delta\eta\right)\eta + (n-2)|\nabla\eta|^2
\end{aligned}
$$

and $g(\eta) = \frac{4}{n}\left(x^k\partial_k\eta\right)\eta$. As easily checked,

$$
\left(x^k\partial_k\hat{u}_i\right)\eta\left(\nabla\eta\nabla\hat{u}_i\right) = \frac{1}{r}\eta\frac{d\eta}{dr}\left(x^k\partial_k\hat{u}_i\right)^2
$$

Since η was chosen to be nonnegative and nonincreasing, $\left(x^k\partial_k\hat{u}_i\right)\eta\left(\nabla\eta\nabla\hat{u}_i\right)$ is nonpositive, and we get that

$$
\begin{aligned}
&2\int_B \eta^2 h_i \hat{u}_i^2 dx + \int_B \eta^2 \left(x^k\partial_k h_i\right)\hat{u}_i^2 dx \\
&+\int_B f(\eta)\hat{u}_i^2 dx + \int_B g(\eta)\hat{u}_i^{2^\star}dx \le 0
\end{aligned}
$$

As easily checked,

$$
\begin{aligned}
2h_i + \left(x^k\partial_k h_i\right) &= \left(\alpha_i - \frac{n-2}{4(n-1)}S_g\right)\left(2\varphi^{2-2^\star} + x^k\partial_k\varphi^{2-2^\star}\right) \\
&\quad -\frac{n-2}{4(n-1)}\varphi^{2-2^\star}\left(x^k\partial_k S_g\right)
\end{aligned}
$$

Choosing $\delta > 0$ sufficiently small, there exists $c_1 > 0$ such that

$$
2\varphi^{2-2^\star} + x^k\partial_k\varphi^{2-2^\star} \ge c_1
$$

in B. Since we assumed that $\alpha > \alpha_0$, α the limit of the α_i's, we get that for i sufficiently large, $2h_i + \left(x^k\partial_k h_i\right) \ge c_2$ for some $c_2 > 0$ independent of i. We then get that there exists some constant $C > 0$, independent of i, such that

$$
\int_{B_0(\delta)} \hat{u}_i^2 dx \le C\int_{B_0(2\delta)\setminus B_0(\delta)} \hat{u}_i^2 dx
$$

Coming back to the manifold, it follows that for any $\delta_0 > 0$, there exists $0 < \delta < \delta_0$ and a constant $C > 0$, independent of i, such that

$$
\int_{B_{x_0}(\delta)} u_i^2 dv_g \le C\int_{B_{x_0}(\delta_0)\setminus B_{x_0}(\delta)} u_i^2 dv_g
$$

Repeating the argument for the other geometrical blow-up points in S, and summing the different inequalities we get, leads to the following: for $\delta > 0$ small, there exists a constant $C > 0$, independent of i, such that

$$\int_{B_\delta} u_i^2 dv_g \leq C \int_{M \setminus B_\delta} u_i^2 dv_g$$

where B_δ is as in step 2. By step 2,

$$\lim_{i \to +\infty} \frac{\int_{M \setminus B_\delta} u_i^2 dv_g}{\int_M u_i^2 dv_g} = 0$$

when $n \geq 4$, and the contradiction follows. This ends the proof of Theorem 6.1.

As a remark, it follows from the above proof that there always exists $u \in S_\alpha$ such that $E(u) = E_m(\alpha)$ when $\alpha > \alpha_0$.

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