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Critical Point Theory

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Chapter 1

Mountain pass theorem

1.1 Differentiable functionals

Let us recall some notions of differentiability.

Definition 1.1. *Let $\varphi : U \to \mathbb{R}$ where U is an open subset of a Banach space X. The functional φ has a Gateaux derivative $f \in X'$ at $u \in U$ if, for every $h \in X$,*

$$\lim_{t \to 0} \frac{1}{t}[\varphi(u+th) - \varphi(u) - \langle f, th \rangle] = 0.$$

The Gateaux derivative at u is denoted by $\varphi'(u)$.

The functional φ has a Fréchet derivative $f \in X'$ at $u \in U$ if

$$\lim_{h \to 0} \frac{1}{||h||}[\varphi(u+h) - \varphi(u) - \langle f, h \rangle] = 0.$$

The functional φ belongs to $C^1(U, \mathbb{R})$ if the Fréchet derivative of φ exists and is continuous on U.

If X is a Hilbert space and φ has a Gateaux derivative at $u \in U$, the gradient of φ at u is defined by

$$(\nabla \varphi(u), h) := \langle \varphi'(u), h \rangle.$$

Remarks 1.2. a) The Gateaux derivative is given by

$$\langle \varphi'(u), h \rangle := \lim_{t \to 0} \frac{1}{t}[\varphi(u+th) - \varphi(u)].$$

b) Any Fréchet derivative is a Gateaux derivative. Using the mean value theorem, it is easy to prove the following result:

Proposition 1.3. *If φ has a continuous Gateaux derivative on U then $\varphi \in C^1(U, \mathbb{R})$.*

Definition 1.4. *Let $\varphi \in C^1(U,\mathbb{R})$. The functional φ has a second Gateaux derivative $L \in \mathcal{L}(X,X')$ at $u \in U$ if, for every $h, v \in X$,*

$$\lim_{t\to 0}\frac{1}{t}\langle \varphi'(u+th)-\varphi'(u)-Lth, v\rangle = 0.$$

The second Gateaux derivative at u is denoted by $\varphi''(u)$.

The functional φ has a second Fréchet derivative $L \in \mathcal{L}(X,X')$ at $u \in U$ if

$$\lim_{h\to 0}\frac{1}{||h||}[\varphi'(u+h)-\varphi'(u)-Lh] = 0.$$

The functional φ belongs to $C^2(U,\mathbb{R})$ if the second Fréchet derivative of φ exists and is continuous on U.

Remarks 1.5. a) The second Gateaux derivative is given by

$$\langle \varphi''(u)h, v\rangle := \lim_{t\to 0}\frac{1}{t}\langle \varphi'(u+th)-\varphi'(u), v\rangle.$$

b) Any second Fréchet derivative is a second Gateaux derivative. Using the mean value theorem, it is easy to prove the following:

Proposition 1.6. *If φ has a continuous second Gateaux derivative on V then $\varphi \in C^2(U,\mathbb{R})$.*

We will use the following function spaces.

Definition 1.7. *The space*

$$H^1(\mathbb{R}^N) := \{u \in L^2(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$$

with the inner product

$$(u,v)_1 := \int_{\mathbb{R}^N}[\nabla u\cdot\nabla v + uv]$$

and the corresponding norm

$$||u||_1 := \left(\int_{\mathbb{R}^N}|\nabla u|^2 + |u|^2\right)^{1/2}$$

is a Hilbert space. Let Ω be an open subset of $\mathbb{R}^N$. The space $H_0^1(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $H^1(\mathbb{R}^N)$.

Let $N \geq 3$ and $2^* := 2N/(N-2)$. The space

$$\mathcal{D}^{1,2}(\mathbb{R}^N) := \{u \in L^{2^*}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$$

with the inner product

$$\int_{\mathbb{R}^N}\nabla u\cdot\nabla v$$

and the corresponding norm

$$\left(\int_{\mathbb{R}^N}|\nabla u|^2\right)^{1/2}$$

is a Hilbert space. The space $\mathcal{D}_0^{1,2}(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. For simplicity of notations, we shall write $2^* = \infty$ when $N = 1$ or $N = 2$.

For the following results, see [20] or [90].

Theorem 1.8. (Sobolev imbedding theorem). *The following imbeddings are continuous:*

$$\begin{array}{ll} H^1(\mathbb{R}^N) \subset L^p(\mathbb{R}^N), & 2 \leq p < \infty, N = 1,2, \\ H^1(\mathbb{R}^N) \subset L^p(\mathbb{R}^N), & 2 \leq p \leq 2^*, N \geq 3, \\ D^{1,2}(\mathbb{R}^N) \subset L^{2^*}(\mathbb{R}^N), & N \geq 3. \end{array}$$

In particular, the Sobolev inequality holds:

$$S := \inf_{\substack{u\in\mathcal{D}^{1,2}(\mathbb{R}^N)\\ |u|_{2^*}=1}} |\nabla u|_2^2 > 0.$$

Theorem 1.9. (Rellich imbedding theorem). *If $|\Omega| < \infty$, the following embeddings are compact:*

$$H_0^1(\Omega) \subset L^p(\Omega), \quad 1 \leq p < 2^*.$$

Corollary 1.10. (Poincaré inequality). *If $|\Omega| < \infty$, then*

$$\lambda_1(\Omega) := \inf_{\substack{u\in H_0^1(\Omega)\\ |u|_2=1}} |\nabla u|_2^2 > 0$$

is achieved.

Remarks 1.11. a) It is clear that $H_0^1(\Omega) \subset \mathcal{D}_0^{1,2}(\Omega)$.

b) If $|\Omega| < \infty$, Poincaré inequality implies that $H_0^1(\Omega) = \mathcal{D}_0^{1,2}(\Omega)$.

Proposition 1.12. *Let Ω be an open subset of $\mathbb{R}^N$ and let $2 < p < \infty$. The functionals*

$$\psi(u) := \int_\Omega |u|^p, \quad \chi(u) := \int_\Omega |u^+|^p$$

are of class $C^2(L^p(\Omega),\mathbb{R})$ and

$$\langle \psi'(u), h\rangle = p\int_\Omega |u|^{p-2}uh, \quad \langle \chi'(u), h\rangle = p\int_\Omega (u^+)^{p-1}h.$$

Proof. **Existence of the Gateaux derivative.** We only consider ψ. The proof for χ is similar. Let $u, h \in L^p$. Given $x \in \Omega$ and $0 < |t| < 1$, by the mean value theorem, there exists $\lambda \in]0,1[$ such that

$$\begin{aligned} \big||u(x)+th(x)|^p - |u(x)|^p\big|/|t| &= p|u(x)+\lambda th(x)|^{p-1}|h(x)| \\ &\leq p[|u(x)|+|h(x)|]^{p-1}|h(x)|. \end{aligned}$$

The Hölder inequality implies that

$$[|u(x)|+|h(x)|]^{p-1}|h(x)| \in L^1(\Omega).$$

It follows then from the Lebesgue theorem that

$$\langle \psi'(u), h\rangle = p\int_\Omega |u|^{p-2}uh.$$

Continuity of the Gateaux derivative. Let us define $f(u) := p|u|^{p-2}u$. Assume that $u_n \to u$ in L^p. Theorem A.2 or A.4 implies that $f(u_n) \to f(u)$ in L^q when $q := p/(p-1)$. We obtain, by the Hölder inequality,

$$|\langle \psi'(u_n) - \psi'(u), h\rangle| \leq |f(u_n) - f(u)|_q |h|_p,$$

and so

$$||\psi'(u_n) - \psi'(u)|| \leq |f(u_n) - f(u_n)|_q \to 0, n \to \infty.$$

Existence of the second Gateaux derivative. Let $u, h, v \in L^p(\Omega)$. Given $x \in \Omega$ and $0 < |t| < 1$, by the mean value theorem, there exists $\lambda \in]0,1[$ such that

$$\begin{aligned} &|[f(u(x)+th(x)) - f(u(x))]v(x)|/|t| \\ &\quad = p(p-1)|u(x)+\lambda th(x)|^{p-2}|h(x)|\,|v(x)| \\ &\quad \leq p(p-1)[|u(x)|+|h(x)|]^{p-2}|h(x)|\,|v(x)|. \end{aligned}$$

The Hölder inequality implies that

$$[|u(x)|+|h(x)|]^{p-2}|h(x)|\,|v(x)| \in L^1(\Omega).$$

It follows then from the Lebesgue theorem that

$$\langle \psi''(u)h, v\rangle = p(p-1)\int_\Omega |u|^{p-2}hv.$$

Continuity of the second Gateaux derivative. Let us define $g(u) := p(p-1)\,|u|^{p-2}$. Assume that $u_n \to u$ in L^p. Theorem A.2 or A.4 implies that $g(u_n) \to g(u)$ in L^r where $r := p/(p-2)$. We obtain, by the Hölder inequality,

$$|\langle(\psi''(u_n) - \psi''(u))h, v\rangle \leq |g(u_n) - g(u)|_r |h|_p |v|_p,$$

and so

$$||\psi''(u_n) - \psi''(u)|| \leq |g(u_n) - g(u_n)|_r \to 0, \quad n \to \infty. \qquad \square$$

Corollary 1.13. *a) Let $2 < p < \infty$ if $N = 1, 2$ and $2 < p \leq 2^*$ if $N \geq 3$. The functionals ψ and χ are of class $\mathcal{C}^2(H_0^1(\Omega), \mathbb{R})$.*

b) Let $N \geq 3$ and $p = 2^*$. The functional ψ and χ are of class $\mathcal{C}^2(\mathcal{D}_0^{1,2}(\Omega), \mathbb{R})$.

Proof. The result follows directly from the Sobolev theorem. $\square$

1.2 Quantitative deformation lemma

We will prove a simple case of the quantitative deformation lemma. The general version will be given in the next chapter. Let us recall that $\varphi^d := \varphi^{-1}(]-\infty, d])$.

Lemma 1.14. *Let X be a Hilbert space, $\varphi \in \mathcal{C}^2(X, \mathbb{R})$, $c \in \mathbb{R}$, $\varepsilon > 0$. Assume that*

$$(\forall u \in \varphi^{-1}([c-2\varepsilon, c+2\varepsilon])) : ||\varphi'(u)|| \geq 2\varepsilon.$$

Then there exists $\eta \in \mathcal{C}(X, X)$ such that
(i) $\eta(u) = u, \forall u \notin \varphi^{-1}([(c-2\varepsilon, c+2\varepsilon])$,
(ii) $\eta(\varphi^{c+\varepsilon}) \subset \varphi^{c-\varepsilon}$.

Proof. Let us define

$$\begin{aligned} A &:= \varphi^{-1}([c-2\varepsilon, c+2\varepsilon]), \\ B &:= \varphi^{-1}([c-\varepsilon, c+\varepsilon]), \\ \psi(u) &:= \operatorname{dist}(u, X\backslash A)(\operatorname{dist}(u, X\backslash A) + \operatorname{dist}(u, B))^{-1}, \end{aligned}$$

so that ψ is locally Lipschitz continuous, $\psi = 1$ on B and $\psi = 0$ on $X\backslash A$. Let us also define the locally Lipschitz continuous vector field

$$\begin{aligned} f(u) &:= -\psi(u)||\nabla\varphi(u)||^{-2}\nabla\varphi(u), & u \in A, \\ &:= 0, & u \in X\backslash A. \end{aligned}$$

It is clear that $||f(u)|| \leq (2\varepsilon)^{-1}$ on X. For each $u \in X$, the Cauchy problem

$$\begin{aligned} \frac{d}{dt}\sigma(t,u) &= f(\sigma(t,u)), \\ \sigma(0,u) &= u, \end{aligned}$$

has a unique solution $\sigma(.,u)$ defined on $\mathbb{R}$. Moreover, σ is continuous on $\mathbb{R}\times X$ (see e.g. [78]). The map η defined on X by $\eta(u) := \sigma(2\varepsilon, u)$ satisfies (i). Since

$$\begin{aligned}\frac{d}{dt}\varphi(\sigma(t,u)) &= \Big(\nabla\varphi(\sigma(t,u)), \frac{d}{dt}\sigma(t,u)\Big)\\ &= (\nabla\varphi(\sigma(t,u)), f(\sigma(t,u)))\\ &= -\psi(\sigma(t,u))\end{aligned} \tag{1.1}$$

$\varphi(\sigma(.,u))$ is nonincreasing. Let $u \in \varphi^{c+\varepsilon}$. If there is $t \in [0, 2\varepsilon]$ such that $\varphi(\sigma(t,u)) < c-\varepsilon$, then $\varphi(\sigma(2\varepsilon,u)) < c-\varepsilon$ and (ii) is satisfied. If

$$\sigma(t,u) \in \varphi^{-1}([c-\varepsilon, c+\varepsilon]), \forall t \in [0, 2\varepsilon],$$

then we obtain from (1.1),

$$\begin{aligned}\varphi(\sigma(2\varepsilon,u)) &= \varphi(u) + \int_0^{2\varepsilon}\frac{d}{dt}\varphi(\sigma(t,u))dt\\ &= \varphi(u) - \int_0^{2\varepsilon}\psi(\sigma(t,u))dt\\ &\le c+\varepsilon-2\varepsilon = c-\varepsilon,\end{aligned}$$

and (ii) is also satisfied. □

1.3 Mountain pass theorem

The mountain pass theorem is the simplest and one of the most useful minimax theorems.

Theorem 1.15. *Let X be a Hilbert space, $\varphi \in \mathcal{C}^2(X,\mathbb{R})$, $e \in X$ and $r > 0$ be such that $\|e\| > r$ and*

$$b := \inf_{\|u\|=r} \varphi(u) > \varphi(0) \ge \varphi(e). \tag{1.2}$$

Then, for each $\varepsilon > 0$, there exists $u \in X$ such that
a) $c - 2\varepsilon \le \varphi(u) \le c + 2\varepsilon$,
b) $\|\varphi'(u)\| < 2\varepsilon$,
where

$$c := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\varphi(\gamma(t))$$

and

$$\Gamma := \{\gamma \in \mathcal{C}([0,1],X) : \gamma(0) = 0, \gamma(1) = e\}.$$

Proof. Assumption (1.2) implies that

$$b \le \max_{t\in[0,1]}\varphi(\gamma(t)),$$

and so

$$b \le c \le \max_{t\in[0,1]}\varphi(te).$$

Suppose that, for some $\varepsilon > 0$, the conclusion of the theorem is not satisfied. We may assume

$$c - 2\varepsilon \ge \varphi(0) \ge \varphi(e). \tag{1.3}$$

By the definition of c, there exists $\gamma \in \Gamma$ such that

$$\max_{t\in[0,1]}\varphi(\gamma(t)) \le c + \varepsilon. \tag{1.4}$$

Consider $\beta := \eta\circ\gamma$, where η is given by the preceding lemma. We have, using (i) and (1.3),

$$\beta(0) = \eta(\gamma(0)) = \eta(0) = 0,$$

and similarly $\beta(1) = e$, so that $\beta \in \Gamma$. It follows from (ii) and (1.4) that

$$c \le \max_{t\in[0,1]}\varphi(\beta(t)) \le c - \varepsilon.$$

This is a contradiction. □

In order to prove that c is a critical value of φ, we need the following compactness condition.

Definition 1.16. (Brézis-Coron-Nirenberg, 1980). *Let X be a Banach space, $\varphi \in \mathcal{C}^1(X,\mathbb{R})$ and $c \in \mathbb{R}$. The function φ satisfies the $(PS)_c$ condition if any sequence $(u_n) \subset X$ such that*

$$\varphi(u_n) \to c, \quad \varphi'(u_n) \to 0 \tag{1.5}$$

has a convergent subsequence.

Theorem 1.17. (Ambrosetti-Rabinowitz, 1973). *Under the assumption of Theorem 1.15, if φ satisfies the $(PS)_c$ condition, then c is a critical value of φ.*

Proof. Theorem 1.15 implies the existence of a sequence $(u_n) \subset X$ satisfying (1.5). By $(\mathrm{PS})_c$, (u_n) has a subsequence converging to $u \in X$. But then $\varphi(u) = c$ and $\varphi'(u) = 0$. □

Example 1.18. (Brézis-Nirenberg, 1991). *Under the assumptions of Theorem 1.15, c is not, in general, a critical value of φ. Let us define $\varphi \in \mathcal{C}^\infty(\mathbb{R}^2,\mathbb{R})$ by*

$$\varphi(x,y) := x^2 + (1-x)^3y^2.$$

Clearly φ satisfies the assumptions of Theorem 1.15. But 0 is the only critical value of φ.

1.4 Semilinear Dirichlet problem

In this section, we consider the model problem

$$(\mathcal{P}_1) \qquad \begin{cases} -\Delta u + \lambda u = |u|^{p-2}u, \\ u \geq 0, u \in H_0^1(\Omega), \end{cases}$$

where Ω is a domain of $\mathbb{R}^N$. The main result is the following:

Theorem 1.19. *Assume that* $|\Omega| < \infty$ *and* $2 < p < 2^*$. *Then problem* $(\mathcal{P}_1)$ *has a nontrivial solution if and only if* $\lambda > -\lambda_1(\Omega)$.

Proof. **Necessary condition.** Suppose u is a nontrivial solution of $(\mathcal{P}_1)$. Let $e_1 \in H_0^1$ be an eigenfunction of $-\Delta$ corresponding to $\lambda_1 = \lambda_1(\Omega)$ with $e_1 > 0$ on Ω (see [90]). We have

$$\lambda \int_\Omega u\, e_1 = \int_\Omega (u^{p-1} + \Delta u) e_1 > \int_\Omega \Delta u\, e_1 = -\lambda_1 \int_\Omega u\, e_1$$

and thus $\lambda > -\lambda_1$.

Sufficient condition. Suppose $\lambda > -\lambda_1$, so that $c_1 := 1+\min(0, \lambda/\lambda_1) > 0$. On H_0^1 we have, by the Poincaré inequality,

$$|\nabla u|_2^2 + \lambda |u|_2^2 \geq c_1 |\nabla u|_2^2.$$

On H_0^1 we choose the norm $||u|| := \sqrt{|\nabla u|_2^2 + \lambda |u|_2^2}$. Let us define $f(u) := (u^+)^{p-1}$ and $F(u) := (u^+)^p/p$.

By Corollary 1.13, the functional

$$\varphi(u) := \int_\Omega \Big[\frac{|\nabla u|^2}{2} + \lambda \frac{u^2}{2} - F(u)\Big]$$

is of class $\mathcal{C}^2(H_0^1, \mathbb{R})$. We will verify the assumptions of the mountain pass theorem. The $(\mathrm{PS})_c$ condition follows from the next lemma. By the Sobolev theorem, $c_2 > 0$ such that, on H_0^1,

$$|u|_p \leq c_2 ||u||.$$

Hence we obtain

$$\begin{aligned} \varphi(u) &\geq \frac{1}{2}||u||^2 - \frac{1}{p}|u|_p^p \\ &\geq \frac{1}{2}||u||^2 - \frac{c_2^p}{p}||u||^p \end{aligned}$$

and there exists $r > 0$ such that

$$b := \inf_{||u||=r} \varphi(u) > 0 = \varphi(0).$$

Let $u \in H_0^1$ with $u > 0$ on Ω. We have, for $t \geq 0$,

$$\varphi(tu) = \frac{t^2}{2}(|\nabla u|_2^2 + \lambda |u|_2^2) - \frac{t^p}{p}|u|_p^p.$$

Since $p > 2$, there exists $e := tu$ such that $||e|| > r$ and $\varphi(e) \leq 0$.

By the mountain pass theorem, φ has a positive critical value and problem

$$\begin{aligned} -\Delta u + \lambda u &= f(u), \\ u &\in H_0^1(\Omega), \end{aligned}$$

has a nontrivial solution u. Multiplying the equation by u^- and integrating over Ω, we find

$$0 = |\nabla u^-|_2^2 + \lambda |u^-|_2^2 = ||u^-||^2.$$

Hence $u^- = 0$ and u is a solution of $(\mathcal{P}_1)$. □

Lemma 1.20. *Under the assumptions p of Theorem 1.19, if* $\lambda > -\lambda_1$ *any sequence* $(u_n) \subset H_0^1$ *such that*

$$d := \sup_n \varphi(u_n) < \infty, \varphi'(u_n) \to 0$$

contains a convergent subsequence.

Proof. 1) For n big enough, we have

$$\begin{aligned} d + 1 + ||u_n|| &\geq \varphi(u_n) - p^{-1}\langle \varphi'(u_n), u_n \rangle \\ &= \Big(\frac{1}{2} - \frac{1}{p}\Big)(|\nabla u_n|_2^2 + \lambda |u_n|_2^2) \\ &= \Big(\frac{1}{2} - \frac{1}{p}\Big)||u_n||^2. \end{aligned}$$

It follows that $||u_n||$ is bounded.

2) Going if necessary to a subsequence, we can assume that $u_n \rightharpoonup u$ in H_0^1. By the Rellich theorem, $u_n \to u$ in L^p. Theorem A.2 implies that $f(u_n) \to f(u)$ in L^q where $q := p/(p-1)$. Observe that

$$||u_n - u||^2 = \langle \varphi'(u_n) - \varphi'(u), u_n - u \rangle + \int_\Omega (f(u_n) - f(u)(u_n - u)).$$

It is clear that

$$\langle \varphi'(u_n) - \varphi'(u), u_n - u \rangle \to 0, n \to \infty.$$

It follows from the Hölder inequality that

$$\Big|\int_\Omega (f(u_n) - f(u))(u_n - u)\Big| \leq |f(u_n) - f(u)|_q |u_n - u|_p \to 0, n \to \infty.$$

Thus we have proved that $||u_n - u|| \to 0, n \to \infty$. □

1.5 Symmetry and compactness

Symmetry plays a basic role in variational problems. For example, the imbedding $H^1(\mathbb{R}^N) \subset L^2(\mathbb{R}^N)$ is noncompact because of the action of translations. If Ω is bounded, the embedding $H_0^1(\Omega) \subset L^{2^*}(\Omega)$ is noncompact because of the action of dilations. When the problem is invariant by a group of orthogonal transformations, the situation is different. In some cases, it suffices to consider invariant functions in order to recover compactness. We will also see in chapter 3, that, in other cases, symmetry implies multiplicity.

We will use the following lemma.

Lemma 1.21. (P.L. Lions, 1984). *Let $r > 0$ and $2 \le q < 2^*$. If (u_n) is bounded in $H^1(\mathbb{R}^N)$ and if*

$$\sup_{y\in\mathbb{R}^N} \int_{B(y,r)} |u_n|^q \to 0, n \to \infty,$$

then $u_n \to 0$ in $L^p(\mathbb{R}^N)$ for $2 < p < 2^*$.

Proof. We consider the case $N \ge 3$. Let $q < s < 2^*$ and $u \in H^1(\mathbb{R}^N)$. Hölder and Sobolev inequalities imply that

$$\begin{aligned} |u|_{L^s(B(y,r))} &\le |u|^{1-\lambda}_{L^q(B(y,r))}|u|^{\lambda}_{L^{2^*}(B(y,r))} \\ &\le c|u|^{1-\lambda}_{L^q(B(y,r))}\left[\int_{B(y,r)}(|u|^2 + |\nabla u|^2)\right]^{\lambda/2} \end{aligned}$$

where $\lambda := \frac{s-q}{2^*-q}\frac{2^*}{s}$. Choosing $\lambda = 2/s$, we obtain

$$\int_{B(y,r)} |u|^s \le c^s |u|^{(1-\lambda)s}_{L^q(B(y,r))} \int_{B(y,r)}(|u|^2 + |\nabla u|^2).$$

Now, covering $\mathbb{R}^N$ by balls of radius r, in such a way that each point of $\mathbb{R}^N$ is contained in at most $N+1$ balls, we find

$$\int_{\mathbb{R}^N} |u|^s \le (N+1)c^s \sup_{y\in\mathbb{R}^N}\left[\int_{B(y,r)} |u|^q\right]^{(1-\lambda)s/q} \int_{\mathbb{R}^N}(|u|^2 + |\nabla u|^2).$$

Under the assumption of the lemma, $u_n \to 0$ in $L^s(\mathbb{R}^N)$. Since $2 < s < 2^*$, $u_n \to 0$ in $L^p(\mathbb{R}^N)$ for $2 < p < 2^*$, by Sobolev and Hölder inequalities. □

Definition 1.22. *Let G be a subgroup of $\mathbf{O}(N)$, $y \in \mathbb{R}^N$ and $r > 0$. We define*

$$m(y,r,G) := \sup\{n \in \mathbb{N} : \exists g_1, \dots, g_n \in G : j \ne k \Rightarrow B(g_j y, r) \cap B(g_k y, r) = \phi\}.$$

An open subset Ω of $\mathbb{R}^N$ is invariant if $g\Omega = \Omega$ for every $g \in G$. An invariant subset Ω of $\mathbb{R}^N$ is compatible with G if, for some $r > 0$,

$$\lim_{\substack{|y|\to\infty \\ \mathrm{dist}(y,\Omega)\le r}} m(y,r,G) = \infty.$$

Definition 1.23. *Let G be a subgroup of $\mathbf{O}(N)$ and let Ω be an invariant open subset of $\mathbb{R}^N$. The action of G on $H_0^1(\Omega)$ is defined by*

$$gu(x) := u(g^{-1}x).$$

The subspace of invariant functions is defined by

$$H^1_{0,G}(\Omega) := \{u \in H_0^1(\Omega) : gu = u, \forall g \in G\}.$$

The following theorem is the main result of this section:

Theorem 1.24. *If Ω is compatible with G, the following embeddings are compact:*

$$H^1_{0,G}(\Omega) \subset L^p(\Omega), 2 < p < 2^*.$$

Proof. Assume that $u_n \rightharpoonup 0$ in $H^1_{0,G}(\Omega)$. It is clear that, for every n,

$$\int_{B(y,r)} |u_n|^2 \le \sup_n |u_n|_2^2 / m(y,r,G).$$

Let $\varepsilon > 0$. If Ω is compatible with G, there exists $R > 0$ such that, for every n,

$$\sup_{|y|\ge R} \int_{B(y,r)} |u_n|^2 \le \varepsilon.$$

It follows from the Rellich theorem that

$$\int_{B(0,R+r)} |u_n|^2 \to 0, n \to \infty,$$

and so

$$\sup_{|y|\le R} \int_{B(y,r)} |u_n|^2 \to 0, n \to \infty.$$

By the preceding lemma, $u_n \to 0$ in $L^p(\Omega)$ for $2 < p < 2^*$. □

Corollary 1.25. (P.L. Lions, 1982). *Let $N_j \ge 2$, $j = 1, \dots, k$, $\sum_{j=1}^k N_j = N$ and*

$$G := \mathbf{O}(N_1) \times \mathbf{O}(N_2) \times \dots \times \mathbf{O}(N_k).$$

Then the following embeddings are compact:

$$H^1_G(\mathbb{R}^N) \subset L^p(\mathbb{R}^N), 2 < p < 2^*.$$

Proof. It is easy to verify that $\mathbb{R}^N$ is compatible with G. □

Corollary 1.26. (Strauss, 1977). *Let $N \geq 2$. Then the following embeddings are compact:*

$$H^1_{O(N)}(\mathbb{R}^N) \subset L^p(\mathbb{R}^N), 2 < p < 2^*.$$

Proof. It suffices to apply the preceding result. □

1.6 Symmetric solitary waves

This section is devoted to the problem

$$(\mathcal{P}_2) \qquad \begin{cases} -\Delta u + u = |u|^{p-2}u, \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

where $N \geq 2$ and $2 < p < 2^*$.

We will apply the mountain pass theorem to the functional

$$\varphi(u) := \int_{\mathbb{R}^N} \left[\frac{|\nabla u|^2}{2} + \frac{u^2}{2} - F(u)\right]$$

where $F(u) := (u^+)^p/p$. In fact it suffices to find the critical points of φ restricted to a subspace of invariant functions.

Definition 1.27. *The action of a topological group G on a normed space X is a continuous map*

$$G \times X \to X : [g, u] \to gu$$

such that

$$\begin{aligned} &1 \cdot u = u, \\ &(gh)u = g(hu), \\ &u \mapsto gu \text{ is linear.} \end{aligned}$$

The action is isometric if

$$||gu|| = ||u||.$$

The space of invariant points is defined by

$$Fix(G) := \{u \in X : gu = u, \forall g \in G\}.$$

A set $A \subset X$ is invariant if $gA = A$ for every $g \in G$. A function $\varphi : X \to \mathbb{R}$ is invariant if $\varphi \circ g = \varphi$ for every $g \in G$. A map $f : X \to X$ is equivariant if $g \circ f = f \circ g$ for every $g \in G$.

Theorem 1.28. (Principle of symmetric criticality, Palais, 1979). *Assume that the action of the topological group G on the Hilbert space X is isometric. If $\varphi \in \mathcal{C}^1(X, \mathbb{R})$ is invariant and if u is a critical point of φ restricted to $Fix(G)$ then u is a critical point of φ.*

Proof. 1) Since φ is invariant, we have

$$\begin{aligned} \langle \varphi'(gu), v\rangle &= \lim_{t\to 0} \frac{\varphi(u + tg^{-1}v) - \varphi(u)}{t} \\ &= \langle \varphi'(u), g^{-1}v\rangle. \end{aligned}$$

2) Since the action is isometric, we obtain

$$(\nabla\varphi(gu), v) = (\nabla\varphi(u), g^{-1}v) = (g\nabla\varphi(u), v)$$

and so $\nabla\varphi$ is equivariant.

3) Assume that u is a critical point of φ restricted to Fix(G). It is clear that

$$g\nabla\varphi(u) = \nabla\varphi(gu) = \nabla\varphi(u)$$

and so $\nabla\varphi(u) \in$ Fix(G). Hence

$$\nabla\varphi(u) \in \text{Fix}(G) \cap \text{Fix}(G)^\perp = \{0\}.$$

□

Theorem 1.29. (Strauss, 1977). *If $N \geq 2$ and $2 < p < 2^*$, there exists a radially symmetric, positive, classical solution of $(\mathcal{P}_2)$.*

Proof. 1) Consider the functional φ restricted to $X := H^1_{O(N)}(\mathbb{R}^N)$. We shall verify the assumptions of the mountain pass theorem. As in the proof of Theorem 1.19, there exists $e \in X$ and $r > 0$ such that $||e||_1 > r$ and

$$b := \inf_{||u||_1 = r} \varphi(u) > 0 = \varphi(0) \geq \varphi(e).$$

2) It remains to prove the Palais-Smale condition. Consider a sequence $(u_n) \subset X$ such that

$$\sup_n \varphi(u_n) < \infty, \varphi'(u_n) \to 0 \quad \text{in} \quad X'.$$

As in the proof of Lemma 1.20, $||u_n||_1$ is bounded. Going if necessary to a subsequence, we can assume that $u_n \rightharpoonup u$ in X. By Corollary 1.26, $u_n \to u$ in L^p. As in the proof of Lemma 1.20, it follows that $||u_n - u||_1 \to 0$.

3) Using the mountain pass theorem, we obtain a nontrivial critical point u of φ restricted to X. By the principle of symmetric criticality, we have

$$-\Delta u + u = (u^+)^{p-2}u.$$

Multiplying the equation by u^- and integrating over $\mathbb{R}^N$, we find

$$0 = |\nabla u^-|_2^2 + |u^-|_2^2 = ||u^-||_1^2.$$

Hence $u^- = 0$ and u is a nonnegative solution of $(\mathcal{P}_2)$.

4) The next lemma implies that $u \in \mathcal{C}^2(\mathbb{R}^N)$. By the strong maximum principle u is positive. □

Lemma 1.30. *If u is a solution of $(\mathcal{P}_2)$ then $u \in \mathcal{C}^2(\mathbb{R}^N)$.*

Proof. Since
$$-\Delta u = au$$
where $a := |u|^{p-2} - 1 \in L^{N/2}_{\text{loc}}(\mathbb{R}^N)$, the Brézis-Kato theorem implies that $u \in L^p_{\text{loc}}(\mathbb{R}^N)$ for all $1 \leq p < \infty$. Thus $u \in W^{2,p}_{\text{loc}}(\mathbb{R}^N)$ for all $1 \leq p < \infty$. By elliptic regularity theory, $u \in \mathcal{C}^2(\mathbb{R}^N)$. □

The existence of a nonradial solution of $(\mathcal{P}_2)$ has been an open problem for some time.

Theorem 1.31. (Bartsch-Willem, 1993). *If $N = 4$ or $N \geq 6$ and $2 < p < 2^*$ then problem $(\mathcal{P}_2)$ has a nonradial solution.*

Proof. Let $2 \leq m \leq N/2$ be a fixed integer different from $(N-1)/2$. The action of
$$G := \mathbf{O}(m) \times \mathbf{O}(m) \times \mathbf{O}(N-2m)$$
on $H^1(\mathbb{R}^N)$ is defined by
$$gu(x) := u(g^{-1}x).$$
By Corollary 1.25, the embedding $H^1_G(\mathbb{R}^N) \subset L^p(\mathbb{R}^N)$ is compact. Let τ be the involution defined on $\mathbb{R}^N = \mathbb{R}^m \oplus \mathbb{R}^m \oplus \mathbb{R}^{N-2m}$ by
$$\tau(x_1, x_2, x_3) := (x_2, x_1, x_3).$$
The action of $H := \{\text{id}, \tau\}$ on $H^1_G(\mathbb{R}^N)$ is defined by
$$\begin{aligned} hu(x) &:= u(x), & h &= \text{id}, \\ &:= -u(h^{-1}x), & h &= \tau. \end{aligned}$$
It is clear that 0 is the only radial function of
$$X := \{u \in H^1_G(\mathbb{R}^N) : hu = u, \forall h \in H\}.$$
Moreover the embedding $X \subset L^p(\mathbb{R}^N)$ is compact. As in the proof of Theorem 1.29, we apply the mountain pass theorem. We obtain a nontrivial critical point u of φ restricted to X. By the principle of symmetric criticality, u is a nontrivial critical point of φ. □

1.7 Subcritical Sobolev inequalities

Let $N \geq 2$ and $2 < p < 2^*$. The Sobolev theorem implies that
$$S_p := \inf_{\substack{u \in H^1(\mathbb{R}^N) \\ |u|_p = 1}} ||u||_1^2 > 0.$$

In order to prove that the infimum is achieved, we consider a minimizing sequence $(u_n) \subset H^1(\mathbb{R}^N)$:
$$(1.8) \qquad |u_n|_p = 1, \quad ||u_n||_1^2 \to S_p, \quad n \to \infty.$$
Going if necessary to a subsequence, we may assume $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$, so that
$$||u||_1^2 \leq \underline{\lim}\, ||u_n||_1^2 = S_p.$$
Thus u is a minimizer provided $|u|_p = 1$. But we know only that $|u|_p \leq 1$. Indeed, for any $v \in H^1$ and $y \in \mathbb{R}^N$ the translated function
$$v^y(x) := v(x+y)$$
satisfies
$$||v^y||_1 = ||v||_1, \quad |v^y|_p = |v|_p.$$
Hence the problem is invariant by the noncompact group of translations. In order to overcome this difficulty, we will use the following result.

Lemma 1.32. (Brézis-Lieb Lemma, 1983). *Let Ω be an open subset of $\mathbb{R}^N$ and let $(u_n) \subset L^p(\Omega)$, $1 \leq p < \infty$. If*
a) (u_n) is bounded in $L^p(\Omega)$,
b) $u_n \to u$ almost everywhere on Ω, then
$$\lim_{n \to \infty} (|u_n|_p^p - |u_n - u|_p^p) = |u|_p^p.$$

Proof. Fatou's Lemma yields
$$|u|_p \leq \underline{\lim}\, |u_n|_p < \infty.$$
Fix $\varepsilon > 0$. There exists $c(\varepsilon)$ such that, for all $a, b \in \mathbb{R}$,
$$\Big| |a+b|^p - |a|^p \Big| \leq \varepsilon |a|^p + c(\varepsilon)|b|^p.$$
Hence we obtain
$$\begin{aligned} f_n^\varepsilon &:= \left(\Big| |u_n|^p - |u_n - u|^p - |u|^p \Big| - \varepsilon |u_n - u|^p \right)^+ \\ &\leq (1 + c(\varepsilon))|u|^p. \end{aligned}$$
By the Lebesgue theorem, $\int_\Omega f_n^\varepsilon \to 0$, $n \to \infty$. Since
$$\Big| |u_n|^p - |u_n - u|^p - |u|^p \Big| \leq f_n^\varepsilon + \varepsilon |u_n - u|^p,$$
we obtain
$$\overline{\lim_{n \to \infty}} \int_\Omega \Big| |u_n|^p - |u_n - u|^p - |u|^p \Big| \leq c\,\varepsilon$$
where $c := \sup_n |u_n - u|_p^p < \infty$. Now let $\varepsilon \to 0$. □

Remarks 1.33. a) The preceding lemma is a refinement of Fatou's Lemma.

b) Under the assumptions of the lemma, $u_n \rightharpoonup u$ weakly in $L^p(\Omega)$. However, weak convergence in $L^p(\Omega)$ is not sufficient to obtain the conclusion, except when $p = 2$.

c) In any Hilbert space

$$u_n \rightharpoonup u \Rightarrow \lim_{n\to\infty}(|u_n|^2 - |u_n - u|^2) = |u|^2.$$

Theorem 1.34. (P.L. Lions, 1984). *Let $(u_n) \subset H^1(\mathbb{R}^N)$ be a minimizing sequence satisfying* (1.8). *Then there exists a sequence $(y_n) \subset \mathbb{R}^N$ such that $u_n^{y_n}$ contains a convergent subsequence. In particular there exists a minimizer for S_p.*

Proof. Since $|u_n|_p = 1$, Lemma 1.21 implies that

$$\delta := \varliminf_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,1)} |u_n|^2 > 0.$$

Going if necessary to a subsequence, we may assume the existence of $(y_n) \subset \mathbb{R}^N$ such that

$$\int_{B(y_n,1)} |u_n|^2 > \delta/2.$$

Let us define $v_n := u_n^{y_n}$. Hence $|v_n|_p = 1$, $||v_n||_1^2 \to S_p$ and

$$\int_{B(0,1)} |v_n|^2 > \delta/2. \tag{1.9}$$

Since (v_n) is bounded in $H^1(\mathbb{R}^N)$, we may assume, going if necessary to a subsequence

$$\begin{aligned} v_n &\rightharpoonup v && \text{in } H^1(\mathbb{R}^N),\\ v_n &\to v && \text{in } L^2_{\text{loc}}(\mathbb{R}^N),\\ v_n &\to v && \text{a.e. on } \mathbb{R}^N. \end{aligned}$$

By the preceding lemma,

$$1 = |v|_p^p + \lim |w_n|_p^p,$$

where $w_n := v_n - v$. Hence we have

$$\begin{aligned} S_p = \lim ||v_n||_1^2 &= ||v||_1^2 + \lim ||w_n||_1^2 \\ &\geq S_p[(|v|_p^p)^{2/p} + (1 - |v|_p^p)^{2/p}]. \end{aligned}$$

Since, by (1.9), $v \neq 0$, we obtain $|v|_p^p = 1$, and so

$$||v||_1^2 = S_p = \lim ||v_n||_1^2.$$ □

Theorem 1.35. *There exists a radially symmetric, positive, C^2 minimizer for S_p.*

Proof. 1) By the preceding theorem, there exists a minimizer $u \in H^1(\mathbb{R}^N)$ for S_p. By Theorem C.4, u is radially symmetric. Replacing u by $|u|$, we may also assume that u is non-negative.

2) It follows from Lagrange multiplier rule that, for some $\lambda > 0$, u is a solution of

$$-\Delta u + u = \lambda u^{p-1}.$$

By Lemma 1.30, $u \in C^2(\mathbb{R}^N)$. The strong maximum principle implies that u is positive. □

1.8 Non symmetric solitary waves

This section is devoted to the problem

$$(\mathcal{P}_3) \qquad \begin{cases} -\Delta u + u = Q(x)|u|^{p-2}u, \\ u \geq 0, u \in H^1(\mathbb{R}^N), \end{cases}$$

where $N \geq 2$, $2 < p < 2^*$ and $Q \in \mathcal{C}(\mathbb{R}^N)$ satisfies

$$1 = \lim_{|x|\to\infty} Q(x) = \inf_{x\in\mathbb{R}^N} Q(x). \tag{1.10}$$

By scaling, it is easy to replace 1 by any positive number. Let us define as before $f(u) := (u^+)^{p-1}$ and $F(u) := (u^+)^p/p$. By a variant of Corollary 1.13, the functional

$$\varphi(u) := \int_{\mathbb{R}^N} \Big[\frac{|\nabla u|^2}{2} + \frac{u^2}{2} - Q(x)F(u)\Big]dx$$

is of class $\mathcal{C}^2(H^1(\mathbb{R}^N), \mathbb{R})$. Let $v > 0$ be a minimizing function for S_p and let $(a_n) \subset \mathbb{R}^N$ be such that $|a_n| \to \infty$, $n \to \infty$. It is easy to verify that

$$\varphi'\big(S_p^{\frac{1}{p-2}} v^{a_n}\big) \to 0, \varphi\big(S_p^{\frac{1}{p-2}} v^{a_n}\big) \to (\frac{1}{2} - \frac{1}{p})S_p^{\frac{p}{p-2}}, n \to \infty.$$

Hence condition $(PS)_c$ is not satisfied for $c = (\frac{1}{2} - \frac{1}{p})S_p^{\frac{p}{p-2}}$.

Lemma 1.36. *Under assumption* (1.10), *any sequence $(u_n) \subset H^1(\mathbb{R}^N)$ such that*

$$d := \sup_n \varphi(u_n) < c^* := (\frac{1}{2} - \frac{1}{p})S_p^{\frac{p}{p-2}}, \varphi'(u_n) \to 0$$

contains a convergent subsequence.

Proof. 1) As in the proof of Lemma 1.20, $(||u_n||_1)$ is bounded. Going if necessary to a subsequence, we can assume that

$$\begin{aligned} u_n &\rightharpoonup u && \text{in } H^1(\mathbb{R}^N), \\ u_n &\to u && \text{in } L^p_{\text{loc}}(\mathbb{R}^N), \\ u_n &\to u && \text{a.e. on } \mathbb{R}^N. \end{aligned}$$

It follows that

$$f(u_n) \to f(u) \quad \text{in } L^{p/(p-1)}_{\text{loc}}(\mathbb{R}^N),$$

and so

$$-\Delta u + u = Q(x)|u|^{p-2}u,$$

$$\varphi(u) = \frac{||u||_1^2}{2} - \int Q(x)F(u)dx = (\frac{1}{2} - \frac{1}{p})||u||_1^2 \geq 0. \tag{1.11}$$

2) We write $v_n := u_n - u$. The Brézis-Lieb Lemma leads to

$$\begin{aligned} \int Q(x)F(u_n)dx &= \int Q(x)F(u)dx + \int Q(x)F(v_n)dx + o(1) \\ &= \int Q(x)F(u)dx + \int \frac{(v_n^+)^p}{p}dx + o(1). \end{aligned}$$

Assuming $\varphi(u_n) \to c \leq d$, we obtain

$$\varphi(u) + \frac{||v_n||_1^2}{2} - \int \frac{(v_n^+)^p}{p}dx \to c. \tag{1.12}$$

Since $\langle \varphi'(u_n), u_n \rangle \to 0$, we also obtain

$$\begin{aligned} ||v_n||_1^2 - \int (v_n^+)^p dx &\to p\int Q(x)F(u)dx - ||u||_1^2 \\ &= -\langle \varphi'(u), u \rangle \\ &= 0. \end{aligned}$$

We may therefore assume that

$$||v_n||_1^2 \to b, \quad \int (v_n^+)^p \to b.$$

By the Sobolev inequality, we have

$$||v_n||_1^2 \geq S_p |v_n^+|_p^2,$$

and so $b \geq S_p b^{2/p}$. Either $b = 0$ or $b \geq S_p^{\frac{p}{p-2}}$. If $b = 0$, the proof is complete. Assume $b \geq S_p^{\frac{p}{p-2}}$. We obtain from (1.11) and (1.12)

$$c^* = (\frac{1}{2} - \frac{1}{p})S_p^{\frac{p}{p-2}} \leq (\frac{1}{2} - \frac{1}{p})b \leq c \leq d < c^*,$$

a contradiction. □

Theorem 1.37. (Ding-Ni, 1986). *Under assumption* (1.10), *if* $N \geq 2$ *and* $2 < p < 2^*$, *problem* $(\mathcal{P}_3)$ *has a nontrivial solution.*

Proof. 1) It suffices to apply the mountain pass theorem with a value $c < c^*$. Let $v > 0$ be a minimizing function for S_p. If $Q \equiv 1$, the result follows from Theorem 1.29. We may assume that $Q \not\equiv 1$. Hence we obtain $\int Q(x)v^p dx > \int v^p dx$. It follows that

$$\begin{aligned} 0 < \max_{t\geq 0} \varphi(tv) &= \max_{t\geq 0}\Big(\frac{t^2}{2}||v||_1^2 - \frac{t^p}{p}\int Q(x)v^p dx\Big) \\ &= (\frac{1}{2} - \frac{1}{p})\Big[||v||_1^2/(\int Q(x)v^p dx)^{\frac{2}{p}}\Big]^{\frac{p}{p-2}} \\ &< (\frac{1}{2} - \frac{1}{p})[||v||_1^2/|v|_p^2]^{\frac{p}{p-2}} \\ &= (\frac{1}{2} - \frac{1}{p})S_p^{\frac{p}{p-2}} = c^*. \end{aligned}$$

2) Since

$$\begin{aligned} \varphi(u) &\geq \frac{||u||_1^2}{2} - \frac{M}{p}|u|_p^p \\ &\geq \frac{||u||_1^2}{2} - \frac{M}{pS_p^{p/2}}||u||_1^p, \end{aligned}$$

where $M := \max_{\mathbb{R}^N} Q$, there exists $r > 0$ such that

$$b := \inf_{||u||_1 = r} \varphi(u) > 0 = \varphi(0).$$

There exists $t_0 > 0$ such that $||t_0 v||_1 > r$ and $\varphi(t_0 v) < 0$. It follows from the preceding step that

$$\max_{t\in[0,1]} \varphi(tt_0 v) < c^*.$$

By the preceding lemma and the mountain pass theorem, φ has a critical value $c \in [b, c^*[$ and problem

$$\begin{cases} -\Delta u + u = Q(x)f(u), \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

has a nontrivial solution u. Multiplying the equation by u^- and integrating, we find $u^- = 0$ and u is a solution of $(\mathcal{P}_3)$. □

1.9 Critical Sobolev inequality

Let $N \geq 3$. The optimal constant in the Sobolev inequality is given by

$$S := \inf_{\substack{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \\ |u|_{2^*}=1}} |\nabla u|_2^2 > 0.$$

In order to prove that the infimum is achieved, we consider a minimizing sequence $(u_n) \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$:

$$(1.13) \qquad |u_n|_{2^*} = 1, |\nabla u_n|_2^2 \to S, n \to \infty.$$

Going if necessary to a subsequence, we may assume $u_n \rightharpoonup u$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, so that

$$|\nabla u|_2^2 \leq \underline{\lim} |\nabla u_n|_2^2 = S.$$

Thus u is a minimizer provided $|u|_{2^*} = 1$. But we know only that $|u|_{2^*} \leq 1$. Indeed, for any $v \in \mathcal{D}^{1,2}$, $y \in \mathbb{R}^N$ and $\lambda > 0$, the rescaled function

$$v^{y,\lambda}(x) := \lambda^{(N-2)/2} v(\lambda x + y)$$

satisfies

$$|\nabla v^{y,\lambda}|_2 = |\nabla v|_2, \quad |v^{y,\lambda}|_{2^*} = |v|_{2^*}.$$

Hence the problem is invariant by translations and dilations. In order to exclude noncompactness, we will use some results from measure theory (see [90]).

Definition 1.38. *Let Ω be an open subset of $\mathbb{R}^N$ and define*

$$\mathcal{K}(\Omega) := \{u \in \mathcal{C}(\Omega) : \text{ supp } u \text{ is a compact subset of } \Omega\},$$

$$\mathcal{BC}(\Omega) := \{u \in \mathcal{C}(\Omega) : |u|_\infty := \sup_{x \in \Omega} |u(x)| < \infty\}.$$

The space $\mathcal{C}_0(\Omega)$ is the closure of $\mathcal{K}(\Omega)$ in $\mathcal{BC}(\Omega)$ with respect to the uniform norm. A finite measure on Ω is a continuous linear functional on $\mathcal{C}_0(\Omega)$. The norm of the finite measure μ is defined by

$$||\mu|| := \sup_{\substack{u \in \mathcal{C}_0(\Omega) \\ |u|_\infty = 1}} |\langle \mu, u \rangle|.$$

We denote by $\mathcal{M}(\Omega)$ (resp. $\mathcal{M}^+(\Omega)$) the space of finite measures (resp. positive finite measures) on Ω. A sequence (μ_n) converges weakly to μ in $\mathcal{M}(\Omega)$, written

$$\mu_n \rightharpoonup \mu,$$

provided

$$\langle \mu_n, u \rangle \to \langle \mu, u \rangle, \forall u \in \mathcal{C}_0(\Omega).$$

Theorem 1.39. *a) Every bounded sequence of finite measures on Ω contains a weakly convergent subsequence.*

b) If $\mu_n \rightharpoonup \mu$ in $\mathcal{M}(\Omega)$ then (μ_n) is bounded and

$$||\mu|| \leq \underline{\lim} ||\mu_n||.$$

c) If $\mu \in \mathcal{M}^+(\Omega)$ then

$$||\mu|| = \langle \mu, 1 \rangle = \sup_{\substack{u \in \mathcal{BC}(\Omega) \\ |u|_\infty = 1}} \langle \mu, u \rangle.$$

Following P.L. Lions [51] (inequality 1.15), Bianchi, Chabrowski, Szulkin (inequality 1.16) and Ben-Naoum, Troestler, Willem (equalities 1.17 and 1.18), we describe the lack of compactness of the injection $\mathcal{D}^{1,2}(\mathbb{R}^N) \subset L^{2^*}(\mathbb{R}^N)$.

Lemma 1.40. (Concentration-compactness lemma). *Let $(u_n) \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$ be a sequence such that*

$$\begin{aligned} u_n &\rightharpoonup u && \text{in } \mathcal{D}^{1,2}(\mathbb{R}^N), \\ |\nabla(u_n - u)|^2 &\rightharpoonup \mu && \text{in } \mathcal{M}(\mathbb{R}^N), \\ |u_n - u|^{2^*} &\rightharpoonup \nu && \text{in } \mathcal{M}(\mathbb{R}^N), \\ u_n &\to u && \text{a.e. on } \mathbb{R}^N \end{aligned}$$

and define

$$(1.14) \qquad \mu_\infty := \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{|x|\geq R} |\nabla u_n|^2, \quad \nu_\infty := \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{|x|>R} |u_n|^{2^*}.$$

Then it follows that

$$(1.15) \qquad ||\nu||^{2/2^*} \leq S^{-1} ||\mu||,$$

$$(1.16) \qquad \nu_\infty^{2/2^*} \leq S^{-1} \mu_\infty,$$

$$(1.17) \qquad \overline{\lim_{n\to\infty}} |\nabla u_n|_2^2 = |\nabla u|_2^2 + ||\mu|| + \mu_\infty,$$

$$(1.18) \qquad \overline{\lim_{n\to\infty}} |u_n|_{2^*}^{2^*} = |u|_{2^*}^{2^*} + ||\nu|| + \nu_\infty.$$

Moreover, if $u = 0$ and $||\nu||^{2/2^*} = S^{-1}||\mu||$, then ν and μ are concentrated at a single point.

Proof. 1) Assume first $u = 0$. Choosing $h \in \mathcal{D}(\mathbb{R}^N)$, we infer from the Sobolev inequality that

$$\Big(\int |hu_n|^{2^*} dx\Big)^{2/2^*} \le S^{-1} \int |\nabla(hu_n)|^2 dx.$$

Since $u_n \to 0$ in L^2_{loc}, we obtain

$$\Big(\int |h|^{2^*} d\nu\Big)^{2/2^*} \le S^{-1} \int |h|^2 d\mu. \tag{1.19}$$

Inequality (1.15) then follows.

2) For $R > 1$, let $\psi_R \in C^1(\mathbb{R}^N)$ be such that $\psi_R(x) = 1$ for $|x| > R+1$, $\psi_R(x) = 0$ for $|x| < R$ and $0 \le \psi_R(x) \le 1$ on $\mathbb{R}^N$. By the Sobolev inequality, we have

$$\Big(\int |\psi_R u_n|^{2^*} dx\Big)^{2/2^*} \le S^{-1} \int |\nabla(\psi_R u_n)|^2 dx.$$

Since $u_n \to 0$ in L^2_{loc}, we obtain

$$\overline{\lim_{n\to\infty}} \Big(\int |\psi_R u_n|^{2^*} dx\Big)^{2/2^*} \le S^{-1} \overline{\lim_{n\to\infty}} \int |\nabla u_n|^2 \psi_R^2 dx. \tag{1.20}$$

On the other hand, we have

$$\int_{|x|>R+1} |\nabla u_n|^2 dx \le \int |\nabla u_n|^2 \psi_R^2 dx \le \int_{|x|>R} |\nabla u_n|^2 dx$$

and

$$\int_{|x|>R+1} |u_n|^{2^*} dx \le \int |u_n|^{2^*} \psi_R^{2^*} dx \le \int_{|x|>R} |u_n|^{2^*} dx.$$

We obtain from (1.14)

$$\mu_\infty = \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int |\nabla u_n|^2 \psi_R^2 dx, \quad \nu_\infty = \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int |u_n|^{2^*} \psi_R^{2^*} dx.$$

Inequality (1.16) follows then from (1.20).

3) Assume moreover that $||\nu||^{2/2^*} = S^{-1}||\mu||$. The Hölder inequality and (1.19) imply that, for $h \in \mathcal{D}(\mathbb{R}^N)$,

$$\Big(\int |h|^{2^*} d\nu\Big)^{1/2^*} \le S^{-1/2}||\mu||^{1/N} \Big(\int |h|^{2^*} d\mu\Big)^{1/2^*}$$

We deduce $\nu = S^{-2^*/2}||\mu||^{2/N-2}\mu$. It follows from (1.19) that, for $h \in \mathcal{D}(\mathbb{R}^N)$,

$$\Big(\int |h|^{2^*} d\nu\Big)^{1/2^*} ||\nu||^{1/N} \le \Big(\int |h|^2 d\nu\Big)^{1/2}$$

and so, for each open set Ω,

$$\nu(\Omega)^{1/2^*} \nu(\mathbb{R}^N)^{1/N} \le \nu(\Omega)^{1/2}.$$

It follows that ν is concentrated at a single point.

4) Considering now the general case, we write $v_n := u_n - u$. Since

$$v_n \rightharpoonup 0, \quad \text{in } \mathcal{D}^{1,2}(\mathbb{R}^N),$$

we have

$$|\nabla v_n|^2 \rightharpoonup \mu + |\nabla u|^2, \quad \text{in } \mathcal{M}(\mathbb{R}^N).$$

According to the Brézis-Lieb Lemma, we have for every non negative $h \in \mathcal{K}(\mathbb{R}^N)$,

$$\int h|u|^{2^*} = \lim_{n\to\infty} \Big(\int h|u_n|^{2^*} - \int h|v_n|^{2^*}\Big).$$

Hence we obtain

$$|u_n|^{2^*} \rightharpoonup \nu + |u|^{2^*} \text{ in } \mathcal{M}(\mathbb{R}^N).$$

Inequality (1.15) follows from the corresponding inequality for (v_n).

5) Since

$$\overline{\lim_{n\to\infty}} \int_{|x|>R} |\nabla v_n|^2 = \overline{\lim_{n\to\infty}} \int_{|x|>R} |\nabla u_n|^2 - \int_{|x|>R} |\nabla u|^2,$$

we obtain

$$\lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{|x|>R} |\nabla v_n|^2 = \mu_\infty.$$

By the Brézis-Lieb Lemma, we have

$$\int_{|x|>R} |u|^{2^*} = \lim_{n\to\infty} \Big(\int_{|x|>R} |u_n|^{2^*} - \int_{|x|>R} |v_n|^{2^*}\Big)$$

and so

$$\lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{|x|\ge R} |v_n|^{2^*} = \nu_\infty.$$

Inequality (1.16) follows then from the corresponding inequality for (v_n).

6) For every $R > 1$, we have

$$\begin{aligned} \overline{\lim_{n\to\infty}} \int |\nabla u_n|^2 &= \overline{\lim_{n\to\infty}} \Big(\int \psi_R |\nabla u_n|^2 + \int (1-\psi_R)|\nabla u_n|^2\Big) \\ &= \overline{\lim_{n\to\infty}} \int \psi_R |\nabla v_n|^2 + \int (1-\psi_R) d\mu + \int (1-\psi_R)|\nabla u|^2. \end{aligned}$$

When $R \to \infty$, we obtain, by Lebesgue theorem,

$$\overline{\lim_{n\to\infty}} \int |\nabla u_n|^2 = \mu_\infty + \int d\mu + \int |\nabla u|^2 = \mu_\infty + ||\mu|| + |\nabla u|_2^2.$$

The proof of (1.18) is similar. □

Theorem 1.41. (P.L. Lions, 1985). *Let $(u_n) \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$ be a minimizing sequence satisfying (1.13). Then there exists a sequence $(y_n, \lambda_n) \subset \mathbb{R}^N \times]0, \infty[$ such that $(u_n^{y_n,\lambda_n})$ contains a convergent subsequence. In particular there exists a minimizer for S.*

Proof. Define the Lévy concentration functions

$$Q_n(\lambda) := \sup_{y \in \mathbb{R}^N} \int_{B(y,\lambda)} |u_n|^{2^*}.$$

Since, for every n,

$$\lim_{\lambda \to 0^+} Q_n(\lambda) = 0, \quad \lim_{\lambda \to \infty} Q_n(\lambda) = 1,$$

there exists $\lambda_n > 0$ such that $Q_n(\lambda_n) = 1/2$. Moreover, there exists $y_n \in \mathbb{R}^N$ such that

$$\int_{B(y_n,\lambda_n)} |u_n|^{2^*} = Q_n(\lambda_n) = 1/2,$$

since

$$\lim_{|y| \to \infty} \int_{B(y,\lambda_n)} |u_n|^{2^*} = 0.$$

Let us define $v_n := u_n^{y_n,\lambda_n}$. Hence $|v_n|_{2^*} = 1$, $|\nabla v_n|_2^2 \to S$ and

$$\frac{1}{2} = \int_{B(0,1)} |v_n|^{2^*} = \sup_{y \in \mathbb{R}^N} \int_{B(y,1)} |v_n|^{2^*}. \tag{1.21}$$

Since (v_n) is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, we may assume, going if necessary to a subsequence,

$$\begin{aligned} v_n &\rightharpoonup v && \text{in } \mathcal{D}^{1,2}(\mathbb{R}^N), \\ |\nabla(v_n - v)|^2 &\rightharpoonup \mu && \text{in } \mathcal{M}(\mathbb{R}^N), \\ |v_n - v|^{2^*} &\rightharpoonup \nu && \text{in } \mathcal{M}(\mathbb{R}^N), \\ v_n &\to v && \text{a.e. on } \mathbb{R}^N. \end{aligned}$$

By the preceding lemma,

$$S = \lim |\nabla v_n|_2^2 = |\nabla v|_2^2 + ||\mu|| + \mu_\infty, \tag{1.22}$$

$$1 = |v_n|_{2^*}^{2^*} = |v|_{2^*}^{2^*} + ||\nu|| + \nu_\infty, \tag{1.23}$$

where

$$\mu_\infty := \lim_{R \to \infty} \overline{\lim_{n \to \infty}} \int_{|x|>R} |\nabla v_n|^2, \quad \nu_\infty := \lim_{R \to \infty} \overline{\lim_{n \to \infty}} \int_{|x|>R} |v_n|^{2^*}.$$

We deduce from (1.22), (1.15), (1.16) and Sobolev inequality,

$$S \geq S\Big((|v|_{2^*}^{2^*})^{2/2^*} + ||\nu||^{2/2^*} + \nu_\infty^{2/2^*}\Big).$$

It follows from (1.23) that $|v|_{2^*}^{2^*}$, $||\nu||$ and ν_∞ are equal either to 0 or to 1. By (1.21), $\nu_\infty \leq 1/2$ so that $\nu_\infty = 0$. If $||\nu|| = 1$ then $v = 0$ and $||\nu||^{2/2^*} \geq S^{-1}||\mu||$. The preceding lemma implies that ν is concentrated at a single point z. We deduce from (1.21) the contradiction

$$\frac{1}{2} = \sup_{y \in \mathbb{R}^N} \int_{B(y,1)} |v_n|^{2^*} \geq \int_{B(z,1)} |v_n|^{2^*} \to ||\nu|| = 1.$$

Thus $|v|_{2^*}^{2^*} = 1$ and so

$$|\nabla v|_2^2 = S = \lim |\nabla v_n|_2^2.$$

□

Theorem 1.42. (Aubin, Talenti, 1976). *The instanton*

$$U(x) := \frac{[N(N-2)]^{(N-2)/4}}{[1 + |x|^2]^{(N-2)/2}}$$

is a minimizer for S.

Proof. 1) By the preceding theorem, there exists a minimizer $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ for S. By Theorem C.4, u is radially symmetric. Replacing u by $|u|$, we may also assume that u is non-negative.

2) It follows from Lagrange multiplier rule that, for some $\lambda > 0$, u is a solution of

$$-\Delta u = \lambda u^{\frac{N+2}{N-2}}.$$

By the argument of Lemma 1.30, $u \in \mathcal{C}^2(\mathbb{R}^N)$. The strong maximum principle implies that u is positive.

3) After scaling, we may assume

$$-\Delta u = u^{\frac{N+2}{N-2}}.$$

Moreover we can choose $\varepsilon > 0$ such that

$$U_\varepsilon(x) := \varepsilon^{(2-N)/2} U(x/\varepsilon)$$

satisfies

$$U_\varepsilon(0) = u(0).$$

But then u and U_ε are solutions of the problem

$$\begin{cases} \partial_r(r^{N-1}\partial_r v) = r^{N-1} v^{\frac{N+2}{N-2}}, r > 0, \\ v(0) = u(0) \quad \partial_r v(0) = 0. \end{cases}$$

It follows easily that $u = U_\varepsilon$. By invariance, U is a minimizer for S. □

Proposition 1.43. *For every open subset Ω of $\mathbb{R}^N$,*

$$S(\Omega) := \inf_{\substack{u\in\mathcal{D}_0^{1,2}(\Omega)\\ |u|_{2^*}=1}} |\nabla u|_2^2 = S$$

and $S(\Omega)$ is never achieved except when $\Omega = \mathbb{R}^N$.

Proof. 1) It is clear that $S \le S(\Omega)$. Let $(u_n) \subset \mathcal{D}(\mathbb{R}^N)$ be a minimizing sequence for S. We can choose $y_n \subset \mathbb{R}^N$ and $\lambda_n > 0$ such that

$$u_n^{y_n,\lambda_n} \in \mathcal{D}(\Omega).$$

Hence we obtain $S(\Omega) \le S$.

2) Assume that $\Omega \ne \mathbb{R}^N$ and $u \in \mathcal{D}_0^{1,2}(\Omega)$ is a minimizer for $S(\Omega)$. By the preceding step, u is also a minimizer for S. We may assume that $u \ge 0$, so that u is a solution of

$$-\Delta u = \lambda u^{\frac{N+2}{N-2}}.$$

By the strong maximum principle, $u > 0$ on $\mathbb{R}^N$. This is a contradiction, since $u \in \mathcal{D}_0^{1,2}(\Omega)$. □

1.10 Critical nonlinearities

This section is devoted to the problem

$$(\mathcal{P}_4) \qquad \begin{cases} -\Delta u + \lambda u = |u|^{2^*-2}u, \\ u \ge 0, u \in H_0^1(\Omega), \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \ge 3$ and $\lambda > -\lambda_1(\Omega)$.

Let us define as before $f(u) := (u^+)^{2^*-1}$ and $F(u) := (u^+)^{2^*}/2^*$. By Corollary 1.13, the functional

$$\varphi(u) := \int_\Omega \Big[\frac{|\nabla u|^2}{2} + \lambda\frac{u^2}{2} - F(u)\Big]dx$$

is of class $C^2(H_0^1(\Omega), \mathbb{R})$. On $H_0^1(\Omega)$, we choose the norm $||u|| := \sqrt{|\nabla u|_2^2 + \lambda|u|_2^2}$.

Lemma 1.44. *Any sequence $(u_n) \subset H_0^1(\Omega)$ such that*

$$d := \sup_n \varphi(u_n) < c^* := S^{N/2}/N, \varphi'(u_n) \to 0,$$

contains a convergent subsequence.

Proof. 1) As in the proof of Lemma 1.20, $(||u_n||)$ is bounded. Going if necessary to a subsequence, we can assume that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } H_0^1(\Omega), \\ u_n &\to u \quad \text{in } L^2(\Omega), \\ u_n &\to u \quad \text{a.e. on } \Omega. \end{aligned}$$

Since (u_n) is bounded in $L^{2^*}(\Omega)$, $(f(u_n))$ is bounded in $L^{2N/(N+2)}(\Omega)$ and so (see [90])

$$f(u_n) \rightharpoonup f(u) \quad \text{in} \quad L^{2N/(N+2)}(\Omega).$$

It follows that

$$-\Delta u + \lambda u = f(u)$$

and

$$(1.24) \qquad \varphi(u) = \frac{||u||^2}{2} - \int F(u) = (\frac{1}{2} - \frac{1}{2^*})|u^+|_{2^*}^{2^*} \ge 0.$$

2) We write $v_n := u_n - u$. The Brézis-Lieb Lemma leads to

$$\int F(u_n) = \int F(u) + \int F(v_n) + o(1).$$

Assuming $\varphi(u_n) \to c \le d$, we obtain

$$(1.25) \qquad \varphi(u) + \frac{||v_n||^2}{2} - \int F(v_n) \to c.$$

Since $\langle \varphi'(u_n), u_n\rangle \to 0$, we obtain also

$$\begin{aligned} ||v_n||^2 - 2^* \int F(v_n) &\to 2^* \int F(u) - ||u||^2 \\ &= -\langle \varphi'(u), u\rangle \\ &= 0. \end{aligned}$$

We may therefore assume that

$$||v_n||^2 \to b, \quad 2^* \int F(v_n) \to b.$$

Since $v_n \to 0$ in $L^2(\Omega)$, it follows that $|\nabla v_n|_2^2 \to b$. By Sobolev inequality, we have

$$|\nabla v_n|_2^2 \ge S|v_n^+|_{2^*}^2.$$

and so $b \ge S\, b^{2/2^*}$. Either $b = 0$ or $b \ge S^{N/2}$. If $b = 0$, the proof is complete. Assume $b \ge S^{N/2}$. We obtain, from (1.24) and (1.25),

$$c^* = (\frac{1}{2} - \frac{1}{2^*})S^{N/2} \le (\frac{1}{2} - \frac{1}{2^*})b \le c \le d < c^*,$$

a contradiction. □

Theorem 1.45. (Brézis-Nirenberg, 1983). *Let Ω be a bounded domain of $\mathbb{R}^N$, $N \geq 4$. If $-\lambda_1(\Omega) < \lambda < 0$, then problem $(\mathcal{P}_4)$ has a nontrivial solution.*

Proof. 1) It suffices to apply the mountain pass theorem with a value $c < c^*$. By the next lemma, there exists a nonnegative $v \in H_0^1 \backslash \{0\}$ such that

$$||v||^2/|v|_{2^*}^2 < S.$$

We obtain

$$\begin{aligned} 0 < \max_{t\geq 0} \varphi(tv) &= \max_{t\geq 0}\Big(\frac{t^2}{2}||v||^2 - \frac{t^{2^*}}{2^*}\int v^{2^*}\Big) \\ &= (||v||^2/|v|_{2^*}^2)^{N/2}/N \\ &< S^{N/2}/N = c^*. \end{aligned}$$

2) Since

$$\begin{aligned} \varphi(u) &\geq \frac{||u||^2}{2} - \frac{1}{2^*}|u|_{2^*}^{2^*} \\ &\geq \frac{||u||^2}{2} - \frac{1}{2^* S^{2^*/2}}|\nabla u|_2^{2^*}, \end{aligned}$$

there exists $r > 0$ such that

$$b := \inf_{||u||=r} \varphi(u) > 0 = \varphi(0).$$

There exists also $t_0 > 0$ such that $||t_0 v|| > r$ and $\varphi(t_0 v) < 0$. It follows from the preceding step that

$$\max_{t\in[0,1]} \varphi(t t_0 v) < c^*.$$

By the preceding lemma and the mountain pass theorem, φ has a critical value $c \in [b, c^*[$ and problem

$$\begin{cases} -\Delta u + \lambda u = f(u), \\ u \in H_0^1(\Omega), \end{cases}$$

has a nontrivial solution u. Multiplying the equation by u^- and integrating, we find $u^- = 0$ and u is a solution of $(\mathcal{P}_4)$. □

If U is the instanton, we have, for $\lambda < 0$,

$$\frac{||U||^2}{|U|_{2^*}^2} = \frac{|\nabla U|_2^2 + \lambda |U|_2^2}{|U|_{2^*}^2} < \frac{|\nabla U|_2^2}{|U|_{2^*}^2} = S.$$

Since $U \notin H_0^1(\Omega)$, it is necessary to "concentrate" U near a point of Ω after multiplication by a trunction function.

Lemma 1.46. *Under the assumption of Theorem 1.45, there exists a nonnegative $v \in H_0^1(\Omega)\backslash\{0\}$ such that*

$$||v||^2/|v|_{2^*}^2 < S.$$

Proof. We may assume that $0 \in \Omega$. Let $\psi \in \mathcal{D}(\Omega)$ be a nonnegative function such that $\psi \equiv 1$ on $B(0,\rho)$, $\rho > 0$, and define, for $\varepsilon > 0$,

$$\begin{aligned} U_\varepsilon(x) &:= \varepsilon^{(2-N)/2} U(x/\varepsilon), \\ u_\varepsilon(x) &:= \psi(x) U_\varepsilon(x). \end{aligned}$$

It follows from Theorem 1.42 that

$$|\nabla U_\varepsilon|_2^2 = |U_\varepsilon|_{2^*}^{2^*} = S^{N/2}.$$

As $\varepsilon \to 0^+$, we have that

$$\begin{aligned} \int_\Omega |\nabla u_\varepsilon|^2 &= \int_{\mathbb{R}^N} |\nabla U_\varepsilon|^2 + O(\varepsilon^{N-2}) = S^{N/2} + O(\varepsilon^{N-2}), \\ \int_\Omega |u_\varepsilon|^{2^*} &= \int_{\mathbb{R}^N} |U_\varepsilon|^{2^*} + O(\varepsilon^N) = S^{N/2} + O(\varepsilon^N), \\ \int_\Omega |u_\varepsilon|^2 &= \int_{B(0,\rho)} |U_\varepsilon|^2 + O(\varepsilon^{N-2}) \\ &\geq \int_{B(0,\varepsilon)} \frac{[N(N-2)\varepsilon^2]^{\frac{N-2}{2}}}{[2\varepsilon^2]^{N-2}} + \int_{\varepsilon<|x|<\rho} \frac{[N(N-2)\varepsilon^2]^{\frac{N-2}{2}}}{[2|x|^2]^{N-2}} + O(\varepsilon^{N-2}) \\ &= \begin{cases} d\varepsilon^2|\ell n\varepsilon| + O(\varepsilon^2), & \text{if } N = 4, \\ d\varepsilon^2 + O(\varepsilon^{N-2}), & \text{if } N \geq 5, \end{cases} \end{aligned}$$

where d is a positive constant. If $N = 4$, we obtain

$$\begin{aligned} \frac{||u_\varepsilon||^2}{|u|_{2^*}^2} &\leq \frac{S^2 + \lambda d\varepsilon^2|\ell n\varepsilon| + O(\varepsilon^2)}{(S^2 + O(\varepsilon^4))^{1/2}} \\ &= S + \lambda d\varepsilon^2|\ell n\varepsilon|S^{-1} + O(\varepsilon^2) < S, \end{aligned}$$

for $\varepsilon > 0$ sufficiently small. And similarly, if $N \geq 5$, we have

$$\begin{aligned} \frac{||u_\varepsilon||^2}{|u_\varepsilon|_{2^*}^2} &\leq \frac{S^{N/2} + \lambda d\varepsilon^2 + O(\varepsilon^{N-2})}{(S^{N/2} + O(\varepsilon^N))^{2/2^*}} \\ &= S + \lambda d\varepsilon^2 S^{(2-N)/2} + O(\varepsilon^{N-2}) < S, \end{aligned}$$

for $\varepsilon > 0$ sufficiently small. □

When Ω is a smooth starshaped bounded domain, Theorem 1.45 is sharp.

Proposition 1.47. *Assume that problem* $(\mathcal{P}_4)$ *has a nontrivial solution. Then we have* $\lambda > -\lambda_1(\Omega)$*. Moreover if* Ω *is a smooth starshaped bounded domain, then* $\lambda < 0$.

Proof. As in Theorem 1.19, it is easy to see that $\lambda > -\lambda_1(\Omega)$. Let us prove that any nontrivial solution u of $(\mathcal{P}_4)$ is smooth if Ω is smooth. Since

$$-\Delta u = au$$

where $a := u^{2^*-2} - \lambda \in L^{N/2}(\Omega)$, Brézis-Kato theorem implies that $u \in L^p(\Omega)$ for all $1 \leq p < \infty$. Thus $u \in W^{2,p}(\Omega)$ for all $1 \leq p < \infty$. By elliptic regularity theory, $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\bar{\Omega})$. The Pohozaev identity (Theorem B.1) leads to

$$-\lambda \int_\Omega u^2 = \int_{\partial\Omega} \frac{|\nabla u|^2}{2} \sigma \cdot \nu \, d\sigma.$$

If Ω is starshaped about the origin, we have $s \cdot n > 0$ on $\partial\Omega$. It follows that $\lambda \leq 0$. If $\lambda = 0$, then $\nabla u = 0$ on $\partial\Omega$ and we obtain from $(\mathcal{P}_4)$

$$0 = -\int_\Omega \Delta u = \int_\Omega u^{2^*-1},$$

so that $u = 0$. □

Remarks 1.48. a) It is interesting to compare Propositions 1.43 and 1.47. Under the stronger assumption that the domain Ω is starshaped, Proposition 1.47 gives the stronger conclusion that equation

$$-\Delta u = |u|^{2^*-2} u \tag{1.26}$$

has no positive solution in $H_0^1(\Omega)$.

b) For some domains Ω, equation (1.26) has a positive solution in $H_0^1(\Omega)$ (see [21]). By Proposition 1.43, it is not possible to construct this solution by minimization.

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NON-RADIAL GROUND STATES FOR THE HÉNON EQUATION

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We analyse symmetry breaking for ground states of the Hénon equation [7] in a ball. Asymptotic estimates of the transition are also given when p is close to either 2 or 2^*.

Keywords: Symmetry breaking; Henon's equation.

1. Introduction

Let us consider Hénon's equation (see [7]) with Dirichlet boundary conditions:

$$\begin{cases} -\Delta u = |x|^\alpha u^{p-1}, \ u > 0 & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1}$$

where Ω denotes the unit ball in $\mathbb{R}^N$.

We are interested in the symmetry of ground states solutions of (1). For $\alpha \geq 0$ and p superquadratic but subcritical

$$\begin{cases} 2 < p < 2^* = +\infty\,, & \text{if } N = 2\,, \\ 2 < p < 2^* = \dfrac{2N}{N-2}\,, & \text{if } N \geq 3\,, \end{cases}$$

it is easy to verify that

$$S_{\alpha,p} := \inf_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\int_\Omega |\nabla u|^2 dx}{(\int_\Omega |x|^\alpha |u|^p dx)^{2/p}} \tag{2}$$

is achieved at least by a positive function. By ground state, we mean any minimizer u in (2). After rescaling, u is a solution of (1). Since (1) and (2) are invariant under rotations of Ω, it is natural to consider

$$S^R_{\alpha,p} := \inf_{\substack{u\in H^1_{0,\mathrm{rad}}(\Omega)\\ u\neq 0}} \frac{\int_\Omega |\nabla u|^2 dx}{(\int_\Omega |x|^\alpha |u|^p dx)^{2/p}} \tag{3}$$

where $H^1_{0,\mathrm{rad}}(\Omega)$ denotes the space of radial functions in $H^1_0(\Omega)$. It is also easy to verify that $S^R_{\alpha,p}$ is achieved by a positive function v. By the symmetric criticality principle (see e.g. [12]), after rescaling, v is a solution of (1).

Since the weight $|x|^\alpha$ in front of the nonlinearity is increasing, the celebrated theorem of Gidas, Ni and Nirenberg [6] nor any of its later variants do apply. For the same reason, symmetrization will in general not increase the value of the denominator in (2) (see [9] for a survey on symmetrization). Hence, there is an interest in analyzing the symmetry properties of the ground states.

In a very interesting paper [3] there are numerical computations suggesting that, in some cases,

$$S_{\alpha,p} < S^R_{\alpha,p}. \tag{4}$$

Our main result is that for any $2 < p < 2^*$, there exists $\alpha^* > 0$ such that (4) holds provided $\alpha > \alpha^*$. A monotonicity result can also be proved in dimension 2.

The existence of nonsymmetric ground states of symmetric problems was first proved by Brezis and Nirenberg in [2] for an annulus when p is almost critical. The subcritical case was treated by Coffman [4] for an expanding annulus. The survey [1] by Brezis contains many references. Notice that for an annulus, the Gidas–Ni–Nirenberg theorem does not imply symmetry even in the autonomous case.

In Sec. 2, we prove a general necessary condition for radial ground states, which also holds for local minimizers. This condition is used in Sec. 3 to prove our main result. Section 4 contains an asymptotic analysis:

$$S^R_{\alpha,p} \sim C\left(\frac{\alpha+N}{N}\right)^{1+2/p}, \quad \alpha\to+\infty. \tag{5}$$

Sections 5 and 6 are devoted to ground states for p close to the limit values 2 and 2^*. Section 7 contains some numerical computations concerning the transition from radialicity to symmetry breaking, and in Sec. 8 a generalization to the q-Laplacian is presented.

2. A Necessary Condition for Radial Ground States

Let Ω be a radial bounded domain in $\mathbb{R}^N$, $N\ge 3$, e.g. a ball or an annulus. For $2<p<2^*$ and $\rho\in L^q(\Omega)$, $q=2^*/(2^*-p)$, ρ positive, we define on $H^1_0(\Omega)$ the Rayleigh quotient

$$R(u) = \frac{Z(u)}{N(u)} := \frac{\int_\Omega |\nabla u|^2 dx}{(\int_\Omega \rho(x)|u|^p dx)^{2/p}}. \tag{6}$$

Theorem 2.1. *If ρ is a radial function then any radial local minimizer u of R satisfies*

$$\int_\Omega |\nabla u|^2 dx \le \frac{N-1}{p-2}\int_\Omega \frac{u^2}{|x|^2}dx.$$

Proof. Without loss of generality, we can assume that

$$\int_\Omega \rho(x)|u(x)|^p dx = 1.$$

Any local minimizer u of R satisfies $g''(0)\ge 0$ where $g(\varepsilon)=R(u+\varepsilon h)$ and $h\in H^1_0(\Omega)$ is fixed. Since $g'(0)=0$,

$$g''(0) = \frac{\langle Z''(u)h,h\rangle N(u) - \langle N''(u)h,h\rangle Z(u)}{N^2(u)}, \tag{7}$$

where

$$\langle Z''(u)h,h\rangle = 2\int_\Omega |\nabla h|^2 dx, \tag{8}$$

and

$$\langle N''(u)h,h\rangle = 2\left[(2-p)\left(\int_\Omega \rho(x)|u|^{p-2}uh dx\right)^2 + (p-1)\int_\Omega \rho(x)|u|^{p-2}h^2 dx\right], \tag{9}$$

from which we deduce that

$$\int_\Omega |\nabla u|^2 dx \left[(2-p)\left(\int_\Omega \rho(x)|u|^{p-2}uh dx\right)^2 + (p-1)\int_\Omega \rho(x)|u|^{p-2}h^2 dx\right] \le \int_\Omega |\nabla h|^2 dx. \tag{10}$$

Assume now that u is radial. For h we choose a function of the form $h=u(r)f(\sigma)$ where f is a smooth function defined on the sphere S^{N-1}, with zero mean. Notice that $h\in H^1_0$ since $N\ge 3$.

Since

$$|\nabla h|^2 = \left(\frac{\partial u}{\partial r}\right)^2 f^2 + \frac{1}{r^2}u^2|\nabla_\sigma f|^2,$$

we obtain

$$(p-2)\int_\Omega |\nabla u|^2 dx \int_{S^{N-1}} f^2 d\sigma \le \int_\Omega \frac{u^2}{|x|^2}dx \int_{S^{N-1}} |\nabla_\sigma f|^2 d\sigma.$$

But

$$\inf_{\substack{f\in H^1(S^{N-1})\\ \int_{S^{N-1}} f=0}} \frac{\int_{S^{N-1}}|\nabla_\sigma f|^2 d\sigma}{\int_{S^{N-1}} f^2 d\sigma} = N-1,$$

since the infimum is attained by the first non constant spherical harmonic in dimension N (see [8]). This ends the proof. □

Our argument was suggested in part by [10]. However, it should be pointed out that the function $f(\sigma)$ used in [10] is not in $H^1(S^{N-1})$. Nevertheless, the argument there is easily corrected using the same function $f(\sigma)$ as in Theorem 2.1.

3. Non-Radialicity of the Hénon Equation Ground States

For the Hénon equation, Ω is a ball and ρ is a pure power. By a scaling argument, we can assume that Ω is the unit ball. Let $\alpha > 0$ fixed and u_α denote a minimizer of the problem (3) such that

$$\int_{B(0,1)} |\nabla u_\alpha|^2 dx = 1 .$$

Proposition 3.1.

$$\int_{B(0,1)} \frac{u_\alpha^2}{|x|^2} dx \to 0 \quad \textit{as } \alpha \to +\infty .$$

Proof. Choose $\varepsilon > 0$.

Step 1. There exists $0 < R < 1$ independent of α such that $u_\alpha(R) < \varepsilon$.

Indeed, as $\int_{B(0,1)} |\nabla u_\alpha|^2 dx = 1$, $R := 1 - \varepsilon$ is an admissible value.

Step 2. From the next lemma, we deduce that

$$\int_{B(0,R)} |\nabla u_\alpha|^2 dx \to 0 \quad \text{as } \alpha \to +\infty . \tag{11}$$

Step 3. Split the integral in the following way:

$$\int_{B(0,1)} \frac{u_\alpha^2}{|x|^2} dx = \int_{B(0,R)} \frac{u_\alpha^2}{|x|^2} dx + \int_{A(R,1)} \frac{u_\alpha^2}{|x|^2} dx$$

$$\leq 2 \int_{B(0,R)} \frac{\tilde{u}_\alpha^2}{|x|^2} dx + 2 \int_{B(0,R)} \frac{u_\alpha(R)^2}{|x|^2} dx + \int_{A(R,1)} \frac{u_\alpha^2}{|x|^2} dx , \tag{12}$$

where $\tilde{u}_\alpha := u_\alpha - u_\alpha(R)$.

As $\tilde{u}_\alpha \in H_0^1(B(0,R))$, we deduce from the Hardy inequality that

$$\int_{B(0,R)} \frac{\tilde{u}_\alpha^2}{|x|^2} dx \leq \frac{4}{(N-2)^2} \int_{B(0,R)} |\nabla u_\alpha|^2 dx . \tag{13}$$

Using Step 2, this term must converge to 0 as α goes to $+\infty$.

For the second term,

$$\int_{B(0,R)} \frac{u_\alpha(R)^2}{|x|^2} dx \leq C\varepsilon^2 ,$$

where ε depends only on $N \geq 3$.

And for the third term, we deduce from Step 2 that u_α weakly converges to 0 in H_0^1 as $\alpha \to +\infty$. Hence, by Rellich theorem,

$$\int_{A(R,1)} \frac{u_\alpha^2}{|x|^2} dx \to 0 . \tag{14}$$

Since ε was arbitrary, this ends the proof. □

The next estimate was used in the proof of Proposition 3.1.

Lemma 3.1. *For all* $0 < R < 1$,

$$\int_{B(0,R)} |\nabla u_\alpha|^2 dx \to 0 \quad \textit{as } \alpha \to +\infty .$$

Proof. Fix $\varepsilon > 0$. From the equation

$$-\Delta u_\alpha = \left(\int_{B(0,1)} |x|^\alpha u_\alpha^p dx \right)^{-1} |x|^\alpha u_\alpha^{p-1} , \tag{15}$$

we deduce that for each $r > 0$,

$$\int_{B(0,r)} |\nabla u_\alpha|^2 dx = \int_{\partial B(0,r)} u_\alpha \frac{\partial u_\alpha}{\partial \eta} d\sigma + \left(\int_{B(0,1)} |x|^\alpha u_\alpha^p dx \right)^{-1} \int_{B(0,r)} |x|^\alpha u_\alpha^p dx .$$

Hence, u_α being decreasing with respect to $|x|$,

$$\int_{B(0,r)} |\nabla u_\alpha|^2 dx \leq \frac{\int_{B(0,r)} |x|^\alpha u_\alpha^p dx}{\int_{B(0,1)} |x|^\alpha u_\alpha^p dx} . \tag{16}$$

Let $k \in \mathbb{N}$ such that $2R^{k-1} < \varepsilon$. We define the stretched functions $v_\alpha(|x|) := u_\alpha(|x|^\beta)$, where $\beta := 1 + k/(\alpha + N - k)$. A calculation leads to

$$R(v_\alpha)^{-p/2} = \beta^{-1-p/2} \cdot \frac{\int_{B(0,1)} |x|^{\alpha-k} |u_\alpha|^p dx}{\left(\int_{B(0,1)} |x|^{-\frac{k(N-2)}{\alpha+N}} |\nabla u_\alpha|^2 dx\right)^{p/2}} . \tag{17}$$

From Hölder inequality, it follows that

$$R(v_\alpha)^{-p/2} \geq \beta^{-1-p/2} \cdot \frac{\int_{B(0,1)} |x|^{\alpha-k} |u_\alpha|^p dx}{\int_{B(0,1)} |x|^{-\frac{k(N-2)p}{2(\alpha+N)}} |\nabla u_\alpha|^2 dx} . \tag{18}$$

Notice that $\gamma := \frac{k(N-2)p}{2(\alpha+N)} = o(1)$ and define

$$g(s) := \begin{cases} 1 & \text{if } s \leq 1 \\ s^{-1/\gamma} & \text{if } s \geq 1 \end{cases} .$$

Then,

$$\int_{B(0,1)} |x|^{-\gamma} |\nabla u_\alpha|^2 dx = \omega_{N-1} \int_0^1 r^{-\gamma} (u_\alpha')^2(r) r^{N-1} dr$$

$$= \omega_{N-1} \int_0^{+\infty} \int_0^{g(s)} (u_\alpha')^2(r) r^{N-1} dr ds$$

$$\leq \omega_{N-1} \int_0^{+\infty} \int_0^{g(s)} r^\alpha |u_\alpha|^p r^{N-1} dr ds \cdot \left(\int_{B(0,1)} |x|^\alpha u_\alpha^p dx \right)^{-1}$$

$$= \int_{B(0,1)} |x|^{\alpha-\gamma} u_\alpha^p dx \cdot \left(\int_{B(0,1)} |x|^\alpha u_\alpha^p dx \right)^{-1} \tag{19}$$

Since $R(v_\alpha) \geq R(u_\alpha)$, we deduce from (18) and (19) that

$$\beta^{-1-p/2} \cdot \frac{\int_{B(0,1)} |x|^{\alpha-k} u_\alpha^p dx}{\int_{B(0,1)} |x|^{\alpha-\gamma} u_\alpha^p dx} \leq 1,$$

which clearly imply

$$\beta^{-1-p/2} R^{-k+\gamma} \cdot \frac{\int_{B(0,R)} |x|^{\alpha-\gamma} u_\alpha^p dx}{\int_{B(0,1)} |x|^{\alpha-\gamma} u_\alpha^p dx} \leq 1. \tag{20}$$

Also,

$$\int_{B(0,R)} |x|^{\alpha-\gamma} u_\alpha^p dx \int_{B(0,1)} |x|^{\alpha} u_\alpha^p dx - \int_{B(0,1)} |x|^{\alpha-\gamma} u_\alpha^p dx \int_{B(0,R)} |x|^{\alpha} u_\alpha^p dx$$

$$\geq \int_{B(0,R)} |x|^\alpha u_\alpha^p dx \left(\int_{A(R,1)} (|x|^\alpha - |x|^{\alpha-\gamma}) u_\alpha^p dx \right)$$

$$= o\left(\int_{B(0,1)} |x|^\alpha u_\alpha^p dx \int_{B(0,1)} |x|^{\alpha-\gamma} u_\alpha^p dx \right) \quad \text{as } \alpha \to +\infty. \tag{21}$$

From Eq. (20), we now have

$$\frac{\int_{B(0,R)} |x|^\alpha u_\alpha^p dx}{\int_{B(0,1)} |x|^\alpha u_\alpha^p dx} \leq 2\beta^{1+p/2} R^{k-\gamma} \leq \varepsilon \tag{22}$$

if α is large enough. The conclusion then comes from Eq. (16). □

Using Theorem 2.1 and Proposition 3.1, the next theorem immediately follows.

Theorem 3.1. *Assume $N \geq 3$. For any $2 < p < 2^*$, there exists $\alpha^* > 0$ such that no minimizer of R is radial provided $\alpha > \alpha^*$.*

4. Asymptotic Estimates

The technique used in the two previous sections can be adapted easily to prove equivalent symmetry breaking results in dimension two. In this situation one uses the test function $h := u(r)\varphi(r)f(\sigma)$ where u and f are as before and φ is a cut-off function which is zero in a neighborhood of $r = 0$ and one on a neighborhood of $r = 1$. Thanks to the concentrating behaviour proved in Sec. 3, the contradiction comes the same way.

Nevertheless, we have a different method of proof which is particularly simple when $N = 2$ and which we will present now. It depends on a change of variable that leads to an asymptotic estimate of $S_{\alpha,p}^R$ as $\alpha \to +\infty$. Again, Ω stands for $B(0,1)$.

Theorem 4.1. *If $N \geq 2$, there exists $C > 0$ depending on N and p such that*

$$S_{\alpha,p}^R \sim C\left(\frac{\alpha+N}{N}\right)^{1+2/p}, \quad \alpha \to +\infty.$$

Proof. Let $u \in H_{0,\mathrm{rad}}^1(\Omega)$ and define the rescaled function $v(|x|) = u(|x|^\beta)$, where $\beta = N/(\alpha+N)$. Then

$$\begin{aligned}\int_\Omega |x|^\alpha |u|^p dx &= \omega_{N-1} \int_0^1 |u(r)|^p r^{\alpha+N-1} dr \\ &= \omega_{N-1}\beta \int_0^1 |v(\rho)|^p \rho^{\beta(\alpha+N-1)} \rho^{\beta-1} d\rho \\ &= \omega_{N-1}\beta \int_0^1 |v(\rho)|^p \rho^{N-1} d\rho \\ &= \beta \int_\Omega |v|^p dx\end{aligned} \tag{23}$$

and

$$\begin{aligned}\int_\Omega |\nabla u|^2 dx &= \omega_{N-1} \int_0^1 |u'(r)|^2 r^{N-1} dr \\ &= \omega_{N-1}\beta^{-1} \int_0^1 |v'(\rho)|^2 \rho^{2-2\beta} \rho^{(N-1)\beta} \rho^{\beta-1} d\rho \\ &= \omega_{N-1}\beta^{-1} \int_0^1 |v'(\rho)|^2 \rho^{(2-N)(1-\beta)} \rho^{N-1} d\rho \\ &= \beta^{-1} \int_\Omega |\nabla v|^2 |x|^{(2-N)(1-\beta)} dx.\end{aligned} \tag{24}$$

It follows that

$$R(u) = \frac{1}{\beta^{1+2/p}} \frac{\int_\Omega |\nabla v|^2 |x|^{(2-N)(1-\beta)} dx}{(\int_\Omega |v|^p dx)^{2/p}}.$$

For every $0 \leq \beta \leq 1$,

$$c_\beta = \inf_{v \in H_{0,\mathrm{rad}}^1(\Omega)} \frac{\int_\Omega |\nabla v|^2 |x|^{(2-N)(1-\beta)} dx}{(\int_\Omega |v|^p dx)^{2/p}}$$

is achieved by standard arguments. Since c_β is non-decreasing on $[0,1]$,

$$S_{\alpha,p}^R \sim \frac{C}{\beta^{1+2/p}} = C\left(\frac{\alpha+N}{N}\right)^{1+2/p}, \quad \alpha \to +\infty,$$

where $C = \lim_{\beta \to 0_+} c_\beta$. □

We now derive from Theorem 4.1 another proof of Theorem 3.1, valid from dimension $N = 2$.

Theorem 4.2. *Assume $N \geq 2$. For any $2 < p < 2^*$, there exists $\alpha^* \geq 0$ such that $S_{\alpha,p} < S_{\alpha,p}^R$ provided $\alpha > \alpha^*$.*

Proof. Let $u \in \mathcal{D}(\Omega)$, $u \geq 0$, and define $u_\alpha(x) := u(\alpha(x - x_\alpha))$, where $x_\alpha = (1 - 1/\alpha, 0, \dots, 0)$. One has

$$\int_\Omega |\nabla u_\alpha|^2 dx = \alpha^2 \int_\Omega |\nabla u(\alpha(x - x_\alpha))|^2 dx$$

$$= \alpha^{2-N} \int_\Omega |\nabla u|^2 dx\,, \tag{25}$$

and

$$\int_\Omega |x|^\alpha u_\alpha^p dx \geq \left(1 - \frac{2}{\alpha}\right)^\alpha \int_\Omega u_\alpha^p dx$$

$$= \left(1 - \frac{2}{\alpha}\right)^\alpha \alpha^{-N} \int_\Omega u^p dx\,. \tag{26}$$

Hence by definition, one obtain

$$S_{\alpha,p} \leq R(u_\alpha) \leq \frac{\alpha^{2-N} \int_\Omega |\nabla u|^2 dx}{(\alpha^{-N}(1 - \frac{2}{\alpha})^\alpha \int_\Omega u^p dx)^{2/p}}$$

$$\leq c\alpha^{2-N+\frac{2N}{p}}\,. \tag{27}$$

Since $1 + 2/p > 2 - N + (2N)/p$ whenever $p > 2$, it suffices to use Theorem 4.1. □

5. Analysis for p Close to 2

In Theorems 3.1 and 4.2, we proved the existence of a limiting α^* above which an infimum cannot be radial. At this point, α^* is only an upper bound, and it could be that no minimizer is radial whatever $\alpha > 0$ is. We will now prove that this is not the case, by showing that as p decreases to 2, the limit α^* goes to $+\infty$.

Let $(-\Delta)^{-1}$ denote the inverse of the Laplacian operator with Dirichlet boundary conditions. We define the operator

$$P : [2, 2^*) \times \mathbb{R}_+ \times H_0^1(\Omega) \times \mathbb{R} \longrightarrow H_0^1(\Omega) \times \mathbb{R}$$

$$(p, \alpha, u, \lambda) \longmapsto ((-\Delta)^{-1}(\lambda|x|^\alpha|u|^{p-2}u) - u, \|u\|^2 - 1)\,.$$

Notice that P is well defined,

$$u \in H_0^1 \Rightarrow |u|^{p-2}u \in L^{\frac{2^*}{p-1}} \Rightarrow (-\Delta)^{-1}(\lambda|x|^\alpha|u|^{p-2}u) \in W^{2,\frac{2^*}{p-1}} \cap W_0^{1,\frac{2^*}{p-1}} \subset H_0^1$$

since $(2^*/(p-1))^* > 2$ when $p < 2^*$.

Proposition 5.1. *Let $\alpha_0 > 0$ fixed. There exist $\varepsilon > 0$ and continuous functions*

$$\Lambda : [2, 2 + \varepsilon) \times (\alpha_0 - \varepsilon, \alpha_0 + \varepsilon) \to (S_{\alpha_0,2} - \varepsilon, S_{\alpha_0,2} + \varepsilon)$$
$$U : [2, 2 + \varepsilon) \times (\alpha_0 - \varepsilon, \alpha_0 + \varepsilon) \to B(u_{\alpha_0,2}, \varepsilon) \tag{28}$$

such that, in $[2, 2 + \varepsilon) \times (\alpha_0 - \varepsilon, \alpha_0 + \varepsilon) \times B(u_{\alpha_0,2}, \varepsilon) \times (S_{\alpha_0,2} - \varepsilon, S_{\alpha_0,2} + \varepsilon)$,

$$P(p, \alpha, u, \lambda) = 0 \Longleftrightarrow u = U(p, \alpha) \quad \text{and} \quad \lambda = \Lambda(p, \alpha)\,.$$

Proof. This will follow from the implicit function theorem if we prove that the partial derivative of P with respect to (u, λ) at $(2, \alpha_0, u_{\alpha_0,2}, S_{\alpha_0,2})$ is an homeomorphism on $H_0^1(\Omega) \times \mathbb{R}$. Clearly,

$$\partial P(2, \alpha_0, u_{\alpha_0,2}, S_{\alpha_0,2})(v, t)$$
$$= ((-\Delta)^{-1}(S_{\alpha_0,2}|x|^{\alpha_0}v) - v + t(-\Delta)^{-1}(|x|^{\alpha_0}u_{\alpha_0,2}), 2\langle u_{\alpha_0,2}, v\rangle)\,,$$

so that

$$\partial P(2, \alpha_0, u_{\alpha_0,2}, S_{\alpha_0,2})(v, t) = 0 \Longrightarrow \begin{cases} t = 0\,, \\ (-\Delta)^{-1}(S_{\alpha_0,2}|x|^{\alpha_0}v) - v = 0\,, \\ \langle u_{\alpha_0,2}, v\rangle = 0\,. \end{cases}$$

Since the kernel of $(-\Delta)^{-1}(S_{\alpha_0,2}|x|^{\alpha_0}I) - I$ is reduced to the multiples of $u_{\alpha_0,2}$, it follows that

$$\partial P(2, \alpha_0, u_{\alpha_0,2}, S_{\alpha_0,2})(v, t) = 0 \Longleftrightarrow (v, t) = 0\,.$$

Let $(w, s) \in H_0^1(\Omega) \times \mathbb{R}$. Since $(-\Delta)^{-1}(|x|^{\alpha_0}u_{\alpha_0,2})$ is proportional to $u_{\alpha_0,2}$, there exist $t \in \mathbb{R}$ such that $\tilde{w} := w - t(-\Delta)^{-1}(|x|^{\alpha_0}u_{\alpha_0,2}) \in (u_{\alpha_0,2})^\perp$. By self-adjointness, there exists $\tilde{v} \in H_0^1$ such that $(-\Delta)^{-1}(S_{\alpha_0,2}|x|^{\alpha_0}\tilde{v}) - \tilde{v} = \tilde{w}$. Choose $v := \tilde{v} + (s/2)u_{\alpha_0,2}$. Then,

$$\partial P(2, \alpha_0, u_{\alpha_0,2}, S_{\alpha_0,2})(v, t) = (w, s)\,.$$

Being continuous and bijective, the partial derivative of P is a homeomorphism. This ends the proof. □

Theorem 5.1. *For each $n \in \mathbb{N}$, there exists $\delta_n > 0$ such that the unique minimizer of $S_{\alpha,p}$ is radial provided $\alpha \leq n$ and $p \leq 2 + \delta_n$.*

Proof. By contradiction assume that there exist $n \in \mathbb{N}$ and sequences $(\delta_k) \to 0_+$, $(\alpha_k) \subset [0, n]$ such that a minimizer u_k for $S_{\alpha_k,2+\delta_k}$ is non radial. Without loss of generality, we can assume that $\alpha_k \to \alpha_\infty \leq n$ and $u_k \rightharpoonup u_\infty \in H_0^1$. One has,

$$\int_\Omega |x|^{\alpha_k} u_k^{2+\delta_k} \leq \left(\int_\Omega |x|^{\alpha_k} u_k^2\right)^{\frac{2+\delta_k}{2}\sigma} \left(\int_\Omega u_k^{2^*}\right)^{\frac{2+\delta_k}{2^*}(1-\sigma)}$$

$$\leq C\left(\int_\Omega |x|^{\alpha_k} u_k^2\right)^{\frac{2+\delta_k}{2}\sigma} \|u_k\|^{(2+\delta_k)(1-\sigma)}$$

where $\sigma := N(\frac{1}{2+\delta_k} - \frac{1}{2^*}) \to 1$ as $k \to +\infty$. From this, and the Rellich theorem, we infer that

$$\int_\Omega |x|^{\alpha_\infty} u_\infty^2 dx \geq 1\,,$$

so that in particular $u_\infty \neq 0$. On the other hand,

$$\int_\Omega |\nabla u_\infty|^2 dx \leq \liminf \int_\Omega |\nabla u_k|^2 dx = \liminf S_{\alpha_k,2+\delta_k}\,;$$

which implies that $S_{\alpha_\infty,2} \le \liminf S_{\alpha_k,2+\delta_k}$. By upper semi-continuity, it follows that $S_{\alpha_\infty,2} = \lim S_{\alpha_k,2+\delta_k}$ and by uniform convexity, that $u_k \to u_\infty = u_{\alpha_\infty,2}$ in H_0^1. By the preceding proposition, u_k is unique when k is large, and hence radial, which is a contradiction. □

6. Analysis for p Close to 2^*

In this section, we analyse the case where p is close to 2^*. We will show that for any fixed $\alpha > 0$, the minimizer of R is radial provided $2^* - p$ is sufficiently small.

Lemma 6.1. *If $N \ge 3$, there exists $c_0 > 0$ such that for every $2 \le p \le 2^*$ and for every $\alpha \ge 0$,*

$$c_0 \alpha^{2/p} \le S^R_{\alpha,p}.$$

Proof. It follows from the Ni inequality (see [11]), if $u \in H^1_{0,\mathrm{rad}}(\Omega)$,

$$|u(r)|^p r^{\alpha+N-1} \le c_1 |\nabla u|_2^p r^{\frac{2-N}{2}p+\alpha+N-1}.$$

We obtain

$$\int_\Omega |x|^\alpha |u|^p dx \le c_2 |\nabla u|_2^p \left(\frac{2-N}{2}p + \alpha + N\right)^{-1},$$

or

$$c_0 \left(\frac{2-N}{2}p + \alpha + N\right)^{2/p} \le \frac{\int_\Omega |\nabla u|^2 dx}{(\int_\Omega |x|^\alpha |u|^p dx)^{2/p}}. \tag{29}$$

Since $u \in H^1_{0,\mathrm{rad}}(\Omega)$ is arbitrary,

$$c_0 \left(\frac{2-N}{2}p + \alpha + N\right)^{2/p} \le S^R_{\alpha,p},$$

which ends the proof. □

Let us denote by S the classical Sobolev constant,

$$S = \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx}{(\int_\Omega |u|^{2^*} dx)^{2/2^*}}.$$

It is standard that this Rayleigh quotient is invariant under translations and dilations.

Lemma 6.2. *If $N \ge 3$ and $\alpha > 0$, then*

$$S = S_{\alpha,2^*} < S^R_{\alpha,2^*}.$$

Proof. Using the Ni inequality, it is easy to verify that $S^R_{\alpha,2^*}$ is achieved, so that $S < S^R_{\alpha,2^*}$. By invariance, using a minimizing sequence for S in $\mathcal{D}(\Omega)$ concentrating at $y \in \Omega$, we obtain

$$S \le S_{\alpha,2^*} \le |y|^{-2\alpha/2^*} S.$$

Since $y \in \Omega$ is arbitrary, the proof is complete. □

We can now state the counterpart of Theorem 5.1.

Theorem 6.1. *Assume $N \ge 3$. For any $n \in \mathbb{N}$ there exists $\delta_n > 0$ such that $S_{\alpha,p} < S^R_{\alpha,p}$ provided $\alpha \ge 1/n$ and $2^* - \delta_n < p < 2^*$.*

Proof. By contradiction, assume that there exists $n \in \mathbb{N}$ and sequences $\alpha_k \ge 1/n$ and $\delta_k \to 0$ such that

$$S_{\alpha_k,2^*-\delta_k} = S^R_{\alpha_k,2^*-\delta_k}. \tag{30}$$

By (27), there exists c_3 independent of $2 \le p \le 2^*$, such that

$$S_{\alpha,p} \le c_3 \alpha^{2-N+2N/p}.$$

Lemma 6.1 implies that

$$c_0 \alpha^{2/p} \le S^R_{\alpha,p}.$$

It is then clear that α_k is bounded. We can assume that $\alpha_k \to \alpha \ge 1/n$.

As in the proof of Theorem 5.1,

$$S^R_{\alpha,2^*} = \lim_{k\to+\infty} S^R_{\alpha_k,2^*-\delta_k}. \tag{31}$$

On the other hand, by upper continuity,

$$S_{\alpha,2^*} \ge \limsup_{k\to+\infty} S_{\alpha_k,2^*-\delta_k}. \tag{32}$$

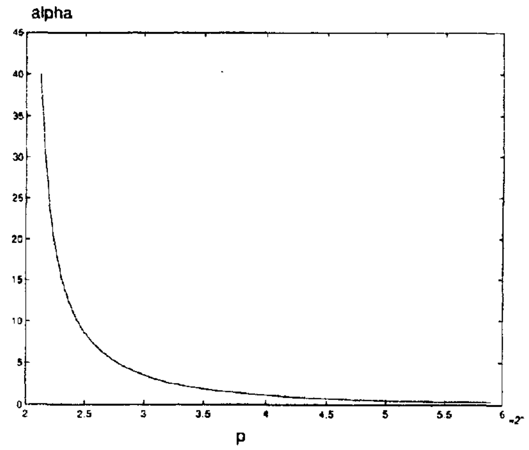
We obtain, from (30), (31), (32), $S_{\alpha,2^*} \ge S^R_{\alpha,2^*}$. But this contradicts Lemma 6.2. □

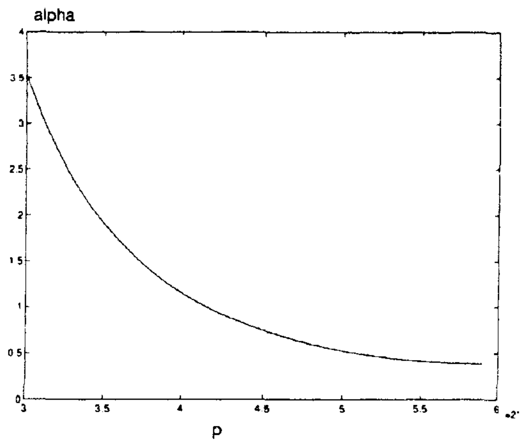
7. Numerics for the Threshold Value of α

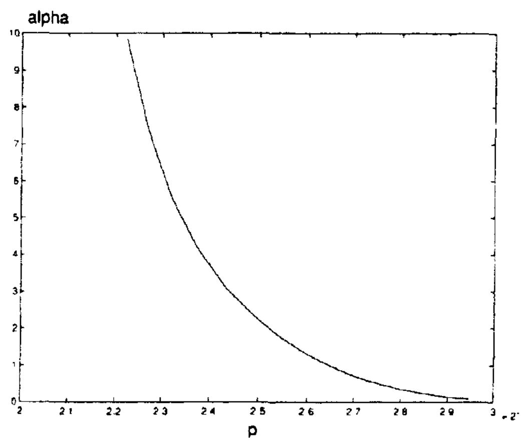
In this section, we briefly present some numerical computations concerning the frontier between radialicity and symmetry breaking. More precisely, the curves in the figures below correspond the limiting α's above which the necessary condition of Theorem 2.1 fails to be satisfied. To compute these curves, we have implemented an algorithm converging, for fixed α and p, to the radial minimizer for $S^R_{\alpha,p}$. Then, p remaining fixed, we determine which value of α yields an equality in the necessary condition, by using a modified secant method. Given the time to compute one minimizer, it is important to make use of the already computed points and minimizers to provide the algorithm with accurate initial guesses.

The first figure here below correspond to $N = 3$, the second is just a zoom of the first one for $p \ge 3$, and the third correspond to $N = 6$.

Notice that, as one would expect, symmetry breaking occurs later in higher dimension.







8. The Case of the q-Laplacian

The necessary condition of Sec. 2 and the symmetry breaking result of Sec. 3 can be generalized in the framework of the case q-laplacian. More precisely, define the minimization problem

$$\min\left\{\int_\Omega |\nabla u|^q dx,\ u \in W_0^{1,q}(\Omega),\ \int_\Omega \rho(x)|u|^p dx = 1\right\}, \tag{33}$$

where $q > 1$ and $q < p < q^* = qN/(N-q)$. Then,

Theorem 8.1. *If ρ is a radial function then any radial local minimizer u of* (33) *satisfies*

$$\int_\Omega |\nabla u|^q dx \le \left(\frac{(N-1)(q-1)}{p-q}\right)^{q/2} \int_\Omega \frac{u^q}{|x|^q} dx\,.$$

Proof. The proof follows the same lines as the one of Theorem 2.1. The equivalent of equation (10) is

$$\int_\Omega |\nabla u|^q dx \left[(q-p)\left(\int_\Omega \rho(x)|u|^{p-2} uh dx\right)^2 + (p-1)\int_\Omega \rho(x)|u|^{p-2}h^2 dx\right]$$
$$\le (q-1)\int_\Omega |\nabla u|^{q-2}|\nabla h|^2 dx \tag{34}$$

from which it follows, with the same h as in Theorem 2.1, that

$$(p-q)\int_\Omega |\nabla u|^q dx \le (q-1)(N-1)\int_\Omega \frac{u^2}{|x|^2}|\nabla u|^{q-2} dx\,. \tag{35}$$

The conclusion comes from the Hölder inequality. □

In the case of the modified Hénon equation, we now have

Theorem 8.2. *Assume $N > q$ and $\rho(x) = |x|^\alpha$ for some $\alpha > 0$. Then, for any $q < p < q^*$, there exists $\alpha^* > 0$ such that no minimizer of* (33) *is radial provided $\alpha > \alpha^*$.*

Proof. The proof of Proposition 3.1 is easily adapted to the present setting. The conclusion follows from the previous theorem. □

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After this work was completed, we were mentioned a result by Ding and Ni [5] where symmetry breaking is proved for a semilinear equation on expanding balls. The situation is however somewhat different since the weight in front of the nonlinearity is unbounded, leaving more chances for symmetry breaking.

References

[1] H. Brezis, *Symmetry in nonlinear PDE's*, Proc. Sympos. Pure Math. **65** (1999) 1–12.

[2] H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. **36** (1983) 437–477.

[3] G. Chen, W.-M. Ni and J. Zhou, *Algorithms and visualization for solutions of nonlinear elliptic equations*, Internat. J. Bifur. Chaos **10** (2000) 1565–1612.

[4] C. V. Coffman, *A non-linear boundary value problem with many positive solutions*, J. Differential Equations **54** (1984) 429–437.

[5] W. Y. Ding and W.-N. Ni, *On the existence of positive entire solutions of a semilinear elliptic equation*, Arch. Rational Mech. Anal. **91** (1986) 283–308.

[6] B. Gidas, W.-M. Ni and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979) 209–243.

[7] M. Hénon, *Numerical experiments on the stability of spherical stellar systems*, Astronomy and Astrophysics **24** (1973) 229–238.

[8] H. Hochstadt, *Les Fonctions de la Physique Mathematique*, Masson et Cie, 1973.

[9] B. Kawohl, *Rearrangements and Convexity of Level Sets in PDE*, Lecture Notes in Mathematics **1150**, Springer-Verlag, Berlin, 1985.

[10] B. Kawohl, *Symmetry results for functions yielding best constants in Sobolev-type inequalities*, Discrete Contin. Dynam. Systems **6** (2000) 683–690.

[11] W.-M. Ni, *A nonlinear Dirichlet problem on the unit ball and its applications*, Indiana Univ. Math. J. **31** (1982) 801–807.

[12] M. Willem, *Minimax Theorems*, Birkhäuser, Boston, 1996.