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Conformal finiteness

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CONFORMAL FINITENESS

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This is a lectures note for lectures delivered at ICTP for the workshop on the development in nonlinear partial differential equations and their applications in differential geometry.

1. INTRODUCTION

How do we understand a closed surface (M, g)? Take an isothermal coordinate chart

$$(\Omega, e^{2u}|dx|^2) \subset (M, g),$$

then compute

$$K = \frac{-\triangle u}{e^{2u}}.$$

By Gauss-Bonnet formula, one knows the Euler number

$$\chi(M) = \frac{1}{2\pi}\int_M K dV_g. \tag{1.1}$$

That gives us a pretty good idea what the surface looks like at least topologically. Because we know that all closed surfaces topologically are just

$$S^2, \quad S^1 \times S^1, \quad S^1 \times S^1 \sharp \ldots \sharp S^1 \times S^1 \quad \cdots$$

and, if denote by g the number of copies of $S^1 \times S^1$ in M, the Euler number $\chi(M) = 2 - 2g$.

How do we understand a complete open surface (M, g)? It turns out that one better considers complete surfaces satisfying

$$\int_M |K| dV_g < \infty, \tag{1.2}$$

in other words, a complete surface with finite total curvature. Cohn-Vossen [CV] showed that a complete surface M with analytic metric and finite total curvature satisfies

$$\int_M K dA \leq 2\pi\chi(M). \tag{1.3}$$

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One should realize that the underlined issue here is finiteness in topology. Huber [H] later extended this inequality to metrics with weaker regularity. More importantly, Huber proved that such surface M is conformally equivalent to a closed surface with finitely many punctures. We will call each noncompact component associated with a puncture by an end for M. The deficit in formula (1.3) has an interpretation as an isoperimetric constant. For a complete surface with finite total curvature, one may represent each end conformally as $R^2 \setminus K$ for some compact set K and consider the following isoperimetric ratio:

$$\nu = \lim_{r \to \infty} \frac{L^2(r)}{4\pi A(r)} \tag{1.4}$$

where $L(r)$ is the length of the boundary circle $\partial B_r = \{|x| = r\}$ and $A(r)$ the area of the annular region $B(r) \setminus K$. For a fairly large class of complete surfaces Finn [F] showed that,

$$\chi(M) - \frac{1}{2\pi} \int_M K dv_M = \sum \nu_j, \tag{1.5}$$

where the sum is taken over each end of M.

what an analogue of the surface theory exists in higher dimension? To start, for a closed LCF 4-manifold with positive scalar curvature, one knows, by a result of Hamilton [Ha], that they are all diffeomorphic to

$$S^4, \quad S^1 \times S^3, \quad S^1 \times S^3 \sharp \ldots \sharp S^1 \times S^3 \quad \cdots \tag{1.6}$$

if we neglect any torsion in the fundamental group. Therefore, again, Euler number $\chi(M)$ is $2 - 2g$, here g is the number of copies of $S^1 \times S^3$. Moreover, in the light of locally conformal flatness, one can take a local conformal coordinate chart

$$(\Omega, e^{2u}|dx|^2) \subset (M, g),$$

and compute

$$Q = \frac{\Delta^2 u}{e^{4u}}. \tag{1.7}$$

By Chern-Gauss-Bonnet formula,

$$\chi(M) = \frac{1}{8\pi^2} \int_M Q dV_g. \tag{1.8}$$

Thus we have a pretty good grip on closed LCF 4-manifolds with positive scalar curvature. Now the question is, what are the complete LCF 4-manifolds with positive scalar curvature that correspond to complete surfaces of a finite number of ends? In this note we will introduce and develop our approach to give a rather complete answer to that question. The organization is: in the first lecture we study the asymptotic behavior of a simple end and show that such end is conformally equivalent to a puncture under certain condition similar to that the total curvature is finite in two dimension. Then in the second lecture we will try to give a full generalization of surface theory mentioned in the above in four dimension. When the fundamental group of the underlined LCF 4-manifold with nonnegative scalar curvature is nontrivial, it is a Kleinian group through the holonomy representation. It turns out that the finiteness of the holonomy representation of the fundamental group plays an important role in our understanding of conformal finiteness. We therefore will introduce a notion of conformal finiteness for a Kleinian group and show that conformal finiteness is equivalent to the finiteness defined in the study of hyperbolic spaces.

The materials covered in this note are mostly from a sequence of joint works with Alice Chang and Paul Yang [CQY1] [CQY2] [CQY3]. The reason that this note is entitled as conformal finiteness is to encourage to consider our works rather a start of the study of conformal finiteness in general.

2. On a simple end

Our motivations are twofold. Firstly we are searching for the analogue of the above mentioned result of Finn (1.5). Secondly we would like to find a geometric interpretation of the fourth order curvature Q. Here we take the initial analytic step and study complete conformal metrics $e^{2u}|dx|^2$ on R^4.

2.1 Spherically symmetric cases

To start we consider a metric $g = e^{2u}|dx|^2$ where u is a radial function, i.e. $u(x) = u(|x|)$. For convenience we use the cylindrical coordinates to rewrite the equation (1.7) as

$$(\frac{\partial^2}{\partial t^2} - 2\frac{\partial}{\partial t})(\frac{\partial^2}{\partial t^2} + 2\frac{\partial}{\partial t})w = Qe^{4w} \tag{2.1}$$

where $|dx|^2 = e^{2t}(dt^2 + g_{S^3})$, $w = u + t$ and $t = \log|x|$. Let us define the isoperimetric ratio as

$$c_{3,4}(s) == \frac{v_3^{\frac{4}{3}}(s)}{(\frac{1}{4}\mathrm{vol}(S^3))^{\frac{1}{3}} v_4(s)} \tag{2.2}$$

where

$$v_4(s) = \mathrm{vol}(\{t < s\}) = \mathrm{vol}(S^3)\int_{-\infty}^{s} e^{4w} dt$$

and

$$v_3(s) = \frac{1}{4}\mathrm{vol}(\{t = s\}) = \frac{1}{4}\mathrm{vol}(S^3)e^{3w(s)}.$$

When both $v_4(s)$ and $v_3(s)$ tend to infinity as s tends to infinity, we have, via the L'Hospital's rule,

$$\lim_{s\longrightarrow\infty} c_{3,4}(s) = \lim_{s\longrightarrow\infty} w'(s). \tag{2.3}$$

Now let us digress to discuss Chern-Gauss-Bonnet formula for 4-manifolds with boundary before we continue. Let us begin with Gauss-Bonnet for surfaces (M^2, g) with boundary

$$\chi(M^2) = \frac{1}{2\pi}\int_M K dv_g + \frac{1}{2\pi}\int_{\partial M} k d\sigma_g. \tag{2.4}$$

We know that Q curvature is an analogue of Gaussian curvature in dimension 4 in Chern-Gauss-Bonnet (1.8), what is the analogue of the geodesic curvature k in (2.4) in dimension 4? It was discovered in [CQ] that, for a 4-manifold (M, g) with boundary, there indeed is a curvature

$$T = \frac{1}{2}\frac{\partial J}{\partial n} + JH - G \cdot L + \frac{1}{3}H^3 - \mathrm{Tr}L^3 + \frac{1}{3}\tilde{\Delta}H, \tag{2.5}$$

where $J = \frac{1}{6}R$, R is the scalar curvature, L is the second fundamental form of ∂M in M, H is the trace of L, $G_{\alpha\beta} = R_{\alpha n\beta n}$ is a part of the Riemannian curvature tensor, n is the outgoing normal direction and $\tilde{\Delta}$ is the Laplacian on the boundary ∂M. Moreover we also have a third order operator P_3 on $(M, \partial M)$ such that

$$P_3[g]\phi + T[g] = T[e^{2\phi}g]e^{3\phi} \tag{2.6}$$

and

$$\chi(M) = \frac{1}{8\pi^2}\int_M (\frac{1}{4}|\mathcal{W}|^2 + Q)dv_g + \frac{1}{4\pi^2}\int_{\partial M}(\mathcal{L} + T)d\sigma_g \tag{2.7}$$

where $\mathcal{W}$ is the Weyl curvature and $\mathcal{L}$ is a boundary curvature that is point-wisely conformally invariant just like $\mathcal{W}$ and measures the umbilicality of ∂M.

Back to our discussion of the metric $e^{2u}|dx|^2$ on R^4, applying formula (2.7) we have

$$8\pi^2\chi(\{t < s\}) - \int_{t\leq s} Qe^{4w}dx = 2\int_{s=t} Te^{3w}dy,$$

where, by (2.6) with the background metric as the Euclidean cylindric coordinates, we have

$$Te^{3w} = P_3 w = -\frac{1}{2}w''' - w'' + 2w'. \tag{2.8}$$

So

$$\chi(\{t < s\}) - \frac{1}{8\pi^2}\int_{t<s} Qe^{4w}dx = w'(s) - \frac{1}{4}(w'''(s) + 2w''(s)). \tag{2.9}$$

Notice that $\mathrm{vol}(S^3) = 2\pi^2$. We would like to show that, under suitable conditions,

$$\lim_{s\to\infty} w'''(s) = \lim_{s\to\infty} w''(s) = 0 \tag{2.10}$$

and (2.3) holds in general. Thus we turn to study the behavior of w. For convenience, we denote Qe^{4w} by F. Equation (2.1) is equivalent to the following ODE

$$v'''' - 4v'' = F, \quad -\infty < t < \infty \tag{2.11}$$

where $\int_{-\infty}^{\infty} |F| dt < \infty$. Our strategy here is to have special solution v such that

$$\begin{aligned} v'(t) = & \frac{1}{8}\{e^{-2t} \int_{-\infty}^{t} F(x)e^{2x}dx - e^{2t} \int_{t}^{\infty} F(x)e^{-2x}dx \\ & + (K_1 - K_2) + \int_{t}^{\infty} F(x)dx - \int_{-\infty}^{t} F(x)dx\} \end{aligned} \tag{2.12}$$

where

$$K_1 = \lim_{t\to-\infty} e^{2t} \int_{t}^{\infty} F(x)e^{-2x}dx, \quad K_2 = \lim_{t\to\infty} e^{-2t} \int_{-\infty}^{t} F(x)e^{2x}dx. \tag{2.13}$$

Then let

$$w(t) = c_0 + c_1 t + c_2 e^{-2t} + c_3 e^{2t} + v(t)$$

for some constants c_0, c_1, c_2, c_3. Then we can fix those coefficients under suitable conditions. For example, we easily prove

Claim A. $K_1 = K_2 = 0$.

Proof.

$$\begin{aligned} & e^{-2t} \int_{-\infty}^{t} F(x)e^{2x}dx \\ = & e^{-2t}\{\int_{-\infty}^{T} F(x)e^{2x}dx + \int_{T}^{t} F(x)e^{2x}dx\} \\ \le & e^{-2(t-T)} \int_{-\infty}^{\infty} |F(x)|dx + \int_{T}^{\infty} |F(x)|dx \end{aligned}$$

So $K_2 = 0$, if take $T = \frac{1}{2}t$ for instance. On the other hand,

$$\begin{aligned} & e^{2t} \int_{t}^{\infty} F(x)e^{-2x}dx \\ = & e^{2s}\{\int_{t}^{T} F(x)e^{-2x}dx + \int_{T}^{\infty} F(x)e^{-2x}dx\} \\ \le & \int_{-\infty}^{T} |F(x)|dx + e^{2(t-T)} \int_{-\infty}^{\infty} |F(x)|dx \end{aligned}$$

Similarly $K_1 = 0$, again, take $T = \frac{1}{2}t$.

Claim B. $c_2 = 0$.

Proof. First from (2.21), one sees that $v''(t)$ is of lower order than e^{-2t} as $t \to -\infty$. Secondly,

$$w''(t) = u''_{rr} r^2 + u'_r r$$

is also of lower order than e^{-2t} as $t \to -\infty$. Therefore it is easily seen that $c_2 = 0$.

Claim C. *$v''(t) \to 0$ and $v'''(t) \to 0$, as $t \to \infty$.*

Proof. This is a consequence of Fact A.

Claim D.

$$\lim_{t \to \infty} v'(t) = -\frac{1}{8} \int_{-\infty}^{\infty} F(x) dx$$

and

$$\lim_{t \to -\infty} v'(t) = \frac{1}{8} \int_{-\infty}^{\infty} F(x) dx.$$

Proof. Again, these are consequences of Fact A.

Therefore

Claim E. $c_1 = 1 - \frac{1}{8} \int_{-\infty}^{\infty} F(x) dx$.

Proof. Since $v'(t) \to 1$ as $t \to -\infty$.

Let us study a simple example before to handle the e^{2t} term.

Example. *Consider $v = e^{2t}$. Then $Q = 0$, and the metric $e^{2(v-t)} g_0$ is complete at infinity. Moreover it can be easily smoothed out at the origin. What we see from this example is that we can't expect the term e^{2t} dropped without any additional condition.*

Claim F. *Suppose, in addition, that the scalar curvature is nonnegative at infinity. Then $c_3 = 0$.*

Proof. This is a consequence of the transform formula for the scalar curvature. Let us denote by R the scalar curvature. Then on the cylinder we have

$$-v'' - (v')^2 + 1 = \frac{1}{6} R e^{2v}.$$

Then from what we know about w, it is easily seen that c_3 has to be zero.

Another way to eliminate the e^{2t} term is to assume some growth condition on Q or v. In summary we have

Theorem 2.1. *Suppose that u is a radial function on R^4, and that $e^{2u}|dx|^2$ is a complete metric satisfying*

$$\int_M |Q|e^{4u}dx < \infty$$

and scalar curvature nonnegative at infinity. Then

$$\chi(R^4) - \frac{1}{8\pi^2}\int_{R^4} Qe^{4u}dx = \lim_{t\to\infty} w'(t) \geq 0. \tag{2.15}$$

Therefore

$$\chi(R^4) - \frac{1}{8\pi^2}\int_{R^4} Qe^{4u}dx = c_{3,4} \geq 0. \tag{2.16}$$

2.2 Normal metrics

Next we consider $(R^4, e^{2u}|dx|^2)$ in general. For this purpose we introduce a notion of normal metrics as Robert Finn did in dimension 2. We will always assume that

$$\int_{R^4} |Q|e^{4u}dx < \infty \tag{2.16}$$

where $Q = e^{-4u}\Delta^2 u$.

Definition 2.2. *A conformal metric $e^{2u}|dx|^2$ satisfying (2.16) is said to be normal if*

$$u(x) = \frac{1}{8\pi^2}\int_{R^4} \log\frac{|y|}{|x-y|}Q(y)e^{4u(y)}dy + C \tag{2.17}$$

for some constant

Clearly the expression on the right side of (2.17) is one solution to

$$\Delta^2 u = Qe^{4u}. \tag{2.18}$$

But, there are many other rather nasty solutions to (2.18). In other words, to be normal the asymptotic behavior of a metric is rather restricted. As before, one defines, for $(R^4, e^{2u}|dx|^2)$,

$$v_4(r) = \int_{B_r} e^{4u}dx, \tag{2.19}$$

$$v_3(r) = \frac{1}{4}\int_{\partial B_r} e^{3u}d\sigma(x), \tag{2.20}$$

and $c_{3,4}$ as in (2.2). In the following, we will adopt some techniques used by Robert Finn in [F] to compare $v_4(r), v_3(r)$ with $\bar{v}_4(r), \bar{v}_3(r)$: the volumes for the metric $e^{2\bar{u}}|dx|^2$ which may be considered to be the average of the metric $e^{2u}|dx|^2$, where $\bar{u}$ is defined as:

$$\bar{u} = \frac{1}{\mathrm{vol}(\partial B_r)}\int_{\partial B_r} ud\sigma(x).$$

Namely,

Lemma 2.3. *Suppose that the metric $e^{2u}|dx|^2$ on R^4 is a normal metric. Then, for any number $k > 0$,*

$$\frac{1}{|\partial B_r|}\int_{\partial B_r} e^{ku} d\sigma(x) = e^{k\bar{u}} e^{o(1)}, \tag{2.21}$$

where $o(1) \to 0$ as $r \to \infty$.

The proof of this lemma is very technical and heavily relies on the fact that $e^{2u}|dx|^2$ is normal. As a direct consequence we have

Corollary 2.4. *Suppose that the metric $e^{2u}|dx|^2$ on R^4 is a normal metric. Then*

$$v_3(r) = \bar{v}_3(r)(1 + o(1)), \tag{2.22}$$

and

$$\frac{d}{dr}V_4(r) = (\frac{d}{dr}\bar{V}_4(r))(1 + o(1)) \tag{2.23}$$

We now study the metric which is an average of the metric $e^{2u}|dx|^2$ on R^4 over the spheres ∂B_r. For convenience, we would like to use the cylindrical coordinates again. Then $\bar{u}(r) = \bar{u}(e^t)$, and $w(t) = \bar{u}(r) + t$. As we have seen in Section 2.1, w satisfies

$$v'''' - 4v'' = \frac{1}{\mathrm{vol}(\partial B_r(0))}\int_{\partial B_r(0)} Qe^{4u} d\sigma(x) = F(t), \quad -\infty < t < \infty \tag{2.24}$$

with

$$\int_{-\infty}^{\infty} F(t)dt = \frac{1}{\mathrm{vol}(S^3)}\int_{R^4} Qe^{4u}dx = \frac{1}{2\pi^2}\int_{R^4} Qe^{4u}dx \tag{2.25}$$

and

$$\int_{-\infty}^{\infty} |F(t)|dt \leq \frac{1}{2\pi^2}\int_{R^4} |Q|e^{4u}dx < \infty \tag{2.26}$$

Lemma 2.5. *Suppose that $(R^4, e^{2u}|dx|^2)$ is a complete normal metric. Then its averaged metric $(R^4, e^{2\bar{u}(r)}|dx|^2)$ is also a complete metric.*

This is basically a consequence of Lemma 2.3. Because

$$\frac{1}{\mathrm{vol}(S^3)}\int_{S^3} e^{u(r\sigma)} d\sigma = e^{\bar{u}(r)} \cdot e^{o(1)}, \tag{2.27}$$

where $o(1) \to 0$ as $r \to \infty$. On the other hand, recall that in Section 2.1 the only reason we need to assume the sign of the scalar curvature in Theorem 2.1 is because we wanted to get rid of the e^{2t} term (2.14). In fact, it is even easier seen that $w''(t) = O(1)$ will also allow us to get rid of the e^{2t} term in (2.14). Fortunately for a normal metric we have

Lemma 2.6. *Suppose that* $(R^4, e^{2u}|dx|^2)$ *is a normal metric. Then*

$$\Delta\bar{u}(r) \le \frac{C}{r^2}. \tag{2.28}$$

Thus we have

Theorem 2.7. *Suppose that* $(R^4, e^{2u}|dx|^2)$ *is a complete normal metric. Then*

$$\chi(R^4) - \frac{1}{8\pi^2}\int_{R^4} Qe^{4u}dx = c_{3,4} \ge 0. \tag{2.29}$$

For the proof of this result, as we indicated previously, we first consider the average metric $e^{2\bar{u}}|dx|^2$, and then apply Lemma 2.3 and Corollary 2.4 to establish (2.29) through (2.16) in Theorem 2.1 in Section 2.1.

2.3 Nonnegative scalar curvature

Let us first look at an example: $u = r^2$, in cylindrical coordinate, $w = e^{2t} + t$ and its curvature $Q = 0$. But clearly this metric $r^2|dx|^2$ is not a normal metric, although it is complete. Therefore one does not expect any complete metric $(R^4, e^{2u}|dx|^2)$ is normal. But we have

Theorem 2.8. *Suppose that* $(R^4, e^{2u}|dx|^2)$ *is complete and with scalar curvature nonnegative at infinity. And suppose that*

$$\int_{R^4} |Q|e^{4u}dx < \infty.$$

Then it is a normal metric.

Proof. The proof of this fact is rather interesting and simple. So let me include it here. First let

$$v(x) = \frac{3}{4\pi^2}\int_{R^4} \log\frac{|y|}{|x-y|}Qe^{4u}dy \tag{2.30}$$

and $w = u - v$. Then we want to show that the biharmonic function w on R^4 has to be a constant. Recall the transform formula for scalar curvature

$$\Delta u + |\nabla u|^2 = -Je^{2u} \tag{2.31}$$

where $6J$ is the scalar curvature for the metric $(R^4, e^{2u}|dx|^2)$. Notice that Δw is a harmonic function. Thus, by mean value property of harmonic functions,

$$\begin{aligned} \Delta w(x_0) &= \frac{1}{|\partial B_r(x_0)|}\int_{\partial B_r(x_0)} \Delta w d\sigma \\ &= -\frac{1}{|\partial B_r(x_0)|}\int_{\partial B_r(x_0)} (|\nabla u|^2 + J)d\sigma - \frac{1}{|\partial B_r(x_0)|}\int_{\partial B_r(x_0)} \Delta v d\sigma. \end{aligned} \tag{2.32}$$

The first term on the right of (2.32) is nonpositive whenever r is large enough by our assumption that J is nonnegative. We now observe that, we have

$$\begin{aligned}\int_{\partial B_r(x_0)} \Delta v d\sigma =& \frac{3}{2\pi^2}\int_{R^4}\{\frac{1}{|S^3|}\int_{S^3}\frac{1}{|r\sigma + x_0 - y|^2}d\sigma\}Q(y)e^{4u(y)}dy. \\ \le& \frac{3}{2\pi^2 r^2}\int_{R^4}|Q|e^{4u}dy.\end{aligned} \tag{2.33}$$

Therefore, taking $r \to \infty$, we have, for each $x_0 \in R^4$,

$$\Delta w(x_0) \le 0. \tag{2.34}$$

Thus $\Delta w = C_0$ for some nonpositive constant by Liouville Theorem for harmonic functions. Thus, any partial derivative of w is harmonic, i.e.

$$\Delta w_{x_i} = 0. \tag{2.35}$$

By the mean value property again,

$$|w_{x_i}(x_0)|^2 = |\frac{1}{|\partial B_r(x_0)|}\int_{\partial B_r(x_0)} w_{x_i} d\sigma|^2 \le \frac{1}{|\partial B_r(x_0)|}\int_{\partial B_r(x_0)}|\nabla w|^2 d\sigma.$$

But

$$|\nabla w|^2 \le 2|\nabla u|^2 + 2|\nabla v|^2 = -2C_0 - 2Je^{2u} + 2|\nabla v|^2$$

and

$$|\nabla v|^2 \le C(\int_{R^4}\frac{1}{|x-y|^2}|Q|e^{4u}dy)(\int_{R^4}|Q|e^{4u}dy) \le C\int_{R^4}\frac{1}{|x-y|^2}|Q|e^{4u}dy.$$

Similarly, we conclude that, for each $x_0 \in R^4$,

$$|w_{x_i}(x_0)|^2 \le -2C_0, \tag{2.36}$$

which implies that all partial derivatives of w are constants. Then $\Delta w = C_0 = 0$, which finally implies that all partial derivatives of w vanish by (2.36). Thus w is a constant.

Theorem 2.9. *Suppose that $(R^4, e^{2u}|dx|^2)$ is a complete metric with its scalar curvature nonnegative at infinity. And*

$$\int_{R^4}|Q|e^{4u}dx < \infty.$$

Then

$$\chi(R^4) - \frac{1}{8\pi^2}\int_{R^4} Qe^{4u}dx = c_{3,4} \ge 0. \tag{2.37}$$

Proof. Simply because the metric has to be a normal metric by the above Theorem 2.8. Then this theorem follows from Theorem 2.7.

3. Conformal compactification

We will first generalize our previous results to a conformally flat simple end of a 4-manifold. Then we will study when an end of a 4-manifold is conformally flat and simple. By a conformally flat simple end we mean that the end is conformally equivalent to a puncture in R^4. Therefore to show a complete 4-manifold has only conformally flat simple ends is to say that the manifold can be conformally compactified by adding points.

3.1 Chern-Gauss-Bonnet integrals

Suppose that (M^4, g) is a complete 4-manifold with only finitely many conformally flat simple ends. We would like to establish the Chern-Gauss-Bonnet formula on M, which will be a full generalization of (1.5) of Finn's works in dimension 2. We first define what is a conformally flat simple end.

Definition 3.1. *Suppose that (M, g) is a complete noncompact 4-manifold, and that E is a connected component of noncompact part of M. We say E is a conformally flat simple end if*

$$(E, g) = (R^4 \setminus B, e^{2w}|dx|^2),$$

for some function w, where B is the unit ball in R^4.

To establish a Chern-Gauss-Bonnet formula for a manifold with only finitely many conformally flat simple ends we first generalize our previous result in Chapter 2. Namely,

Lemma 3.2. *Suppose that w is a radial function on $R^4 \setminus B$ and $e^{2w}|dx|^2$ is a metric complete at infinity with $\int_{R^4\setminus B} |Q|e^{4w} < \infty$ and its scalar curvature nonnegative at infinity. Then*

$$\lim_{t\to\infty} v'(t) = \frac{1}{4\pi^2}\int_{\partial B} Te^{3w} - \frac{1}{8\pi^2}\int_{R^4\setminus B} Qe^{4w} \geq 0. \tag{3.1}$$

Moreover, we have

$$\lim_{r\to\infty} \frac{(vol(\partial B_r))^{\frac{4}{3}}}{4(2\pi^2)^{\frac{1}{3}} vol(B_r \setminus B)} = \frac{1}{4\pi^2}\int_{\partial B} Te^{3w} - \frac{1}{8\pi^2}\int_{R^4\setminus B} Qe^{4w}. \tag{3.2}$$

This basically follows from our discussions in Chapter 2. Next we modify Definition 2.2 as follows:

Definition 3.3. *A conformal metric $e^{2w}|dx|^2$ satisfying condition*

$$\int_{R^4\setminus B} |Q|e^{4w} dx < \infty \tag{3.3}$$

is said to be normal on $E = (R^4 \setminus B, e^{2w}|dx|^2)$ *if*

$$w(x) = \frac{1}{8\pi^2}\int_{R^4\setminus B} \log\frac{|y|}{|x-y|}Q(y)e^{4w(y)}dy + \alpha\log|x| + h(x) \tag{3.4}$$

where α *is some constant and* $\hat{h}(x) = h(\frac{x}{|x|^2})$ *is some biharmonic function on* B.

Similar to Lemma 2.3, Lemma 2.5, Lemma 2.6 we have

Lemma 3.4. *Suppose that the metric* $e^{2w}|dx|^2$ *on* $R^4\setminus B$ *is a normal metric. Then*

$$V_3(r) = |\partial B_r(0)|e^{3\bar{w}(r)} \cdot e^{o(1)} \tag{3.5}$$

and

$$\frac{d}{dr}V_4(r) = |\partial B_r(0)|e^{4\bar{w}(r)} \cdot e^{o(1)}, \tag{3.6}$$

where $\bar{w}(r) = \frac{1}{|\partial B_r(0)|}\int_{\partial B_r(0)} w(y)d\sigma(y)$, *and* $o(1) \to 0$ *as* $|x| \to \infty$.

Lemma 3.5. *Suppose that* $(R^4 \setminus B, e^{2w}|dx|^2)$ *is a complete normal metric. Then its averaged metric* $(R^4 \setminus B, e^{2\bar{w}}|dx|^2)$ *is also a complete metric.*

Lemma 3.6. *Suppose that* $(R^4 \setminus B, e^{2u}|dx|^2)$ *is a normal metric. Then*

$$|\Delta\bar{w}(r)| \leq \frac{C}{r^2}. \tag{3.7}$$

And we may arrive at

Lemma 3.7. *Suppose that* $(R^4 \setminus B, e^{2w}|dx|^2)$ *is a complete normal metric. Then*

$$\lim_{r\to\infty}\frac{(\int_{\partial B_r(0)} e^{3w}d\sigma(x))^{\frac{4}{3}}}{4(2\pi^2)^{\frac{1}{3}}\int_{B_r(0)\setminus B} e^{4w}dx} = \frac{1}{4\pi^2}\int_{\partial B} Te^{3w} - \frac{1}{8\pi^2}\int_{R^4\setminus B} Qe^{4w}dx \geq 0. \tag{3.8}$$

Now the key is to establish the following

Lemma 3.8. *Suppose that* $(R^4 \setminus B, e^{2w}|dx|^2)$ *satisfies*

$$\int_{R^4\setminus B} |Q|e^{4w}dx < \infty,$$

and that its scalar curvature is nonnegative at infinity. Then $e^{2w}|dx|^2$ *is a normal metric.*

Proof. This is a rather interesting proof and will help us to understand Definition 3.3. First set

$$\phi(x) = \frac{3}{4\pi^2}\int_{R^4\setminus B} \log\frac{|y|}{|x-y|}Qe^{4w}dy \tag{3.9}$$

and $\psi = w - \phi$. We will show that the biharmonic function ψ on $R^4 \setminus B$ has to be $\alpha \log|x| + h(x)$ for some constant α and some biharmonic function $h(\frac{x}{|x|^2})$ on B. Recall the transform formula for the scalar curvature function

$$\Delta w + |\nabla w|^2 = -Je^{2w}.$$

And notice that $\Delta\psi$ is a harmonic function on $R^4 \setminus B$. Thus

$$\begin{aligned} \Delta\psi(x_0) &= \frac{1}{|\partial B_r(x_0)|}\int_{\partial B_r(x_0)} \Delta\psi d\sigma \\ &= -\frac{1}{|\partial B_r(x_0)|}\int_{\partial B_r(x_0)} (|\nabla w|^2 + J)d\sigma - \frac{1}{|\partial B_r(x_0)|}\int_{\partial B_r(x_0)} \Delta\phi d\sigma \end{aligned} \tag{3.10}$$

as long as $B_r(x_0) \subset R^4 \setminus B$. The first term on the right of (3.10) is certainly nonpositive since $J \geq 0$ when $|x_0|$ is large enough. For the second term, we have

$$\begin{aligned} \int_{\partial B_r(x_0)} \Delta\phi d\sigma =& \frac{3}{2\pi^2}\int_{R^4\setminus B}\{\frac{1}{|S^3|}\int_{S^3}\frac{1}{|r\sigma + x_0 - y|^2}d\sigma\}Q(y)e^{4w(y)}dy. \\ \leq& \frac{3}{2\pi^2 r^2}\int_{R^4\setminus B}|Q|e^{4w}dy. \end{aligned}$$

Therefore, taking $r = \frac{1}{2}|x_0|$ for $|x_0| > 2$, for instance, we have, for any $x_0 \in R^4 \setminus B$,

$$\Delta\psi(x_0) \leq \frac{C}{|x_0|^2}$$

for some constant C. Thus (it may depend on ψ but not $x \in R^4 \setminus B$.)

$$\Delta(\psi + \frac{C}{2}\log|x|) \leq 0.$$

So, if we set $g(x) = -\psi - \frac{C}{2}\log|x|$, then, $\Delta g(\frac{x}{|x|^2}) \geq 0$ and is harmonic on $B \setminus \{0\}$. By Bôcher's theorem (Theorem 3.9 in [ABR]), we have

$$\Delta g(\frac{x}{|x|^2}) = \beta\frac{1}{|x|^2} + b(x),$$

for some positive constant β and some harmonic function $b(x)$ on B. This implies that $g(\frac{x}{|x|^2}) + \frac{1}{2}\beta\log|\frac{x}{|x|^2}|$ is a biharmonic function on B. In other words, if denoted by $h(x) = \psi + \frac{C}{2}\log|x| - \frac{1}{2}\beta\log|x|$, then, $h(\frac{x}{|x|^2})$ is biharmonic on B and

$$\psi(x) = \frac{1}{2}(\beta - C)\log|x| + h(x) \quad \text{on } R^4 \setminus B.$$

We have thus finished the proof of the lemma.

Combining Lemma 3.7 and Lemma 3.8 we then have

Corollary 3.9. *Suppose that* $(R^4 \setminus B, e^{2w}|dx|^2)$ *is a complete metric satisfying*

$$\int_{R^4\setminus B} |Q|e^{4w}dx < \infty$$

and that its scalar curvature is nonnegative at infinity. Then

$$\lim_{r\to\infty} \frac{(\int_{\partial B_r(0)} e^{3w}d\sigma(x))^{\frac{4}{3}}}{4(2\pi^2)^{\frac{1}{3}} \int_{B_r(0)\setminus B} e^{4w}dx} = \frac{1}{4\pi^2}\int_{\partial B} Te^{3u} - \frac{1}{8\pi^2}\int_{R^4\setminus B} Qe^{4u}dx \geq 0. \tag{3.11}$$

From Definition 3.1, we understand that a complete manifold (M, g) has only finitely many conformally flat ends means that

$$M = N \bigcup \{\bigcup_{i=1}^{k} E_i\}$$

where (N, g) is a compact Riemannian manifold with boundary

$$\partial N = \bigcup_{i=1}^{k} \partial E_i,$$

and each E_i is a conformally flat simple end of M, i.e.

$$(E_i, g) = (R^4 \setminus B, e^{2w_i}|dx|^2),$$

for some function w_i. Then Lemma 3.9 implies the following rather full generalization of (1.5) of Finn's.

Theorem 3.10. *Suppose that* (M, g) *is a complete 4-manifold with finite number of conformally flat simple ends. And suppose that*

$$\int_M |Q|dv_M < \infty,$$

and that the scalar curvature of g *is nonnegative at each end. Then*

$$\chi(M) - \frac{1}{8\pi^2}\int_M \{\frac{1}{4}|\mathcal{W}|^2 + Q\}dv_M = \sum_{i=1}^{k} \mu_i, \tag{3.12}$$

where the summation is taken for every end $(E_i, g) = (R^4 \setminus B, e^{2w_i}|dx|^2)$ *and*

$$\mu_i = \lim_{r\to\infty} \frac{(\int_{\partial B_r(0)} e^{3w_i}d\sigma(x))^{\frac{4}{3}}}{4(2\pi^2)^{\frac{1}{3}} \int_{B_r(0)\setminus B} e^{4w_i}dx}. \tag{3.13}$$

3.2 Simply connected cases

Given a simply connected, locally conformally flat, complete manifold M of dimension $n \geq 3$, there always exists an immersion $\Phi : M \to S^n$ such that the locally conformally flat structure of M is induced by Φ. This immersion Φ is called the developing map of M. Existence of such developing map is due to the fact that conformal transformation is determined locally. By a well known result of Schoen and Yau (cf. [SY, Chapter 6]), under the assumption that the scalar curvature $R_g \geq 0$, then Φ is injective. Therefore, any such manifold can be considered as a subdomain of S^n with a complete metric $g = \nu^{\frac{4}{n-2}} g_c$ where g_c is the standard metric on the sphere S^n. For convenience, we choose a point P in M and use stereographic projection which maps $S^n \setminus \{P\}$ to R^n and P to infinity; then we may identify (M, g) as $(\Omega, u^{\frac{4}{n-2}}|dx|^2)$, where $\Omega \subset R^n$. Our goal is to measure the size of $\partial\Omega \subset R^n$, which in someway is to estimate the size of the singular set of the conformal factor u. For this purpose, first, we have the following lower bound estimate from ([SY, Theorem 2.12, Chapter VI])

Lemma 3.11. *Suppose $(\Omega, u^{\frac{4}{n-2}}|dx|^2)$ is a complete Riemannian manifold with scalar curvature bounded, i.e. $|R| < k$, covariant derivatives of scalar curvature bounded, i.e. $|\nabla_g R| < k$, and the Ricci curvature bounded from below, i.e. $Ric > -k$. Then there exists a constant $C > 0$ such that*

$$u(x) \geq C d(x)^{-\frac{n-2}{2}} \quad \textit{for all } x \in \Omega \ . \tag{3.14}$$

The proof of this estimate is an application of gradient estimates. It turns out that, in dimension 4, we can also establish the upper bound of u in terms of the distance function d too.

Lemma 3.12. *Suppose $(\Omega, u^2|dx|^2)$ is a complete manifold such that*
(a) its scalar curvature R satisfies $0 < R_0 \leq R \leq R_1$ and $|\nabla R|_g \leq k$, where R_0, R_1, k are constants, and
(b) $\int_\Omega |Q| u^4 dx < \infty$.
Then there exists some constant C so that

$$u(x) \leq C d(x)^{-1} \quad \textit{for all } x \in \Omega. \tag{3.15}$$

Our proof of the above lemma uses a blow-up argument, which in the case the scalar curvature function R is a constant has been applied by Schoen (presented in [Po]) to obtain the same upper bound estimate (3.15). Thus what we have done here is to replace the constant scalar condition by the integral bound of the Q curvature and the condition (a).

The proof we have below for Lemma 3.12 depends on the following simple result, which is a consequence of Theorem 2.8 in Chapter 2.

Lemma 3.13. *On R^4, every metric $u^2|dx|^2$ with $Q[u^2|dx|^2] \equiv 0$ and $R[u^2|dx|^2] \geq 0$ at infinity is isometric to the Euclidean space $(R^4, |dx|^2)$.*

We will now estimate the size of the integral of Q over a subset of Ω in terms of the integral of the boundary curvature T via the Chern-Gauss-Bonnet formula. To do so, we will first derive a formula for the boundary integral. We will use the following notations. We consider $(\Omega, u^2|dx|^2)$ where $\Omega \subset R^4$, and denote the level set for the conformal factor $u = e^w$ by

$$U_\lambda = \{x : 1 \leq u \leq \lambda\}, \quad \text{and} \quad S_\lambda = \{x : u = \lambda\}. \tag{3.16}$$

Also, ∂_n denotes the normal derivative (chosen so that $\frac{\partial w}{\partial n} \geq 0$) .

Lemma 3.14. *Suppose that $(\Omega, u^2 g_0)$ is a complete Riemannian manifold. Then on the level set S_λ where λ is a regular value for u, we have:*
(3.17)

$$\begin{aligned} -\int_{S_\lambda} \partial_n \Delta w d\sigma =& \lambda \frac{d}{d\lambda} \left(\int_{S_\lambda} (\partial_n w)^3 d\sigma + \int_{S_\lambda} J \partial_n w e^{2w} d\sigma + 2 \int_{U_\lambda} J |\nabla u|^2 dx \right) \\ &+ \int_{S_\lambda} (\partial_n w)^3 d\sigma + \int_{S_\lambda} J^2 \frac{e^{4w}}{\partial_n w} d\sigma \end{aligned}$$

The proof of this identity is simply a calculation. The idea behind this is to realize what we believe that the integral of Q curvature should in someway control the growth of the volume towards the ends of the manifold. This idea is believed to be the one that enabled Cohn-Vossen to obtain his result in [CV] in dimension 2. Now the final ingredient is the following simple covering lemma, namely,

Lemma 3.15. *Suppose that Λ is a compact subset of R^4. Then*
(3.18)

$$|\{x : dist(x, \Lambda) = s\}| \geq \begin{cases} Ns^3, & \textit{for any } N > 0 \textit{ if } \dim(\Lambda) = 0 \textit{ and } H^0(\Lambda) = \infty \\ Cs^{3-\alpha}, & \textit{for } \alpha = \frac{3}{4}\beta \;\; \textit{if } \dim(\Lambda) = \beta > 0. \end{cases}$$

Where $dim(\Lambda)$ denotes the Hausdorff dimension of the set Λ, and H^β denotes the Hausdorff measure of exponent β on R^4.

Now we are ready to state and prove the main theorem of this section.

Theorem 3.16. *Suppose that (M, g) is a simply connected, complete, LCF, 4-manifold satisfies:*
(a) The scalar curvature is bounded between two positive constants; $|\nabla_g R|$ is bounded with respect to g; and the Ricci curvature of the metric g has a lower bound;
(b) the Q curvature is absolutely integrable, i.e.

$$\int_M |Q| dv_g < \infty.$$

Then M is conformally equivalent to S^4 with finitely many punctures.

Proof. We identify (M, g) as $(\Omega, u^2|dx|^2)$ for some subset Ω of R^4 as in the above. Notice that all ends of M is in bounded region in R^4. Apply integration by parts, we get

$$\int_{U_\lambda} Qe^{4w}dx = \int_{U_\lambda} \Delta^2 w dx = -\int_{S_1} \partial_n \Delta w d\sigma + \int_{S_\lambda} \partial_n \Delta w d\sigma \tag{3.19}$$

Apply formula (3.17) in Lemma 3.14, we obtain

$$\begin{aligned} -\int_{U_\lambda} Qe^{4w}dx - \int_{S_1} \partial_n \Delta w d\sigma = & -\int_{S_\lambda} \partial_n \Delta w d\sigma \\ = & \lambda \frac{d}{d\lambda}\Big(\int_{S_\lambda} (\partial_n w)^3 d\sigma + \int_{S_\lambda} J(\partial_n w)e^{2w} d\sigma \\ & +2\int_{U_\lambda} J|\nabla u|^2 dx\Big) + \int_{S_\lambda} (\partial_n w)^3 d\sigma + \int_{S_\lambda} J^2 \frac{e^{4w}}{\partial_n w} d\sigma \\ \geq & \lambda \frac{d}{d\lambda} V(\lambda) \end{aligned} \tag{3.20}$$

Where $V(\lambda)$ is defined as:

$$V(\lambda) = \int_{S_\lambda} (\partial_n w)^3 d\sigma + \int_{S_\lambda} J(\partial_n w)e^{2w} d\sigma + 2\int_{U_\lambda} J|\nabla u|^2 dx. \tag{3.21}$$

We recall the scalar equation

$$-\Delta u = Ju^3 \quad \text{in } \Omega.$$

Thus

$$\begin{aligned} \int_{U_\lambda} J|\nabla u|^2 dx \geq J_0 \int_{U_\lambda} |\nabla u|^2 dx & \\ \geq J_0\Big(\int_{U_\lambda} Ju^4 dx - \int_{S_1} u\partial_n u d\sigma + \int_{S_\lambda} u\partial_n u d\sigma\Big) & \\ \geq J_0^2 \int_{U_\lambda} u^4 dx, & \end{aligned} \tag{3.22}$$

where $J \geq J_0 > 0$ as assumed in (a). And

$$V(\lambda) \geq 2J_0^2 \int_{U_\lambda} u^4 dx. \tag{3.23}$$

To estimate the growth of V we use the lower and upper bound estimates of the conformal factor u as in Lemma 3.11 and Lemma 3.12. Thus we may replace the region U_λ by

$$D_\lambda = \{x : C_1 \geq d(x, \partial\Omega) \geq C_2\lambda^{-1}\} \subset U_\lambda \tag{3.24}$$

Therefore we have

$$V(\lambda) \geq \int_{D_\lambda} u^4 = \int_{\frac{C_2}{\lambda}}^{C_1} \int_{\{x:d(x,\partial\Omega)=s\}} u^4 d\sigma ds$$

by the co-area formula. Hence

$$V(\lambda) \geq C \int_{\frac{C_2}{\lambda}}^{C_1} |\{x : d(x, \partial\Omega) = s\}| s^{-4} ds. \tag{3.25}$$

We now estimate the size of the set $\partial\Omega$ by lemma 2.6. In the case $\dim(\partial\Omega) = \beta$ is positive, we have from (3.18) and (3.25) that

$$V(\lambda) \geq \frac{C}{\alpha}\left(\left(\frac{\lambda}{C_2}\right)^{\alpha} - \frac{1}{C_1^{\alpha}}\right), \tag{3.26}$$

for $\alpha = \frac{3}{4}\beta$ which is positive. In the case when $\dim(\partial\Omega)$ is zero, we have either the zero Hausdorff measure of the set (i.e. number of points in the set) is finite; then we have proved the theorem; or we have

$$|\{x : d(x, \partial\Omega) = s\}| \geq N s^3 \tag{3.27}$$

for any number $N > 0$. Hence

$$V(\lambda) \geq N \int_{\frac{C_2}{\lambda}}^{C_1} \frac{1}{s} ds = N \log \lambda - C. \tag{3.28}$$

In either case, we conclude that there exists at least a sequence of $\lambda_i \to \infty$ as $i \to \infty$ such that λ_i are all regular values (due to Sard's theorem) and

$$\lambda_i \frac{d}{d\lambda} V(\lambda_i) \geq N \tag{3.29}$$

for any number $N > 0$. But in view of the equality (3.20), this contradicts with our assumption (c) that Q is integrable. We have thus finished the proof of the theorem.

3.3 General Cases

Suppose (M^4, g) is a locally conformally flat 4-manifold with positive scalar curvature, then by the result of Schoen-Yau, the universal cover $\tilde{M}$ can be embedded as a domain in the 4-sphere. Hence the fundamental group Γ acts on S^4 as a discrete group of conformal transformations with a domain of discontinuity $\Omega(\Gamma)$ which contains $\tilde{M}$. (Here as in the rest of this section, we refer to [Ra] for standard notations and definitions of Kleinian groups.) The limit set L of a Kleinian group Γ consists of accumulation points of orbits of Γ. The discrete group Γ also acts as hyperbolic isometries on the interior B^5 of S^4. We recall the following definitions of limit points.

A point $p \in S^4$ is called a conical limit point of the group Γ if there is a point $x \in B$, a sequence $\{g_i\} \subset \Gamma$, a hyperbolic ray γ in B ending at p and a positive number r such that $\{g_i(x)\}$ converges to p within hyperbolic distance r from the geodesic γ.

A point p is a cusped limit point of a discrete group Γ if it is a fixed point of a parabolic element of Γ that has a cusped region. To explain the notion of a cusped region, we identify S^4 as R^4 and conjugate the point p to infinity in the upper half space $\mathbb{R}^5_+$, and consider the stabilizer Γ_∞ of ∞. Γ_∞ is a discrete subgroup of isometries of R^4 of rank $1 \leq m \leq 4$. Let E be the maximal Γ_∞-invariant subspace such that E/Γ_∞ is compact. Denote N a neighborhood of E in $\bar{\mathbb{R}}^5_+$ and set $U = \bar{\mathbb{R}}^5_+ - \bar{N}$. Then U is an open Γ_∞-invariant subset of $\bar{\mathbb{R}}^5_+$. The set U is said to be a cusped region for Γ based at ∞ if and only if for all g in $\Gamma \setminus \Gamma_\infty$, we have $U \cap gU = \emptyset$. In other words, a fundamental domain of Γ_∞ in a cusped region U is a part of the fundamental domain for Γ, that is to say, the hyperbolic manifold B^5/Γ has a cusped end as $\dot{U}/\Gamma_\infty$ and the Kleinian manifold $\Omega(\Gamma)/\Gamma$ has a conformal cusped end as $(U \setminus \mathbb{R}^5_+)/\Gamma_\infty$. The understanding of a conformal cusped end is essential to our following discussions.

Definition 3.17. *A Kleinian group Γ is said to be geometrically finite if its set of limit points consists of only conical limit points or cusped limit points.*

One also knows that, if a Kleinian group has only conical limit points, then the Kleinian manifold $\Omega(\Gamma)/\Gamma$ is compact. Now given a complete, LCF, 4-manifold M with positive scalar curvature, in the light of the above discussion, we know that $M \subset \Omega(\Gamma)/\Gamma$ where Γ is the holonomy representation of $\pi_1(M)$. Our strategy is to take $\Omega(\Gamma)/\Gamma$ as the candidate for a compactification of M and show that $M = \Omega(\Gamma)/\Gamma \setminus \{p_k\}_{k=1}^{l}$. Then the first question is whether $\Omega(\Gamma)/\Gamma$ is compact. It turns out that the strictly positive scalar curvature assumption is exactly used again to eliminate all cusped limit points. Thus we establish

Theorem 3.18. *Suppose M is a locally conformally flat complete 4-manifold which satisfies:*
(a) The scalar curvature is bounded between two positive constants, $|\nabla_g R|$ is bounded with respect to g, and the Ricci curvature has a lower bound;
(b) The Q curvature is absolutely integrable, i.e.

$$\int_M |Q| dv_g < \infty;$$

(c) The fundamental group of M acting as deck transformation group is a geometrically finite Kleinian group.
Then $M = \bar{M} \setminus \{p_i\}_{i=1}^{k}$ where $\bar{M}$ is compact manifold with a locally conformally flat structure.

Proof. Denote by

$$\begin{array}{ll} & \Phi : \tilde{M} \hookrightarrow S^4 \\ (3.30) & \pi : \downarrow \\ & \quad M \end{array}$$

where Φ is the developing map from the universal covering $\tilde{M}$ of M into S^4, which is an embedding. The holonomy representation Γ of the fundamental group of M then becomes a Kleinian group of the conformal transformation group of S^4, which is assumed to be geometrically finite. Let $\Omega = \Phi(\tilde{M})$. Clearly any point in Ω is a ordinary point for Γ, that is $\Omega \subset \Omega(\Gamma)$: the domain of discontinuity of Γ. We are interested in the set $\partial\Omega = S^4 \setminus \Omega$. If M is compact, then $\partial\Omega = L(\Gamma)$ is the set of all limit points of Γ and $\Omega = \Omega(\Gamma)$, therefore $M = \Omega(\Gamma)/\Gamma$. But, in our case, we have

$$\partial\Omega = (\partial\Omega \cap \Omega(\Gamma)) \bigcup L(\Gamma), \tag{3.31}$$

and $L(\Gamma)$ consists of only conical limit points and cusped limit points. we claim

Claim 1. *Every point in $\partial\Omega \cap \Omega(\Gamma)$ is an isolated point in S^4,*

and

Claim 2. *There is no cusped limit points except possibly cusped limit points of rank four in which case the closure of the fundamental region for the cusp does not meet the limit set.*

First we apply the proof of Theorem 3.16 in previous section to prove Claim 1. According to Theorem 2.9 and Theorem 2.11 in Chapter VI of [SY], $\dim(\partial\Omega) \leq d(M) < 1$, hence is a totally disconnected set.(cf Lemma 4.1, Chapter 4 in [Fa]). Therefore, for any $x \in \partial\Omega \cap \Omega(\Gamma)$, there exists a ball $B(r,x)$ such that $\gamma B(r,x) \bigcap B(r,x) = \emptyset$ for all $\gamma \in \Gamma$ and $\partial\bar{B}(r,x) \bigcap \partial\Omega = \emptyset$. Since the Q curvature is absolutely integrable over $\Omega \cap B(r,x)$, therefore we can restrict the conformal metric to $B(r,x) \cap \Omega$ and apply the argument in the proof of Theorem 2.1 to conclude that $\partial\Omega \cap B(r,x)$ consists of at most finite number of points including x, thus in particular x is an isolated boundary point.

To prove Claim 2, we recall, from the definition of the cusped limit points, a cusped limit point p is a fixed point of a parabolic element γ_p in Γ and there is a cusped region U for Γ based at p. The cusped region based at p restricted to the 4-sphere gives a conformal coordinates chart for M at the end E_p around p of the following form (cf: Chapter 12 in [Ra]):

$$(M,g) \supset (E_p, g) = (S_m, \phi^2 g_m)$$

for some $S_m, 1 \leq m \leq 4$, where $S_1 = T^1 \times \{x \in R^3 : |x| \geq K\}$, $S_2 = T^2 \times \{x \in R^2 : |x| \geq K\}$, $S_3 = T^3 \times \{x \in R : |x| \geq K\}$, $S_4 = T^4$, where T^k is a flat manifold of dimension k, $K > 0$ is a large positive number, g_m is the product metric on each S_m, and ϕ is a positive smooth function. Now we will show that such ends E_p can not exist in our case.

Lemma 3.19. *There is no complete conformal metric $\phi^2 g_m$ on S_m with its scalar curvature bounded from below by a positive number, for $m = 1, 2, 3$.*

Proof. Let us illustrate the proof in case $m = 1$. Recall the scalar curvature equation on S_1:

$$-(\partial_r\partial_r\phi + \frac{2}{r}\partial_r\phi + \frac{1}{r^2}\Delta_{S^2}\phi + \Delta_{T^1}\phi) = J\phi^3$$

for $r = |x|$, $r \in [K, \infty)$ the norm of a point x in R^3. We take the average of ϕ over $S^2 \times T^1$ for each r and get

$$-(\partial_r\partial_r\bar{\phi} + \frac{2}{r}\partial_r\bar{\phi}) \geq J_0(\bar{\phi})^3$$

where $J \geq J_0$ as assumed. Now take a change of variables

$$\begin{cases} e^t = r \\ \psi = r\bar{\phi} \end{cases}, \tag{3.32}$$

therefore

$$-\psi_{tt} + \psi_t \geq J_0\psi^3 \tag{3.33}$$

We will show that ψ attains zero or infinity at some finite t, which will be a contradiction. First we observed that, if $\psi_t(t_0) \leq 0$ at certain t_0, then $\psi_{tt}(t_0) < 0$. Therefore we have $\psi_t \leq -\alpha < 0$ and $\psi_{tt} \leq 0$ for all $t \geq t_1$ for some $t_1 > t_0$, which implies ψ has to be zero at some finite t. Thus, we may assume $\psi_t > 0$ for all t. Next we observed that, if $\psi_t - J_0\psi^3 \leq 0$ at some t_0, then $\psi_{tt}(t_0) \leq 0$ and therefore $\psi_{tt}(t) \leq -\beta < 0$ for all $t \geq t_2$ for some $t_2 > t_0$; then ψ_t can not be positive for all t. Thus, we may assume

$$\psi_t - J_0\psi^3 > 0$$

for all t. But this implies

$$\frac{d}{dt}(\frac{1}{\psi^2}) < -2J_0,$$

which is impossible unless ψ goes to infinity at some finite t. This finished the proof of case $m = 1$.

To continue the proof of Theorem 3.18, we apply Lemma 3.19 to conclude that the limit set consists of either conical limit points or cusp of rank four.

Since fixed points of either hyperbolic or parabolic elements in Γ are all in the limit set $L(\Gamma)$ of Γ, Γ acts on $\Omega(\Gamma)$ has no fixed points. Thus $\Omega(\Gamma)/\Gamma = \bar{M}$ is a manifold with a locally conformally flat structure. We consider a fundamental domain F which satisfies:

$$\Omega(\Gamma) = \bigcup_{\gamma\in\Gamma} \gamma F \quad \text{and} \quad F \bigcap \gamma F = \emptyset, \ \forall \gamma \in \Gamma.$$

And let $C = \bar{F}$ the closure of F in S^4. Since all points in $L(\Gamma)$ are either conical or cusps of rank four, the closure of F does not meet the limit set, hence $C \subset \Omega(\Gamma)$. This proves that $\bar{M} = C/\Gamma$ is compact since C is compact. In the mean while, we have $M = (C \setminus (\partial\Omega \cap \Omega(\Gamma))$. Now since $\partial\Omega \cap \Omega(\Gamma)$ are all isolated points without accumulation points in $\Omega(\Gamma)$, $C \cap (\partial\Omega \cap \Omega(\Gamma))$ must be a finite point set. Therefore

$$M = \bar{M} \setminus \{p_i\}_{i=1}^k$$

for some $k < \infty$. We have thus finished the proof of the theorem.

4. Finiteness of Kleinian groups in general dimension

In Theorem 3.18, in order to have a compactification $\Omega(\Gamma)/\Gamma$, we assume that the holonomy representation Γ of the fundamental group $\pi_1(M)$ to be a geometrically finite Kleinian group. Obviously it is much more desirable to only assume that $\pi_1(M)$ is algebraically finite, i.e. finitely generated. With this particular motivation we would like to understand the finiteness of Kleinian groups in dimension higher than two.

4.1 Finiteness of Kleinian groups in dimension smaller than 3

When dimension is one, things are very simple. A Fuchsian group (conventional name for Kleinian group in dimension one) is algebraically finite if and only if it is geometrically finite. Basically this is because any end of a hyperbolic 2-manifold is either a funnel or a cusp.

In dimension two, it is already very interesting. Another finiteness is introduced, namely, analytic finiteness. We say a Kleinian group Γ in dimension two is analytically finite if $\Omega(\Gamma)/\Gamma$ is a collection of finitely many Riemann surfaces of finite type. A Riemann surface is of finite type if it is a compact closed Riemann surface with finitely many punctures. The celebrated finiteness theorem of Ahlfors [Ah] and Bers [Be], which states that a finitely generated Kleinian group in dimension two is analytically finite. Meanwhile, though much later, Bishop and Jones [BJ] showed that, if the Hausdorff dimension of the set of limit points of a Kleinian group Γ in dimension 2 is less than 2, then Γ is geometrically finite if and only if it is analytically finite, therefore, by Ahlfors's finiteness theorem, if and only if it is algebraically finite. Then the interesting nasty case is that Γ is algebraically finite but with the set of limit points of Hausdorff dimension 2. In this case there is a famous open conjecture of Ahlfors, that is, the area of the set of limit points is still zero, which turns out to be intimately related to Thurston's approach to the study of hyperbolic 3-manifolds. For our purpose, the overwhelming fact in dimension two is the finiteness theorem of Ahlfors.

4.2 Conformal finiteness

As we mentioned in the above that algebraic finiteness is not equivalent to geometric finiteness even in dimension two. But Bishop and Jones in [BJ] made it clear that algebraic finiteness does not imply geometric finiteness unless the Hausdorff dimension of the limit set is less than 2. In this section we will formulate an analogue in higher dimension $n \geq 3$. But first we introduce standard conformal cusped ends. Of course, a standard conformal cusped end C_m is the ideal boundary of the standard hyperbolic cusped end. Suppose that Γ_∞ is the stabilizer of a parabolic fixed point. Then Γ_∞ is a discrete subgroup of the group of Euclidean isometries of R^n (a maximum parabolic subgroup in a Keinian group Γ), let R^m be the maximal invariant subspace so that R^m/Γ_∞ is compact. Suppose that $N(R^m, \epsilon)$ is an ϵ-neighborhood of R^m in R^n. Then $N(R^m, \epsilon)$ is also invariant under Γ_∞ and a standard conformal cusp end is of the form

$$(R^n \setminus N(R^m, \epsilon))/\Gamma_\infty. \tag{4.1}$$

Therefore, a standard conformal cusp end is conformal to $(R^{n-m} \setminus B_\epsilon(0)) \times K$ where K is a compact locally flat manifold of dimension m. We will call by m the rank of the conformal cusped end C_m. We remark that without a cusped region for a cusped limit point one would not be able to recover the whole cusped end (the hyperbolic $(n+1)$-dimensional end) from a given conformal cusped end, which is at least on the surface the distinguishing property here. Interestingly, on a standard conformal cusped end, there is a nice complete metric

$$ds^2 = \frac{|dx|^2 + |dy|^2}{|y|^2}, \quad \text{for } x \in R^m \text{ and } y \in R^{n-m} \setminus N(R^m, \epsilon) \tag{4.2}$$

which has a finite volume

$$\frac{1}{m}\epsilon^{-m}\text{vol}(S^{n-m-1})\text{vol}(K) \tag{4.3}$$

and the scalar curvature

$$R = (n-1)(n-2-2m). \tag{4.4}$$

Note that the metric in (4.2) on $R^m \times (R^{n-m} \setminus \{0\})$ gives $H^{m+1} \times S^{n-m-1}$. Now we are ready to give the following definition.

Definition 4.1. *Suppose that Γ is a Kleinian group. Then we say Γ is conformally finite if $\Omega(\Gamma)/\Gamma$ consists of finitely many components and each component is a disjoint union of a compact set and a finite number of standard conformal cusped ends.*

By definition, geometric finiteness implies conformal finiteness. In this terminology, our goal is to investigate when conformal finiteness implies geometric finiteness. It is clear that the notion of conformal finiteness is a higher dimension analogue of the analytic finiteness. On the same note we give a name to each component of $\Omega(\Gamma)/\Gamma$ with the induced LCF structure by Klainian manifold as the analogue of Riemann surface in dimension two. Therefore by a Klainian manifold of finite type we mean it has at most finitely many conformal ends for the noncompact parts.

Theorem 4.2. *Suppose that Γ is a nonelementary, conformally finite Kleinian group on S^n, then Γ is geometrically finite if and only if the limit set of Γ has Hausdorff dimension strictly smaller than n.*

4.3 Construction of the Lipschitz graph

In our point of view, an important part of the work of [BJ] is the construction of an invariant Lipschitz graph which serves to relate the geometry of the hyperbolic manifold B^3/Γ to the geometry of the Riemann surface $\Omega(\Gamma)/\Gamma$. Our following construction, by completely elementary means, of the invariant Lipschitz graph over a domain of discontinuity $\Omega(\Gamma)$ of a Kleinian group Γ in higher dimension is based on the idea of Bishop and Jones in [BJ]. Take a small positive number ϵ_0 and consider a collection of balls $\{B_\alpha\}$ such that

$$B_\alpha = B(x_\alpha, d_\alpha) \quad \text{and} \quad d_\alpha = \epsilon_0 \cdot \text{dist}(x_\alpha, L(\Gamma)) \tag{4.5}$$

for each point $x_\alpha \in \Omega(\Gamma)$, where diameters and distances are all measured on S^n with the standard metric g_0. To construct an invariant graph we would enlarge the collection to take in all images of B_α under the group Γ and denote the collection of balls by $B(\Gamma)$. Set

$$G(\Gamma) = \partial(\bigcup_\beta H_\beta) \bigcap B^{n+1} \tag{4.6}$$

where H_β is the hyperbolic half space over each ball B_β in $B(\Gamma)$, i.e. the dome whose boundary intersects perpendicularly at ∂B_β with S^n. Clearly $G(\Gamma)$ is a graph over $\Omega(\Gamma)$ in the following sense

$$G(\Gamma) = \{f(x)x : x \in \Omega(\Gamma)\} \tag{4.7}$$

where $f(x) : \Omega(\Gamma) \longrightarrow (0,1)$. In fact $G(\Gamma)$ is a Lipschitz graph in the sense that

$$|f(x) - f(y)| \leq M\text{dist}(x,y)$$

for some $M > 0$ and all $x, y \in \Omega(\Gamma)$. Therefore

Lemma 4.3. *Given a nonelementary Kleinian group Γ and a small positive number ϵ_0, the above constructed graph $G(\Gamma)$ is a Γ-invariant Lipschitz graph. Moreover*

$$0 < C_1 \leq \frac{1 - f(x)}{dist(x, L(\Gamma))} \leq C_2 \tag{4.8}$$

for all $x \in \Omega(\Gamma)$, where C_1, C_2 only depend on ϵ_0.

The proof of this lemma is based on a generalized distortion estimate. It is a rather elementary fact about nonelementary Kleinian groups. Namely,

Lemma 4.4. *Suppose that Γ is a nonelementary Kleinian group, $\Omega(\Gamma)$ is its domain of discontinuity and $L(\Gamma) = \partial\Omega(\Gamma)$ is its set of limit points. Then there exists a positive number C such that*

$$\frac{1}{C}\frac{dist(\gamma(x), L(\Gamma))}{dist(x, L(\Gamma))} \leq |\gamma'(x)|_s \leq C\frac{dist(\gamma(x), L(\Gamma))}{dist(x, L(\Gamma))} \tag{4.9}$$

for all $x \in \Omega(\Gamma)$ and all $\gamma \in \Gamma$.

Remark 4.5. *For $L(\Gamma) = \{\infty\}$ and Γ is simply generated by a translation $\gamma x = x + h$, we have*

$$|\gamma'(x)|_s = \frac{1 + |x|^2}{1 + |\gamma x|^2} = \frac{q(\gamma x, \infty)^2}{q(x, \infty)^2},$$

where

$$q(x, y) = \frac{2|x - y|}{\sqrt{1 + |x|^2}\sqrt{1 + |y|^2}}, \quad \textit{and} \quad q(x, \infty) = \frac{2}{\sqrt{1 + |x|^2}}$$

which is called the chordal metric and is equivalent to the spherical distance $d(x, y)$. For $L(\Gamma) = \{0, \infty\}$ and Γ is simply generated by an inversion $\gamma x = \frac{x}{|x|^2}$, we also have

$$|\gamma'(x)|_s = \frac{q(\gamma x, \infty)^2}{q(x, 0)^2}.$$

Let us discuss, when the Klainian manifold $\Omega(\Gamma)/\Gamma$ is compact, how the constructed invariant Lipschitz graph serves to relate the geometry of the hyperbolic manifold B^{n+1}/Γ and the geometry of the Klainina manifold $\Omega(\Gamma)/\Gamma$. More precisely, since the Lipschitz graph has an induced metric from the hyperbolic metric, we would like to compare it with a suitable conformal metric on the Kleinian manifold $\Omega(\Gamma)/\Gamma$. In the case when the Kleinian quotient is compact, this is a relatively simple matter.

Proposition 4.6. *Suppose that Γ is a nonelementary Kleinian group, and that the Kleinian manifold $\Omega(\Gamma)/\Gamma$ is compact. Then, for any metric in the conformal class on $\Omega(\Gamma)/\Gamma$, we have a complete Γ-invariant metric $e^{2u}g_0$ on $\Omega(\Gamma)$ and*

$$\frac{1}{K}\frac{1}{dist(x, L(\Gamma))} \leq e^{u(x)} \leq K\frac{1}{dist(x, L(\Gamma))}, \tag{4.10}$$

for all $x \in \Omega(\Gamma)$ and some positive number K.

Proof. This basically is a consequence of Lemma 4.4. Due to the invariance of the metric $e^{2u}g_0$, we have

$$e^{u(x)} = e^{u(\gamma x)}|\gamma'(x)|_s.$$

Now, fix a fundamental region D, whose closure $\bar{D}$ is compact in $\Omega(\Gamma)$, and for any $x \in \Omega(\Gamma)$, there exists $\gamma \in \Gamma$ such that $\gamma^{-1}x = y \in \bar{D}$. Then

$$e^{u(y)} = e^{u(x)}|\gamma'(y)|_s$$

where, by Lemma 4.4.

$$\frac{1}{C}\frac{\mathrm{dist}(x, L(\Gamma))}{\mathrm{dist}(y, L(\Gamma))} \le |\gamma'(y)|_s \le C\frac{\mathrm{dist}(x, L(\Gamma))}{\mathrm{dist}(y, L(\Gamma))}.$$

Thus

$$\frac{1}{C}\frac{e^{u(y)}\mathrm{dist}(y, L(\Gamma))}{\mathrm{dist}(x, L(\Gamma))} \le e^{u(x)} \le C\frac{e^{u(y)}\mathrm{dist}(y, L(\Gamma))}{\mathrm{dist}(x, L(\Gamma))},$$

that is

$$\frac{1}{K}\frac{1}{\mathrm{dist}(x, L(\Gamma))} \le e^{u(x)} \le K\frac{1}{\mathrm{dist}(x, L(\Gamma))}$$

for some positive constant K.

Remark 4.7. *In particular, we may consider the Yamabe metric on $\Omega(\Gamma)/\Gamma$ in Proposition 4.6. Therefore, under the assumptions of Proposition 4.6, there is a complete Γ-invariant metric on $\Omega(\Gamma)$ with constant scalar curvature and satisfying (4.9).*

We point out that the natural bounds (4.9) on the invariant metric on $\Omega(\Gamma)$ is the key to relate the hyperbolic geometry inside B/Γ to the conformal geometry at infinity $\Omega(\Gamma)/\Gamma$ through the constructed Γ-invariant Lipschitz graph $G(\Gamma)$. Namely,

Proposition 4.8. *Suppose that Γ is a nonelementary Kleinian group and that the Kleinian manifold $\Omega(\Gamma)/\Gamma$ is compact. Then the map*

$$F(x) = f(x)\, x : \Omega(\Gamma) \to G(\Gamma)$$

is a Γ-invariant bi-Lipschitz map with respect to the induced hyperbolic metric on the graph $G(\Gamma)$ and any metric on $\Omega(\Gamma)$ which is induced by a metric on $\Omega(\Gamma)/\Gamma$ in the conformal class.

Next, when Γ is only assumed to be conformally finite, we can still obtain the conclusions in Proposition 4.6 and 4.8 in the above. Let us fix a conformal metric on $\Omega(\Gamma)/\Gamma$ which agrees with the g_h on each conformal cusp end and arbitrary on the compact part. Let us denote it by g_Γ (this is not intended to signify g_Γ is an any way canonical).

Proposition 4.9. *Suppose that Γ is nonelementary, conformally finite Kleinian group, and that g_Γ is a metric constructed as the above. Then the metric $e^{2u}g_0$ on $\Omega(\Gamma)$ lifted from g_Γ satisfies*

$$\frac{1}{C}\frac{1}{\mathit{dist}(x, L(\Gamma))} \le e^{u(x)} \le C\frac{1}{\mathit{dist}(x, L(\Gamma))} \tag{4.11}$$

for some constant $C > 0$ and all $x \in \Omega(\Gamma)$.

To relate the geometry of the hyperbolic manifold B^{n+1}/Γ and the Kleinian manifold $\Omega(\Gamma)/\Gamma$, we find

Proposition 4.10. *Suppose that Γ is nonelementary, conformally finite, and that g_Γ is a metric constructed as the above. Then the map*

$$F(x) = f(x)x : \Omega(\Gamma) \to G(\Gamma)$$

is a Γ-invariant bi-Lipschitz map with respect to the induced hyperbolic metric on the graph $G(\Gamma)$ and the above metric g_Γ on $\Omega(\Gamma)$. Moreover the hypersurface $G(\Gamma)/\Gamma$ in B^{n+1}/Γ has finite volume.

4.4. Sketch of the Proof of Theorem 4.2

Let us consider the hyperbolically harmonic function on B^{n+1} with the boundary condition

$$u|_{S^n} = \chi_{\Omega(\Gamma)} = \begin{cases} 1 & \text{on } \Omega(\Gamma) \\ 0 & \text{on } L(\Gamma) \end{cases}$$

Lemma 4.11.

$$u(x) = c_n \int_{G(\Gamma)/\Gamma} \frac{\partial g_\Gamma}{\partial n}(x, y) d\sigma(y) \tag{4.12}$$

where g_Γ is the positive minimal Green function on B^{n+1}/Γ.

To illustrate, let us compute $u(0)$:

$$u(0) = c_n \int_{\Omega(\Gamma)} d\omega = \frac{\text{vol}(\Omega(\Gamma))}{\text{vol}(S^n)}.$$

Recall that, for the Green function on the hyperbolic ball B^{n+1},

$$-\frac{\partial g}{\partial n} d\sigma|_{\partial B_r(0)} = \frac{2^{n-1}}{r^n} d\omega|_{B_r(0)}.$$

Let Ω_r be the part of $\partial B_r(0)$ which is between $G(\Gamma)$ and $\Omega(\Gamma)$ and $G_r = G(\Gamma) \cap B_r$. Clearly

$$\begin{aligned} u(0) &= \frac{1}{\text{vol}(S^n)} \lim_{r\to 1} \frac{\text{vol}(\Omega_r)}{r^n} \\ &= \frac{1}{2^{n-1}\text{vol}(S^n)} \lim_{r\to 1} \int_{\Omega_r} (-\frac{\partial g}{\partial n}) d\sigma \\ &= \frac{1}{2^{n-1}\text{vol}(S^n)} \lim_{r\to 1} \int_{G_r} (-\frac{\partial g}{\partial n}) d\sigma \\ &= c_n \int_{G(\Gamma)} \frac{\partial g}{\partial n} d\sigma. \end{aligned}$$

Then, due to the fact that $G(\Gamma)$ is Lipschitz,

$$\int_{G(\Gamma)} |\frac{\partial g}{\partial n}| d\sigma < \infty.$$

If let S be any fundamental region for $G(\Gamma)$,

$$\begin{aligned}\int_{G(\Gamma)} \frac{\partial g}{\partial n} d\sigma &= \sum_{\gamma\in\Gamma} \int_{\gamma S} \frac{\partial g}{\partial n} d\sigma \\ &= \sum_{\gamma\in\Gamma} \int_{S} \frac{\partial g(\gamma 0, \gamma y)}{\partial n} d\sigma \\ &= \int_{S} \frac{\partial \sum_{\gamma\in\Gamma} g(\gamma 0, \gamma y)}{\partial n} d\sigma \\ &= \int_{G(\Gamma)/\Gamma} \frac{\partial g_\Gamma(0, y)}{\partial n} d\sigma.\end{aligned}$$

which proves this lemma for $x = 0$. For other points one may use the transformations to compute. It is clear that, under the assumption that $\dim(L(\Gamma)) < n$, the harmonic function constructed in the above is constant 1. But, on the other hand, we are going to use (4.12) to show that, if Γ is conformally finite but geometrically infinite, in addition to $\dim(L(\Gamma)) < n$, then $u(x)$ can be small for some choices of x on B^{n+1}/Γ (x will be chosen to be far into the geometrically infinite end). But first, we have

Theorem 4.12. *(Sullivan) Suppose that Γ is a nonelementary Kleinain group. Then*

$$\lambda_0(B^{n+1}/\Gamma) = \begin{cases} (\frac{n}{2})^2 & \text{if } \delta \leq \frac{n}{2} \\ \delta(n-\delta) & \text{if } \delta \geq \frac{n}{2} \end{cases} \tag{4.13}$$

where δ is the Poincaré exponent.

and

Theorem 4.13. *(Bishop and Jones) Suppose that Γ is a nonelementary Kleinain group. Then*

$$\delta = dim(L_c(\Gamma)). \tag{4.14}$$

Therefore, combining with estimates for the Green functions, we have

Proposition 4.14. *If $dim(L(\Gamma)) < n$, then*

$$0 < g_\Gamma(x, y) \leq C vol(B_1(x))^{-\frac{1}{2}} vol(B_1(y))^{-\frac{1}{2}} e^{-\epsilon\rho(x,y)}, \tag{4.15}$$

where g_Γ is the positive minimum Green function on B^{n+1}/Γ.

Meanwhile, as observed by Bishop and Jones in [BJ], we have the following for us to make the choice of the x to evaluate the Γ-invariant harmonic function u. Namely,

Lemma 4.15. *Suppose that Γ is nonelementary, conformally finite, and that Γ is geometrically infinite. Then, there exist $\epsilon > 0$ and a sequence of points $x_n \in C(B^{n+1}/\Gamma)$ such that $d_H(x_n, G(\Gamma)/\Gamma) \to \infty$ and the injectivity radius $inj(x_n) > \epsilon$ for all n.*

We recall that equivalently Γ is geometrically finite if and only if the thick part of the convex core $C(B^{n+1}/\Gamma)$ is compact. Thus, if Γ is geometrically infinite, there must be a sequence of points $\{p_i\} \in C(B^{n+1}/\Gamma)$ for which the injective radius of B^{n+1}/Γ at p_i is bounded from below and p_i tends to infinity in the convex core $C(B^{n+1}/\Gamma)$. Finally we have to use the so-called thick-thin decomposition for hyperbolic manifolds to prove

lemma 4.16. *Suppose that Γ is a nonelementary and conformally finite Kleinian group. Then*

$$\int_{G(\Gamma)/\Gamma} vol(B_1(y))d\sigma(y) < \infty. \tag{4.16}$$

Outline of proof of Theorem 4.2. Let assume otherwise that Γ is geometrically infinite. First, by (4.12) in Lemma 4.11 and (4.15) in Lemma 4.14, we have

$$u(x) \leq c_n \text{vol}(B_1(x))^{-\frac{1}{2}} e^{-\epsilon d_H(x, G(\Gamma)/\Gamma)} \int_{G(\Gamma)/\Gamma} \text{vol}(B_1(y))^{-\frac{1}{2}} d\sigma(y)$$

In the light of (4.16) in Lemma 4.16, if we consider x_n given in Lemma 4.15 we have $u(x_n)$ gets smaller and smaller when x_n goes far and far away from the hypersurface $G(\Gamma)/\Gamma$ in B^{n+1}/Γ, which contradicts with the fact that $u(x)$ is constant.

4.5 Combination Theorems

To understand how algebraic finiteness relates to our conformal finiteness, we propose to use combination theorems of Kleinian groups. Probably the first combination theorem for Kleinian groups was the combination theorem of Klein:

Theorem 4.17. *(Klein) let Γ_1 and Γ_2 be finitely generated Kleinian groups, and let D_1 and D_2 be fundamental domain for Γ_1 and Γ_2, respectively. Suppose that the interior of D_1 contains the boundary and exterior of D_2, and that the interior of D_2 contains the boundary and the exterior of D_1. Then the group generated by Γ_1 and Γ_2 is a free product $\Gamma_1 \star \Gamma_2$ and also a Kleinian groups with the fundamental domain $D_1 \cap D_2$.*

Maskit in a series of papers [Mt] further well developed combination theory for Kleinian groups, particularly for Kleinian groups in two dimension. One of his combination theorem that is particularly interesting to us is the following:

Theorem 4.18. *(Maskit) Let* $(\Gamma_1, D_1, H, \Delta, \gamma, B_1)$, $(\Gamma_2, D_2, H, \Delta, \gamma, B_2)$ *be two big conglomerates where* $B_1 \cap B_2 = \emptyset$. *Let* Γ *be the group generated by* Γ_1 *and* Γ_2 *and* $D = D_1 \cap D_2$. *Then* Γ *is a Kleinian group and is a free product of* Γ_1 *and* Γ_2 *with amalgamated subgroup* H. *And* D *is a fundamental set of* Γ.

Here a big conglomerate $(\Gamma, D, H, \Delta, \gamma, B)$ is a collection of a Kleinian group Γ in two dimension, its fundamental set D, a subgroup $H \subset \Gamma$, a fundamental set Δ for H such that $D \subset \Delta$, a simple curve γ in the interior of D and invariant under H, and a topological disk bounded by γ.

Our approach to combination theorem for Kleinian groups is different. In fact, we are interested in not only decomposition of Kleinian groups but also decomposition of Kleinian manifolds. But it appears that the above combination theorem of Maskit gave a nice setting for us. Luckily but not surprisingly, the above combination theorem of Maskit takes a simpler form in dimension higher than two. Because we may replace the simple curve by an embedded sphere (simply connected) and we may take the trivial subgroup H, therefore Γ is simply a free product of Γ_1 and Γ_2.

Let us consider a connected Kleinian manifold $M^n = \Omega(\Gamma)/\Gamma$ of dimension $n \geq 3$. Suppose that there is an embedded (n-1)-sphere Σ in M, and that the embedded sphere Σ separates M into N_1 and N_2, i.e. N_1 and N_2 are the two connected components in $M \setminus \Sigma$. Let $M_1 = N_1 \# B^n$ and $M_2 = N_2 \# B^n$ where B^n is a n-ball. Therefore, one sees that M is a connected sum of M_1 and M_2, i.e. $M = M_1 \# M_2$. Let Σ_α be all the lifts of Σ in $\Omega(\Gamma)$ and

$$\Omega(\Gamma) \setminus \{\Sigma_\alpha\} = \bigcup_\beta \Omega_\beta = (\bigcup_i \Omega_i) \bigcup (\bigcup_j \Omega_j)$$

where each Ω_α is a connected component, and Ω_i covers N_1 and Ω_j covers N_2.

Take a fundamental set D of Γ which includes a lift Σ_0 of Σ in its interior. Let $\Omega_1 \in \{\Omega_i\}$ and $\Omega_2 \in \{\Omega_j\}$ such that $D \subset \Omega_1 \cup \Omega_2$. Let Γ_1 is the subgroup of Γ which is the stabilizer of Ω_1, i.e.

$$\Gamma_1 = \{g \in \Gamma : g : \Omega_1 \longrightarrow \Omega_1\}.$$

And let Γ_2 is the subgroup of Γ which is the stabilizer of Ω_2, i.e.

$$\Gamma_2 = \{g \in \Gamma : g : \Omega_2 \longrightarrow \Omega_2\}.$$

Since Σ_0 bounds two n-balls B_1, B_2 in S^n (B_1 is at the same side of Σ_0 as $\Omega_1 \cap D$ is), we let

$$D_1 = (D \bigcap \Omega_1) \bigcup B_2$$

and

$$D_2 = (D \bigcap \Omega_2) \bigcup B_1.$$

Notice that

$$\Omega_1 = \Omega(\Gamma) \setminus (\bigcup_{g \in \Gamma_1} gB_2).$$

Then it is not hard to see that, if denote by $\Omega(\Gamma_1)$ the domain of discontinuity of Γ_1, then

$$\Omega(\Gamma_1) = \Omega(\Gamma) \bigcup (\bigcup_{g \in \Gamma_1} gB_2)$$

and D_1 is a fundamental set for Γ_1. Similarly,

$$\Omega(\Gamma_2) = \Omega(\Gamma) \bigcup (\bigcup_{g \in \Gamma_2} gB_1)$$

and D_2 is a fundamental set for Γ_2. Thus we obtain a Kleinian structure for M_1 as $\Omega(\Gamma_1)/\Gamma_1$ and a Kleinian structure for M_2 as $\Omega(\Gamma_2)/\Gamma_2$. Moreover, M as a Kleinian manifold is a connected sum of Kleinian manifolds M_1 and M_2. In summary

Theorem 4.19. *Suppose that $M = \Omega(\Gamma)/\Gamma$ is a connected Kleinian manifold of dimension $n \geq 3$, and there is an embedded (n-1)-sphere Σ in M, which separates M. Then there are two subgroups Γ_1 and Γ_2 of Γ such that $\Gamma = \Gamma_1 \star \Gamma_2$ and $M = M_1 \# M_2$ where $M_1 = \Omega(\Gamma_1)/\Gamma_1$ and $M_2 = \Omega(\Gamma_2)/\Gamma_2$.*

The picture we present here was also observed by Kulkarni, when he found that a connected sum of two Kleinian manifolds is still a Kleinian manifold. In the rest of this section we will focus on $n = 3$. The reason we would rather focus on $n = 3$ is because the following sphere theorem which apparently is not available in any higher dimension.

Theorem 4.20. *Suppose that M^3 is an oriented and connected 3-manifold, and that $\pi_2(M) \neq 0$. Then there exists an embedding*

$$F : S^2 \longrightarrow M^3$$

where $[F] \neq 0 \in \pi_2(M)$.

Now we are ready to state and prove a finiteness theorem as follows.

Theorem 4.21. *Suppose that Γ is a finitely generated Kleinian group acting on B^4. And suppose that the limit set of Γ is of Hausdorff dimension less than one. Then it is geometrically finite.*

Proof. First, since the limit set of Γ is of Hausdorff dimension less than one, the set $\Omega(\Gamma)$ of ordinary point on S^3 is connected and simply connected. Therefore the fundamental group of the Kleinian manifold $\Omega(\Gamma)/\Gamma$ is Γ. Now, by Theorem 4.19, we may decompose Γ into the free product of two subgroups Γ_1 and Γ_2 as long as we can find an embedded 2-sphere in $\Omega(\Gamma)/\Gamma$ which separates $\Omega(\Gamma)/\Gamma$. Next we observe that, as long as the limit point set contains more than one point, $\pi_2(\Omega(\Gamma)/\Gamma) \neq 0$. Therefore, by sphere theorem as stated in the above, we find an embedded 2-sphere in $\Omega(\Gamma)/\Gamma$. Then, if this embedded 2-sphere separates $\Omega(\Gamma)/\Gamma$, we may decompose $\Gamma = \Gamma_1 \star \Gamma_2$ and $\Omega(\Gamma)/\Gamma = \Omega(\Gamma_1)/\Gamma_1 \sharp \Omega(\Gamma_2)/\Gamma_2$; if this embedded 2-sphere does not separate $\Omega(\Gamma)/\Gamma$, then we have $\Omega(\Gamma)/\Gamma = M_1 \sharp S^1 \times S^2$, where $M_1 = \Omega(\Gamma_1)/\Gamma_1$ and $\Gamma = \Gamma_1 \star Z$. Thus, in the end, we may conclude that

$$\Omega(\Gamma)/\Gamma = \Omega(\Gamma_1)/\Gamma_1 \sharp \cdots \sharp \Omega(\Gamma_k)/\Gamma_k \sharp S^1 \times S^2 \sharp \cdots \sharp S^1 \times S^2. \tag{4.17}$$

and

$$\Gamma = (Z^{n_1}) \star \cdots \star (Z^{n_k}) \star Z \star \cdots \star Z. \tag{4.18}$$

The key here is to notice that, indeed, $\Omega(\Gamma)/\Gamma$ is compact except a finitely many conformal ends, that is to say that Γ is conformally finite. Therefore, by Theorem 4.2, we conclude that Γ is geometrically finite.

To end this section we would like to pose as an open question that how we find an analogue of Theorem 4.21 in dimension higher than 3.

4.6 Conformal finiteness revisited

In this section we would like to give some geometric criteria for a Kleinian group to be conformally finite. The idea still is that the hypersurface $G(\Gamma)/\Gamma$ with the metric induced from the hyperbolic metric is the right geometric representative for the Kleinian manifold $\Omega(\Gamma)/\Gamma$. We first observe:

Theorem 4.22. *Suppose that Γ is a nonelementary Kleinian group. Then Γ is conformally finite if and only if the volume of the hypersurface $G(\Gamma)/\Gamma$ in the hyperbolic manifold B^{n+1}/Γ is finite.*

Proof. Let us begin with the thick-thin decomposition of hyperbolic manifolds B^{n+1}/Γ with respect to a small number ϵ which is smaller than the Margulis constant in the same dimension. The hypersurface $G(\Gamma)/\Gamma$ is also decomposed into thick part W_ϵ and thin part S_ϵ. Clearly at each point in the thick part W_ϵ there is the hyperbolic geodesic ball $B_{\frac{1}{2}\epsilon}$ where $G(\Gamma)\bigcap B_{\frac{1}{2}\epsilon}$ belongs to some fundamental domain for Γ on the graph $G(\Gamma)$. Because $G(\Gamma)$ is Lipschitz graph over the unit sphere (or any sphere with the same center), the volume of $G(\Gamma)\bigcap B_{\frac{1}{2}\epsilon}$ under the metric induced from the hyperbolic metric on B^{n+1} is bounded from below by some constant only depending on ϵ. Therefore, if $G(\Gamma)/\Gamma$ has a finite volume with the metric, then the thick part W_ϵ has to be compact. Now we may conclude that the number of the noncompact connected components has to be finite. Because the finite boundary of each noncompact component, which is the connecting region of the end to the thick part, has a size again bounded from below (depending on ϵ). Notice that, each of those noncompact thin ends corresponds to a maximum parabolic subgroup P_i whose fixed point is p_i. Then, we find that, for some fundamental domain for Γ in its domain of discontinuity $\Omega(\Gamma)$, there are only finite number of limit points p_i on its boundary. Moreover, those parabolic fixed points therefore have to be bounded, i.e. have to be so-called cusped limit points. This means precisely that the Kleinian manifold $\Omega(\Gamma)/\Gamma$ is a disjoint union of a compact part and a finite number of standard conformal cusp ends. So Γ is conformally finite.

On the other hand if Γ is conformally finite, it follows from Theorem 3.5 that the hypersurface $G(\Gamma)/\Gamma$ has finite volume. So the proof is completed.

As a consequence we have the following criterion to tell when a Kleinian group is conformally finite.

Theorem 4.23. *Suppose that Γ is a nonelementary Kleinian group. Then Γ is conformally finite if and only if the Kleinian manifold $\Omega(\Gamma)/\Gamma$ possesses a conformal metric which is of finite geometry.*

Proof. First of all, if Γ is conformally finite, it is clear the Kleinian manifold possesses a metric satisfying (1) and (2), by Proposition 4.9 and 4.10. The converse part of this theorem is a consequence of above Theorem 4.22 and Theorem 2.12, Chapter VI in [SY]. The Harnack estimate in [SY] shows that, denoting the metric by $e^{2u}g_o$ where g_0 is the standard metric on the sphere,

$$u(x) \geq C\frac{1}{d(x)} \quad \text{for all } x \in \Omega(\Gamma) \ .$$

where $d(x) = \text{dist}(x, \partial\Omega(\Gamma))$. Then for the metric induced from the hyperbolic metric on the hypersurface $G(\Gamma)/\Gamma$, its volume is controlled by the volume of the Kleinian manifold with the given metric, therefore is finite. In light of the above Theorem 4.22, the proof is finished.

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