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Fully nonlinear equations in conformal geometry

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1 Background: The curvature tensor

Let (M^n, g) be an n-dimensional Riemannian manifold. We denote the curvature tensor by *Riem*. Contracting the curvature tensor we obtain the Ricci tensor *Ric*; contracting once more gives the scalar curvature R. In this section we want to describe various ways to decompose *Riem* into irreducible components. To this end,it will be helpful to introduce the Kulkarni-Nomizu product of symmetric two-tensors A and B: let X, Y, Z, W be tangent vectors, and define $A \odot B$ by

$$\begin{aligned}(A \odot B)(X,Y,Z,W) =& A(X,Z)B(Y,W) - A(X,W)B(Y,Z) \\ &- A(Y,Z)B(X,W) + A(Y,W)B(X,Z).\end{aligned}$$

This product is commutative, and the resulting tensor shares many of the symmetries of the curvature tensor.

The idea behind our first decomposition of *Riem* is to write it as the sum of a 'trace-less' part and 'pure trace' part. To this end, let $E = Ric - \frac{1}{n}Rg$ denote the trace-free Ricci tensor. Then

$$Riem = W + \frac{1}{(n-2)} g \odot E + \frac{1}{n(n-1)} Rg \odot g, \tag{1.1}$$

where W is the Weyl curvature tensor. For our purposes, we take (1.1) as the definition of the Weyl tensor; see [1] for a more complete description. The point is that W is trace free, while the remaining terms on the right all have a term involving the metric g (i.e., they are 'trace' terms). Geometrically, (1.1) is useful because the vanishing of each component has an interpretation. For example, the last two terms vanish if (and only if) the manifold has constant curvature. If the middle term (involving E) vanishes, then we say the manifold is *Einstein*.

An important property of the Weyl tensor is its *conformal invariance*: if $\tilde{g} = e^{-2u} g$ is a metric conformal to g, then the Weyl tensor of $\tilde{g}$ is given by

$$\tilde{W} = e^{2u} W. \tag{1.2}$$

Keeping this in mind, from the point of view of conformal geometry there is a more natural decomposition of *Riem*.

Definition 1. *The Weyl-Schouten tensor A is*

$$A = \frac{1}{(n-2)}\Big(Ric - \frac{1}{2(n-1)}Rg\Big).$$

In terms of the Weyl-Schouten tensor, we can write

$$Riem = W + g \odot A. \tag{1.3}$$

A consequence of (1.3) and (1.2) is the following: the behavior of the curvature tensor under a conformal change of metric is completely determined by the behavior of the tensor A under this change. This simple observation provides the justification for much of the analysis which follows.

Before we move on to some linear algebra, a final remark about three dimensions. In this case the Weyl tensor always vanishes, therefore

$$Riem = g \odot A. \tag{1.4}$$

In particular, the sectional curvatures of g are determined by A.

2 Some Linear Algebra

In this section we introduce the elementary symmetric polynomials and describe some of their important properties. We only provide a summary of the relevant results; details and proofs can be found in [19], [6], and [14].

Definition 2. *For $1 \leq k \leq n$, the* kth elementary symmetric polynomial $\sigma_k : \mathbf{R}^n \to \mathbf{R}$ *is defined by*

$$\sigma_k(x_1, x_2, \ldots, x_n) = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}. \tag{2.1}$$

The sum in (2.1) is taken over all increasing n-tuples; since there are $\binom{n}{k}$ such terms sometimes a factor of $\binom{n}{k}^{-1}$ is included in the defintion. We prefer the simpler defintion despite the fact that some inequalities involving the symmetric polynomials become more complicated with this convention.

To the functions σ_k we associate the following family of cones in $\mathbf{R}^n$:

Definition 3. *For $1 \leq k \leq n$, let*

$$\Gamma_k^+ = \{x = (x_1, \ldots, x_n) \in \mathbf{R}^n | \forall j \leq k \quad \sigma_j(x) > 0\}. \tag{2.2}$$

Also, $\Gamma_k^- = -\Gamma_k^+$.

It is not difficult to show that the sets Γ_k^+ are open, convex cones; moreover, we have the inclusions

$$\Gamma_n^+ \subset \Gamma_{n-1}^+ \subset \cdots \subset \Gamma_1^+$$

Examples

1. For $k = 1$, Γ_1^+ consists of all vectors in $\mathbf{R}^n$ whose components have a positive sum. For example, in $\mathbf{R}^2$, this is just the set of points (x, y) which lie above the line $x + y = 0$.

2. For $k = n$, Γ_n^+ is the positive cone $\{x = (x_1, \ldots, x_n) \in \mathbf{R}^n | \forall i \quad x_i > 0\}$.

The following result is usually refered to as the *Newton-MacLaurin inequality.*

Lemma 2.1. *If $x \in \Gamma_k^+$, then*

$$\frac{n!k}{(k-1)!(n-k+1)!(n-k+1)}\sigma_k(x)^{k-1} \leq \sigma_{k-1}(x)^k. \tag{2.3}$$

Now, let V be an n-dimensional real inner product space, and $A : V \to V$ a symmetric linear transformation. Then A has n real eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, and we define

$$\sigma_k(A) \equiv \sigma_k(\lambda_1, \lambda_2, \ldots, \lambda_n).$$

Examples

1. $\sigma_1(A) = \lambda_1 + \cdots + \lambda_n = trace(A)$.

2. $\sigma_2(A) = \sum_{i<j} \lambda_i \lambda_j$. If $|A|^2 = \sum_{i=1}^n \lambda_i^2$ denotes the Hilbert-Schmidt norm of A, then

$$\sigma_2(A) = -\frac{1}{2}|A|^2 + \frac{1}{2}trace(A)^2. \tag{2.4}$$

3. $\sigma_n(A) = \lambda_1 \cdots \lambda_n = \det(A)$.

Definition 4. *We say that $A \in \Gamma_k^+$ if $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Gamma_k^+$.*

Since we are viewing the symmetric polynomials as functions on the space of real symmetric matrices, we would like to understand their differentiability properties.

Definition 5. *Let $A : V \to V$ be a symmetric linear transformation. For $0 \leq q \leq n$, the qth* Newton transformation *associated with A is*

$$T_q(A) = \sigma_q(A) \cdot I - \sigma_{q-1}(A) \cdot A + \cdots + (-1)^q A^q. \tag{2.5}$$

Lemma 2.2. *(i) We have the identities*

$$\begin{aligned} &T_{k-1}(A)_i^j A_j^i = k\sigma_k(A), \\ &trace(T_{k-1}(A)) = (n-k+1)\sigma_{k-1}(A). \end{aligned}$$

(ii) If $A \in \Gamma_k^+$ (resp. Γ_k^-) then $T_{k-1}(A)$ is positive (resp. negative) definite.
(iii) If A and B are symmetric linear transformations, then

$$\frac{d}{dt}\sigma_k(A + tB)|_{t=0} = T_{k-1}(A)_i^j B_j^i.$$

Examples

1. The first Newton tranform is

$$T_1(A) = \sigma_1(A) \cdot I - A, \tag{2.6}$$

so that

$$\frac{d}{dt}\sigma_2(A + tB)|_{t=0} = (\sigma_1(A)\delta_i^j - A_i^j)B_j^i.$$

2. When $k = n$, the nth Newton tranform can be expressed in terms of the inverse matrix:

$$T_n(A) = \det(A) \cdot A^{-1} \tag{2.7}$$

Finally, we note the following *convexity* property of the symmetric polynomials:

Lemma 2.3. *For symmetric linear transformations $A, B \in \Gamma_k^+$ and $t \in [0, 1]$ we have*

$$\{\sigma_k\big((1-t)A + tB\big)\}^{1/k} \geq (1-t)\sigma_k(A)^{1/k} + t\sigma_k(B)^{1/k}.$$

3 Fully Nonlinear Equations

Let $\Omega \subset \mathbf{R}^n$ be an open set. A general second order differential equation on Ω can be written

$$F[u] = F(x, u, \nabla u, \nabla^2 u) = 0, \tag{3.1}$$

where $F : S = \Omega \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^{n\times n} \to \mathbf{R}$, $\mathbf{R}^{n\times n}$ denotes the space of real symmetric $n \times n$ matrices, and $\nabla u, \nabla^2 u$ denote respectively the gradient and Hessian of u. We denote points in S by $s = (x, z, p, r)$.

To take a simple example, consider the case of a linear second order equation:

$$F[u] = a^{ij}(x)\partial_i\partial_j u + b^i(x)\partial_i u + c(x)u - g(x) = 0. \tag{3.2}$$

In this case, F is given by

$$F(x, z, p, r) = a^{ij}(x)r_{ij} + b^i(x)p_i + c(x)z - g(x). \tag{3.3}$$

Recall that equation (3.2) is *elliptic* in Ω if the matrix $\{a^{ij}(x)\}$ is positive definite for all $x \in \Omega$; i.e.,

$$a^{ij}(x)\xi_i\xi_j > 0 \quad \forall x \in \Omega, \xi \in \mathbf{R}^n - \{\mathbf{0}\}.$$

If the eigenvalues of $\{a^{ij}\}$ are uniformly bounded above and below by positive constants, then we say that (3.2) is *uniformly elliptic* in Ω.

For more general equations such as (3.1), ellipticity is defined in the following way:

Definition 6. *Given a subset* $\Sigma \subset S$*, we say that the operator* F *is* elliptic *in* Σ *if the matrix*

$$F_{ij}(x, z, p, r) \equiv \frac{\partial F}{\partial r_{ij}}(x, z, p, r) \tag{3.4}$$

is positive definite for all $(x, z, p, r) \in \Sigma$*. If the eigenvalues of* $\frac{\partial F}{\partial r_{ij}}$ *are uniformly bounded above and below by positive constants, then* F *is* uniformly elliptic *in* Σ*.*

Remark 1. *If the matrix in (3.4) is negative definite, we replace* F *with* $-F$*.*

For the linear operator in (3.2), using (3.3) we find

$$\frac{\partial F}{\partial r_{ij}}(x, z, p, r) = a^{ij}(x).$$

So ellipticity in this sense coincides with the usual notion.

Examples

1. **Minimal surface equation** Let $u \in C^2(\Omega)$. Then the graph of u (as a hypersurface in $\mathbf{R}^{n+1}$) is minimal if u satisfies

$$M[u] = \Delta u - \frac{\partial_i u \partial_j u}{(1 + |\nabla u|^2)} \partial_i \partial_j u = 0. \tag{3.5}$$

In this case, the smallest and largest eigenvalues of $\frac{\partial M}{\partial r_{ij}}$ are given by

$$\lambda_{min} = \frac{1}{(1 + |p|^2)}, \quad \lambda_{max} = 1.$$

Notice a crucial point here: while M is clearly elliptic everywhere in $S = \Omega \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^{n \times n}$, it is uniformly elliptic only on subsets of S where the the gradient $|\nabla u|$ is uniformly bounded. Therefore, the results of classical elliptic theory (elliptic regularity, Schauder estimates) are only accesible if one first establishes *a priori* estimates for the first derivatives of a solution.

2. **Prescribed Weingarten curvatures** More generally, if Π denotes the second fundamental form of a hypersurface $M^n \hookrightarrow \mathbf{R}^{n+1}$, then $\sigma_k(\Pi)$ is called the kth *Weingarten curvature.* If M^n is the graph of $u : \Omega \to \mathbf{R}^n$, then the equation which prescribes the kth Weingarten curvature is given by

$$\sigma_k\big(\nabla_i(\frac{\nabla_j u}{\sqrt{1 + |\nabla u|^2}}\big) = f(x). \tag{3.6}$$

Because the relevant operator involves a symmetric polynomial, ellipticity can be described using the ideas developed in the previous section. Since we will be considering similar equations in the context of conformal geometry, we will not give the details here.

3. **Special Lagrangian Equation** In [15], Harvey and Lawson introduced the *Special Lagrangian Equation.* Let $u : \Omega \subset \mathbf{R}^n \to \mathbf{R}$, and consider

$$F[u] = Im\det(I_n + \sqrt{-1}\nabla^2 u) = 0, \tag{3.7}$$

where I_n denotes the identity matrix. This is the basic equation of Calibrated Geometry. The geometric meaning of (3.7) is that the graph $\{(x, u(x))|x \in \Omega\} \subset \mathbf{R}^{2n}$ is minimal.

4 Fully nonlinear equations in conformal geometry

Let us now return to the Riemannian setting, where (M^n, g) is a compact, closed (no boundary) Riemannian manifold of dimension n. Recall A denotes the Weyl-Schouten tensor:

$$A = \frac{1}{(n-2)}\Big(Ric - \frac{1}{2(n-1)}Rg\Big).$$

Being a section of the bundle of symmetric $(0,2)$-tensors, A smoothly assigns to each point $x \in M^n$ a quadratic form on the tangent space T_xM^n. Using the metric we can canonically associate to A a (symmetric) linear transformation of T_xM^n denoted by $g^{-1}A$. This notation is due to the fact that the inverse of the metric tensor is used to identify the T_xM^n and the cotangent space $T_x^*M^n$. In old-fashioned index notation, the components of A with respect to a local coordinate system are A_{ij}, while the components of $g^{-1}A$ are $A^i_j = g^{ik}A_{jk}$. This is classically known as "raising an index".

Let $g_u = e^{-2u}g$ be a conformal metric, and A_u denote the Weyl-Schouten tensor with respect to g_u. Then A_u and A are related by

$$A_u = A + \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g \tag{4.1}$$

Given $f \in C^\infty(M^n)$, we consider the equation

$$\sigma_k^{1/k}(g_u^{-1}A_u) = f(x). \tag{4.2}$$

To simplify our formulas we usually interpret A_u as a bilinear form on the tangent space with inner product g (instead of g_u). That is, $\sigma_k(A_u)$ means $\sigma_k(\cdot)$ applied to the eigenvalues of the $g^{-1}A_u$. With this convention, (4.2) is equivalent to

$$\sigma_k^{1/k}(A + \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g) = f(x)e^{-2u}. \tag{4.3}$$

Note that when $k = 1$, then $\sigma_1(g^{-1}A) = trace(A) = \frac{1}{2(n-1)}R$. Therefore, (4.3) is the classical problem of prescribing scalar curvature.

Based on the results of the preceding two sections, we see that (4.2) is elliptic if the eigenvalues of A_u are in the cone Γ_k^+ (or Γ_k^-). However, as a consequence of the convexity of $\Gamma_k^\pm$, ellipticity can be more easily described:

Lemma 4.1. *(See [21]) If the eigenvalues of $A = A_g$ are everywhere in Γ_k^+, and if u is a solution to (4.3) with $f > 0$, then (4.3) is elliptic.*

Based on this we make the following

Definition 7. *A metric g is k-admissible if the eigenvalues of $A = A_g$ are everywhere in Γ_k^+. We then write $g \in \Gamma_k^+(M^n)$.*

Alternatively, if the eigenvalues of A are everywhere in Γ_k^-, then (4.3) will be elliptic; in this case we would say that g is *negative* k-admissible.

In considering the ellipticity properties of (4.3) we are inevitably lead to two questions:

1. Under what conditions can we verify that a conformal manifold admits an admissible metric? This question should precede any serious analysis of (4.3); if admissibility turns out to be too strong an assumption then (4.3) would only be of limited interest.

2. Given a k-admissible metric g and a function $f > 0$, does (4.3) admit a solution? Does one have *a priori* bounds for solutions? Are solutions unique?

It turns out that these two questions are closely related: admissibility (since it implies ellipticity) is clearly connected to the PDE aspects in Question 2. But it is also true that the apparently geometric question about the existence of admissible metrics can sometimes be addressed by PDE techniques. We will see a striking example of this in Section 6, where we describe some joint work with Chang and Yang in four dimensions. Here, we just want to point out an obstruction to admissibility noted by Guan, Viaclovsky, and Wang:

Proposition 4.1. *(See [7]) Suppose g is k-admissible with $k \geq \frac{n}{2}$. Then the Ricci curvature of g is positive and satisfies*

$$Ric \geq \frac{(2k-n)}{2n(k-1)} Rg. \tag{4.4}$$

Corollary 4.1. *If M^n admits a k-admissible metric with $k \geq \frac{n}{2}$, then the first deRham cohomology group $H^1(M^n) = 0$.*

We will discuss some further consequences of admissibility in four dimensions in Section 6. In the next section, however, we want to begin to lay the groundwork for approaching Question 2.

5 A priori estimates

The first systematic study of (4.3) was carried out in the thesis of Jeff Viaclovsky ([21]). In a subsequent paper, he considered a slightly more general version of (4.3):

$$\sigma_k^{1/k}(A + \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g) = \psi(x, u).$$

Assuming g is admissible and $\psi(x,\cdot)$ satisfies certain growth conditions, Viaclovsky proved various estimates and existence results. To simplify the exposition we will only summarize the results that are relevant to the study of (4.3).

First, the invariance of (4.3) under conformal transformations along with the fact that the round sphere has a non-compact conformal group means that, in general, one should not be able to establish *a priori* L^∞-estimates for solutions. This is a well known problem for the semilinear version of (4.3); i.e., when $k=1$. However, assuming such bounds Viaclovsky proved

Theorem 5.1. *(See [22], Proposition 6 and Proposition 8). Suppose g is admissible and $u \in C^4(M^n)$ is a solution of (4.3) satisfying $-B \le u \le B$. Then there is a constant $C = C(B, g, \|f\|_{C^2})$ such that*

$$|\nabla u| + |\nabla^2 u| \le C. \tag{5.1}$$

Since $\sigma_k(\cdot)$ is a concave function of A_u, a fundamental result of Evans [5] and Krylov [16] says that bounds on the second derivatives of solutions of (4.3) imply Holder norm bounds for the second derivatives. Therefore, we can apply the Schauder estimates and derive bounds for derivatives of all orders.

Corollary 5.1. *Suppose g is admissible and $u \in C^4(M^n)$ is a solution of (4.3) satisfying $-B \le u \le B$. For any $m \ge 1$, there is a constant $C = C(m, B, g, \|f\|_{C^2})$ such that*

$$|\nabla u| + \cdots + |\nabla^m u| \le C. \tag{5.2}$$

Subsequently, local estimates for (4.3) were derived by Guan and Wang in [9]. These estimates imply ϵ-regularity results and are extremely useful when applying blow-up arguments, as we shall see in Section 7.

Lemma 5.1. *(See [9], Proposition 2) Let $u \in C^4$ be an admissible solution of*

$$F(u) = \sigma_k^{1/k}(A + \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g) = f(x)e^{-2u} \tag{5.3}$$

on $B(0,2\rho)$, where $\rho > 0$. Then there is a constant $C(k, n, \rho, \|g\|_{C^3(B(0,\rho))}, \|f\|_{C^2(B(0,\rho))})$ such that

$$|\nabla u|^2(x) + |\nabla^2 u|^2(x) \le C(1 + e^{-2\inf_{B(0,\rho)} u}) \tag{5.4}$$

for all $x \in B(0, \rho/2)$.

Remarks

1. For negative admissible metric, Viaclovsky obtained C^0- and C^1-estimates, but not C^2-estimates. It remains an open question whether such estimates are true.

2. In [4], Chang, Gursky and Yang proved an *a priori* estimate for solutions of (4.3) with $k=2$ on four-manifolds.

3. Once *a priori* estimates of solutions are known there are various methods for establishing existence; e.g., degree theory or the continuity method.

The problem of solving (4.3) with $f(x) = constant$ is referred to as the σ_k-*Yamabe problem*. It will be convenient to normalize the value of this constant, so that the round metric on the sphere is a solution (with no need of rescaling):

$$\sigma_k^{1/k}(A + \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g) = \sigma_k^{1/k}(S^n)e^{-2u}, \tag{5.5}$$

where $\sigma_k^{1/k}(S^n) = \sigma_k^{1/k}(A_0)$, and A_0 is the Weyl-Schouten tensor of the round metric on S^n. We remark that the associated equation is variational when $k = 1$ or $k = 2$, but in general not when $k > 2$ (see [21]). The results of Viaclovsky reduce the question of existence of solutions to that of establishing C^0-bounds, which by our comments above means distinguishing the case of the sphere. This has been accomplished in some cases: in low dimensions, and when the underlying manifold is locally conformally flat. This will be discussed in more detail in subsequent sections.

Turning from general estimates to more specific results, we begin with the work of Chang, Gursky and Yang in four dimensions.

6 Four dimensions

Four dimensions enjoys some special features due to the relationship between (4.3) and the Chern-Gauss-Bonnet formula. The decomposition (1.3) implies a splitting of the Euler form, and consequently we can write

$$4\pi^2\chi(M^4) = \int_{M^4} \|W\|^2 dvol + 2\int \sigma_2(g^{-1}A)dvol. \tag{6.1}$$

Combining this with the signature formula, we obtain the following obstruction to the existence of admissible metrics:

Proposition 6.1. *If M^4 admits a k-admissible metric with $2 \leq k \leq 4$, then the Euler characteristic and signature of M^4 satisfy*

$$\chi(M^4) > \frac{3}{2}|\tau(M^4)|.$$

Another important feature of four dimensions is the conformal invariance of the integral

$$\int_{M^4} \sigma_2(g^{-1}A)dvol. \tag{6.2}$$

In particular, a necessary condition for conformal class of metrics to admit a k-admissible metric ($k \geq 2$) is that the integral in (6.2) is positive and the Yamabe invariant is positive. Remarkably, when $k = 2$ this condition is also *sufficient*:

Theorem 6.1. *(See [3]) If (M^4, g) has positive scalar curvature and if*

$$\int_{M^4} \sigma_2(g^{-1}A) dvol > 0, \tag{6.3}$$

then there is a conformal metric $g_u = e^{-2u}g$ which is 2-admissible; i.e., g_u satisfies $\sigma_1(g_u^{-1}A_u) > 0$ and $\sigma_2(g_u^{-1}A_u) > 0$. Consequently, the Ricci curvature of g_u satisfies

$$0 < Ric_u < \frac{1}{2} R_u \cdot g_u, \tag{6.4}$$

where R_u is the scalar curvature of g_u.

Remark 2. *The inequality (6.4) follows from (4.4) and (2.6).*

Since the hypotheses in Theorem 6.1 are fairly simple to check, by using surgery techniques we are able to construct many examples of manifolds which satisfy (6.3). For example, $\mathbf{CP}^2\#\mathbf{CP}^2$, $\mathbf{CP}^2\#\mathbf{CP}^2\#\mathbf{CP}^2$, and $\mathbf{S}^2 \times \mathbf{S}^2\#\mathbf{S}^2 \times \mathbf{S}^2$, etc. all admit many such metrics.

The proof of Theorem 6.1 is extremely involved, and difficult to even summarize, as it involves techniques from many fields: spectral theory, the calculus of variations, higher order elliptic equations, and fully nonlinear equations. In place of discussing the proof we point out its highly non-trivial geometric consequences.

First, as inequality (6.4) demonstrates, admissibility implies a kind of pinching condition on the Ricci curvature. Thus, this theorem gives a method for constructing metrics with positively pinched Ricci tensor on a large class of conformal 4-manifolds. In contrast, the construction of metrics with just positive Ricci curvature (in four dimensions, anyway) has been limited to either special cases (such as Kähler-Einstein metrics), or to surgery techniques which glued together known examples (see [20]). Moreover, Theorem 6.1 simply requires us to check two conformal invariants: $\int \sigma_2(g^{-1}A)dvol$, and the Yamabe invariant. The hypotheses are therefore stable under fairly dramatic deformations, unlike previous constructions.

In subsequent work, Chang, Gursky, and Yang were able to solve (5.5) when $k = 2$ in four dimensions:

Theorem 6.2. *(See [4]) If g is a 2-admissible metric, then there is a solution $g_u = e^{-2u}g$ of (5.5).*

Recently, Gursky and Viaclovsky treated the cases $k = 3$ and $k = 4$ in four dimensions, by introducing a new conformal invariant. We will describe this work in more detail in the next section.

7 Maximal volume

According to Proposition 4.1, a k-admissible metric with $k > n/2$ has positive Ricci curvature. In fact, if one normalizes such a metric, then by Bishop's Theorem the volume has an explicit upper bound. This observation naturally leads to the following defintion:

Definition 8. *Let (M^n, g) be a compact n-dimensional Riemannian manifold. For $n/2 \leq k \leq n$ we define the* k-maximal volume *of $[g]$ by*

$$\Lambda_k(M^n, [g]) = \sup\{vol(e^{-2u}g) | e^{-2u}g \in \Gamma_k^+(M^n) \ with \ \sigma_k^{1/k}(g_u^{-1}A_u) \geq \sigma_k^{1/k}(S^n)\}. \tag{7.1}$$

If $[g]$ does not admit a k-admissible metric, we set $\Lambda_k(M^n, [g]) = +\infty$.

Proposition 7.1. *If $[g]$ admits a k-admissible metric with $k > n/2$, then there is a constant $C = C(n)$ such that $\Lambda_k(M^n, [g]) < C(n)$.*

In analogy with the classical Yamabe problem, when this invariant is strictly less than the value obtained by the round metric on the sphere we obtain existence of solutions to (5.5):

Theorem 7.1. *(See [12]) Let (M^n, g) be a closed n-dimensional Riemannian manifold satisfying*

$$\Lambda_k(M^n, [g]) < vol(S^n), \tag{7.2}$$

where $vol(S^n)$ denotes the volume of the round sphere. Then $[g]$ admits a solution $g_u = e^{-2u}g$ of (5.5). Furthermore, the set of solutions of (5.5) is compact in the C^m-topology for any $m \geq 0$.

Despite the parallels with the Yamabe problem, Theorem 7.1 can only be considered satisfying if the condition (7.2) is known to be sharp. Although we conjecture this to be the case in general, we can only substantiate it in dimensions three and four. In each case the techniques for proving sharp estimates of $\Lambda_k(M^n, [g])$ are quite different in spirit.

In three dimensions our estimate follows from the volume comparison theorem of Bray ([2]). We will give a precise statement of his result later; for now we simply state the consequence for our invariant.

Theorem 7.2. *Let (M^3, g) be a closed Riemannian three-manifold, and assume $[g]$ admits a k-admissible metric with $k = 2$ or 3. Then*

$$\Lambda_k(M^3, [g]) \leq vol(S^3). \tag{7.3}$$

The proof of this result allows an important refinement of inequality (7.3). As a consequence, we are able to verify the assumptions of Thorem 7.1 whenever M^3 is not simply connected:

Theorem 7.3. *Let (M^3, g) be a closed Riemannian three-manifold, and assume $[g]$ admits a k-admissible metric with $k = 2$ or 3. Let $\pi_1(M^3)$ denote the fundamental group of M^3. Then*

$$\Lambda_k(M^3, [g]) \leq \frac{vol(S^3)}{\|\pi_1(M^3)\|}. \tag{7.4}$$

Corollary 7.1. *Let (M^3, g) be a closed, non-simply connected Riemannian three-manifold. If g is k-admissible with $k = 2$ or 3, then $[g]$ admits a solution $g_u = e^{-2u}g$ of (5.5). Furthermore, the set of solutions of (5.5) is compact in the C^m-topology for any $m \geq 0$.*

In four dimensions, our estimates of Λ_k follow from the sharp integral estimate for $\sigma_2(A)$ due to the first author ([11]).

Theorem 7.4. *Let (M^4, g) be a closed Riemannian four-manifold, and assume $[g]$ admits a k-admissible metric with $2 \leq k \leq 4$. Then*

$$\Lambda_k(M^4, [g]) \leq vol(S^4). \tag{7.5}$$

Furthermore, equality holds in (7.5) if and only if (M^4, g) is conformally equivalent to the round sphere.

Corollary 7.2. *Let (M^4, g) be a closed Riemannian four-manifold, and assume g is a k-admissible metric with $2 \leq k \leq 4$. Then $[g]$ admits a solution $g_u = e^{-2u}g$ of (5.5). Furthermore, if (M^4, g) is not conformally equivalent to the round sphere, then the set of solutions of (5.5) is compact in the C^m-topology for any $m \geq 0$.*

We now briefly outline the proofs of these results. To begin, let M^n be a closed n-dimensional manifold, and suppose g is k-admissible. By rescaling, we assume that g has unit volume. Consider the equation

$$\sigma_k^{1/k}(\lambda_k g + \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g) = \left(\int e^{-(n+1)u}\right)^{\frac{2}{n+1}}, \tag{7.6}$$

where λ_k is given by

$$\lambda_k = \binom{n}{k}^{-1/k}. \tag{7.7}$$

This choice of λ_k implies $\sigma_k(\lambda_k g) = 1$. Consequently, $u \equiv 0$ is a solution of (7.6).

Lemma 7.1. *$u \equiv 0$ is the unique solution of (7.6).*

We now introduce a one-parameter family of equations connecting equation (5.5) with equation (7.6). For $t \in [0, 1]$, consider

$$\begin{aligned} &\sigma_k^{1/k}\left(\lambda_k(1 - \psi(t))g + \psi(t)A + \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g\right) \\ &= (1-t)\left(\int e^{-(n+1)u}\right)^{\frac{2}{n+1}} + \psi(t)\sigma_k^{1/k}(S^n)e^{-2u}, \end{aligned} \tag{7.8}$$

where $\psi(t) \in C^1[0, 1]$ satisfies $0 \leq \psi(t) \leq 1, \psi(0) = 0$, and $\psi(t) \equiv 1$ for $t \geq \frac{1}{2}$. From the properties of $\psi(t)$ we see that if u is a solution of (7.8) with $t \geq \frac{1}{2}$, then $\sigma_k^{1/k}(A_u) \geq \sigma_k^{1/k}(S^n)e^{-2u}$. Therefore,

$$\Lambda_k(M^n, [g]) \geq \sup\{vol(g_u) | u \text{ satisfies (7.8) with } t \geq \frac{1}{2}\}. \tag{7.9}$$

Since (7.8) admits a unique solution when $t = 0$, we would like to use a degree theoretic argument to show that it also admits a solution when $t = 1$. The degree theory developed by Li ([18]) for second order fully nonlinear equations provides a framework for this approach. The first step is to compute the Leray-Schauder degree of the solution $u \equiv 0$ of (7.6). By Lemma 7.1 and the non-degeneracy of the linearization of (7.8), this degree is non-zero. The next step is to appeal to the homotopy invariance of the degree to conclude that (7.8) has a solution when $t = 1$. To justify this, however, we need to establish *a priori* bounds for solutions of (7.8). Basically, when $t < 1$ the integral term in (7.8) imposes L^∞-bounds on solutions, and as we saw before, this suffices.

To establish pointwise bounds, we argue by contradiction, and assume there is a sequence $t_j \to 1$ and solutions $\{u_j\}$ of (7.8) with $\max u_j \to \infty$. We then apply a standard blow-up argument by dilating coordinates. There are two crucial ingredients to this procedure:

1. The local estimates of Guan and Wang (see Proposition 5.1) allow us to prove that a suitably rescaled sequence of solutions converges to a solution on $\mathbf{R}^n$ of (5.5).

2. The classification of such solutions due to Li and Li [17]) imply that all are obtained by pulling back the round metric of the sphere (and its images under the conformal group) by stereographic projection. In particular, the volume of the resulting metric is equal to the volume of the sphere.

If the maximal volume of $[g]$ is strictly less than the volume of the sphere, we conclude that the blow-up could not have happened, and thus all solutions of (7.8) are bounded. This fact, along with the degree theory argument, gives the existence of solutions.

8 Final remarks: the conformally flat case

We close by mentioning some important work by other authors in the conformally flat setting.

In [8], Guan and Wang considered the parabolic version of (5.5). Assuming $k \neq n/2$, they were able to prove the long-time existence and convergence to a solution of (5.5) assuming the initial metric is admissible.

In a separate paper [10], the same authors considered equations involving ratios of symmetric polynomials. They also developed the interesting parallel between these quantities and the classical 'quermassintegrals', resulting in some fascinating Sobolev-type inequalities for admissible metrics.

On the elliptic side, in [17] Li and Li proved the existence of solutions to (5.5) for all $1 \leq k \leq n$, assuming g is admissible, and proved Liouville-type uniqueness theorems. They also considered much more general functions of the eigenvalues of the Weyl-Schouten tensor; these results have been very useful in giving a new proof of the main result in [3] (see [13]).

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