united nations educational, scientific and cultural organization

international atomic energy agency

the
abdus salam
international centre for theoretical physics

SMR1486/8

Workshop and Conference on Recent Trends in Nonlinear Variational Problems

(22 April - 9 May 2003)

Topological defects in Ginzburg-Landau functionals

F. Bethuel
Laboratoire J.L. Lions
Université Paris 6
F-75013 Paris
France

Topological defects in Ginzburg-Landau functionals

F. Bethuel

Abstract

In these notes we survey some results which relate the Ginzburg-Landau equation topological defects. We describe the various concentration phenomena underlying this analysis. The material of these notes is inspired by an earlier survey with L. Almeida and G. Orlandi [3].

1 Introduction

The asymptotic analysis for Ginzburg-Landau equations has been broadly investigated in the last decade. The purpose of this paper is to review some results both in the scalar and complex case. In particular we try to emphasize some analogies and differences between the two theories.

Our main focus will be the elliptic Ginzburg-Landau equation on a smooth, bounded domain Ω of $\mathbb{R}^N$

$$(\mathrm{GL})_\varepsilon \qquad -\Delta u_\varepsilon = -\frac{1}{\varepsilon^2}\nabla_u V(u_\varepsilon) \text{ in } \Omega\,,$$

for functions $u_\varepsilon : \Omega \to \mathbb{R}^d$, $N \geq 1$, $d \geq 1$, and V represents a non-convex smooth non-negative potential on $\mathbb{R}^d$. Here $\varepsilon > 0$ denotes a small parameter (a characterisitc lenght), and we are specially interested in the asymptotic limit $\varepsilon \to 0$.

This equation corresponds to the Euler-Lagrange equation for the Ginzburg-Landau functional

$$\mathcal{E}_\varepsilon(u) = \int_\Omega e_\varepsilon(u) = \int_\Omega \frac{|\nabla u|^2}{2} + \frac{V(u)}{\varepsilon^2} \qquad \text{for } u : \Omega \to \mathbb{R}^d.$$

The set

$$\Sigma = \{y \in \mathbb{R}^d\,,\ V(y) = 0\},$$

which we assume to be non-void, is sometimes called the vacuum manifold in the physical literature and plays an important role in the asymptotic analysis. Indeed, since the potential is non-negative, it achieves its infimum on Σ, and therefore the equation forces u_ε to take values close to Σ for small ε in appropriate energy regimes. This however might not be true uniformly on Ω. We will call defects the points where u_ε is far from Σ. The nature of the defects is essentially topological, and for that reason the topology of Σ will enter directly in the discussion.

The energy $\mathcal{E}_\varepsilon$ has been introduced in the early fifties by Ginzburg and Landau in order to describe phase transitions in condensed matter Physics (more precisely, at low temperature). The nature of the predicted defects (e.g. points, lines, walls) depends crucially on d and Σ (see [23]). Among the many variants of Ginzburg-Landau functionals, there are in particular those including electromagnetic effects, as for instance in superconductivity. Related models have been developed in particle physics (as for examples, Yang-Mills-Higgs theory).

In this paper we will focus on the cases $d = 1$ and $d = 2$ (i.e. u real or complex-valued). Moreover we assume that the potential is given by

$$V(u) = \frac{(1 - |u|^2)^2}{4}.$$

Note that in this case

$$\Sigma = \{-1, 1\} \qquad \text{if } d = 1\,, \qquad\qquad \Sigma = S^1 \qquad \text{if } d = 2\,,$$

where S^1 is the unit circle in $\mathbb{R}^2$. In the first case, i.e. $d = 1$, the non-connectedness of Σ yields typically codimension one defects, whereas in the second case, i.e. $d = 2$, Σ is not simply connected and allows for defects of codimension two. In Section 2 we will briefly show that the typical energy needed to observe a topological defect for $d = 1$ is of order ε^{-1}, whereas it is of order $|\log\varepsilon|$ for $d = 2$.

With this choice of potential, $(\mathrm{GL})_\varepsilon$ writes

$$(\mathrm{GL})_\varepsilon \qquad\qquad -\Delta u_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2)\,.$$

We assume throughout that the solutions u_ε verify the bound

$$(H_0) \qquad\qquad \mathcal{E}_\varepsilon(u_\varepsilon^0) \leq M_0 k_\varepsilon,$$

where M_0 is a fixed positive constant, and the definition of k_ε depends on the dimension d, namely we set

$$k_\varepsilon = \frac{1}{\varepsilon} \qquad \text{if } d = 1\,, \qquad\qquad k_\varepsilon = |\log\varepsilon| \qquad \text{if } d = 2\,.$$

The definition of k_ε in both cases $d = 1$ and $d = 2$ should be related to the energy cost needed for a single defect (we will develop this notion later). In order to analyze the asymptotic properties of solutions to $(\mathrm{GL})_\varepsilon$ we consider two kinds of objects.

The first ones describe the topological defects of u_ε: for $d = 1$ it is simply given by the gradient ∇u_ε, whereas for $d = 2$ it is the jacobian Ju_ε, defined as the 2-form

$$Ju_\varepsilon = du_\varepsilon^1 \wedge du_\varepsilon^2\,.$$

Although this may not be obvious at first glance, they are bounded in suitable norms independently of ε and therefore do not need any kind of renormalization. It can be shown (see Section 2) that in the asymptotic limit $\varepsilon \to 0$ they concentrate on codimension d rectifiable sets in Ω, called respectively the jump set and the vorticity set. This fact is not related to the equation $(\mathrm{GL})_\varepsilon$, but due only to the energy bound (H_0) and properties of the functional $\mathcal{E}_\varepsilon$. Passing to subsequences, the limiting object J_* is a bounded vector measure on Ω. In Section 2 we will discuss in more details the structure of J_*.

The second objects are the renormalized energy densities given by the Radon measures μ_ε, defined on Ω,

$$\mu_\varepsilon = \frac{e_\varepsilon(u_\varepsilon(x))}{k_\varepsilon}\, dx.$$

In view of assumption (H_0), μ_ε is a bounded measure, independently of ε. We may therefore assume, up to a subsequence $\varepsilon_n \to 0$, that there exists a Radon measure μ_* defined on Ω such that

$$\mu_\varepsilon \rightharpoonup \mu_* \qquad \text{as measures.}$$

In the asymptotic limit $\varepsilon \to 0$, there is a simple relation between the quantities introduced so far, namely

$$||J_*|| \le C_d|\mu_*|\,, \tag{1}$$

where $C_1 = \sqrt{2}/3$ and $C_2 = 1$. Moreover these bounds are sharp. This relation will be discussed in Section 2. The structure of μ_* is easier to analyze than that of J_*. Indeed, it is possible to derive directly equations governing μ_*, using $(\mathrm{GL})_\varepsilon$, whereas this is not so clear for J_*. The structure of μ_* can be summarized as follows.

Theorem 1. (Structure of μ_*) *There exists a subset Σ_μ in Ω, such that the following properties hold.*

i) Σ_μ is closed in Ω and for any compact subset $\mathcal{K} \subset \Omega \setminus \Sigma_\mu$

$$|u_\varepsilon(x)| \to 1 \qquad \textit{uniformly on } \mathcal{K} \textit{ as } \varepsilon \to 0.$$

Moroever,

$$\mathcal{H}^{N-d}(\Sigma_\mu) \le KM_0.$$

ii) The measure μ_* can be exactly decomposed as

$$\mu_* = g(x)\mathcal{H}^N + \Theta_*(x)\mathcal{H}^{N-d} \llcorner \Sigma_\mu. \tag{2}$$

iii) In case $d = 1$, $g \equiv 0$, while for $d = 2$, $g = |\nabla \Phi_*|^2$, where the function Φ_* is harmonic on Ω.

iv) the function $\Theta_*(\cdot)$ is bounded, and there exists $\eta > 0$ such that the set Σ_μ is $(N-d)$-rectifiable and

$$\Theta_*(x) = \Theta_{N-d}(\mu_*, x) = \lim_{r\to 0} \frac{\mu_*(B(x,r))}{\omega_{N-d} r^{N-d}} \ge \eta,$$

for $\mathcal{H}^{N-d}$ a.e. $x \in \Sigma_\mu$.

In the case $d = 1$, Theorem 1 has been proved by Hutchinson and Tonegawa [17] (see also Modica-Mortola [25] for the minimizing case and Ilmanen [18] for the corresponding heat flow).

In the case $d = 2$, Theorem 1 as stated has been proved in [8, 10], but the arguments have been developed during the last decade in particular in [7, 32, 11, 27, 21, 22, 8, 20, 1, 13, 9].

In view of the decomposition (2), μ_* can be split into two parts. A diffuse part $|\nabla \Phi_*|^2$, and a concentrated part

$$\nu_* = \Theta_*(x)\mathcal{H}^{N-2} \llcorner \Sigma_\mu \,.$$

An important difference between the scalar and the complex case is that in the scalar case there is no diffuse part (i.e. $g \equiv 0$). The presence of the diffuse term in the complex case is due to the possible oscillating behavior of the phase. This part is harmonic. In other words, in the complex case, the energy has two different modes:

- the linear mode, corresponding to Φ_*;
- the topological mode, corresponding to ν_*.

Concerning J_* we have also, as a consequence of (1),

$$supp\, J_* \subset \Sigma_\mu \,. \tag{3}$$

In some cases the inclusions in (3) are strict. Note also that in the critical dimension $N = d$ the concentration set Σ_μ reduces to a finite set, in particular the measures ν_* are given by a finite sum of Dirac masses with positive coefficients bounded from above and from below.

The next step is to derive further properties for the concentration set Σ_μ. In the critical dimension $N = d$, it turns out that the points of Σ_μ have to be critical for a new renormalized energy. If $N > d$, then we will see that the concentration set Σ_μ is stationary in the sense of varifolds (i.e. it is a generalized minimal surface).

A second class of problems of physical interest are the associated evolution problems, such as the nonlinear heat flow equation

$$(\mathrm{PGL})_\varepsilon \qquad \begin{cases} \frac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2) & \text{in } \Omega \times (0, +\infty), \\ u_\varepsilon(x,0) = u_\varepsilon^0(x) & \text{for a.e. } x \in \Omega\,, \\ u_\varepsilon(x,t) = g_\varepsilon(x) & \text{for a.e. } (x,t) \in \partial\Omega \times (0,+\infty), \end{cases}$$

and the nonlinear Schrödinger equation

$$(\mathrm{NLS})_\varepsilon \qquad \begin{cases} i\frac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2) & \text{in } \Omega \times (0, +\infty), \\ u_\varepsilon(x,0) = u_\varepsilon^0(x) & \text{for a.e. } x \in \Omega\,, \\ u_\varepsilon(x,t) = g_\varepsilon(x) & \text{for a.e. } (x,t) \in \partial\Omega \times (0,+\infty). \end{cases}$$

We will not discuss here the results obtained for these problems (except the existence problem for travelling waves in $(\mathrm{NLS})_\varepsilon$). Let us mention however, that their study relies heavily, for some aspects, on the analysis developed for the elliptic case.

2 Analysis of the topological defects

In this Section we review some results concerning the jumps and vorticity sets. As mentioned, the results here rely only on properties of the Ginzburg-Landau functionals $\mathcal{E}_\varepsilon$ and are completely independent of the equation $(\mathrm{GL})_\varepsilon$.

2.1 The scalar case

The properties of the Ginzburg-Landau functional $\mathcal{E}_\varepsilon$ in the scalar case $d = 1$ have been extensively investigated in the 80's, in particular by the De Giorgi school (starting with the seminal work by Modica-Mortola [25], and [24]) and also, motivated by physical questions, since the works by Gurtin and Sternberg [16, 31]. Let us consider families $\{v_\varepsilon\}_{0<\varepsilon<1}$ of scalar functions defined on Ω, verifying a bound of the form

$$\mathcal{E}_\varepsilon(v_\varepsilon) \leq M_0 k_\varepsilon = \frac{M_0}{\varepsilon}\,, \tag{4}$$

where $M_0 > 0$ is independent on ε. Clearly such a bound does not yield any control on the L^2 norm of the gradient. However, estimate (4) is sufficient to derive some compactness, in particular for the jump set. More precisely, the following holds.

Proposition 2.1. *Let $(v_\varepsilon)_{\varepsilon>0}$ a sequence such that*

$$\mathcal{E}_\varepsilon(v_\varepsilon) \leq \frac{M_0}{\varepsilon}\,.$$

Then, for a subsequence $\varepsilon_n \to 0$,

$$v_{\varepsilon_n} \to v_* \qquad \text{in } L^1(\Omega)\,,$$

where $v_*(x) \in \{-1, 1\}$ for a.e. $x \in \Omega$, and $v_* \in BV(\Omega)$.

Sketch of proof: we have

$$\frac{\varepsilon}{2}\int_\Omega |\nabla v_\varepsilon|^2 + \frac{1}{4\varepsilon}\int_\Omega (1-|v_\varepsilon|^2)^2 = \varepsilon\mathcal{E}_\varepsilon(v_\varepsilon) \leq M_0\,.$$

Hence, from the inequality $ab \leq \frac{1}{2}(a^2+b^2)$, it holds

$$\int_\Omega |\nabla v_\varepsilon||1-|v_\varepsilon|^2| \leq \frac{\varepsilon}{\sqrt{2}}\int_\Omega |\nabla v_\varepsilon|^2 + \frac{\sqrt{2}}{4\varepsilon}\int_\Omega (1-|v_\varepsilon|^2)^2 \leq \sqrt{2}M_0\,, \tag{5}$$

that is

$$\int_\Omega |\nabla \zeta(v_\varepsilon)| \leq \sqrt{2}M_0\,, \tag{6}$$

where $\zeta(t) = t - t^3/3$. This yields a uniform bound in $W^{1,1}$ for $\zeta(v_\varepsilon)$, and a subsequence $\zeta(v_{\varepsilon_n})$ converges therefore weakly in $BV(\Omega)$, hence strongly in L^1. So does $v_{\varepsilon_n} = \zeta^{-1}(\zeta(v_{\varepsilon_n}))$. Moreover, since $\zeta(v_*) = \frac{2}{3}v_*$, we have

$$\int_\Omega |\nabla v_*| = \frac{3}{2}\int_\Omega |\nabla\zeta(v_*)| \leq \frac{3\sqrt{2}}{2}M_0\,, \tag{7}$$

by (6) and lower semicontinuity of total variation. Hence $v_* \in BV(\Omega)$.

□

Notice that Proposition 2.1 states that ∇v_{ε_n} converges in $W^{-1,1}$ to $J_* = \nabla v_*$, and that the limiting jump set J_* is a bounded measure.

Remark 2.1. i) Let us emphasize that condition (4) does not imply that the sequence v_ε is bounded in BV. A simple example in dimension one is given by

$$v_\varepsilon(x) = 1 + \varepsilon^{1/2} \sin(\frac{x}{\varepsilon}), \qquad \text{for } -1 \le x \le 1.$$

Clearly the maps v_ε satisfy (4) but they are not equibounded in BV.

ii) On the other hand, one may prove that given any sequence v_ε satisfying (4) there exists another sequence $\tilde{v}_\varepsilon$ verifying also (4) which is equibounded in BV and which is close to the original sequence v_ε in the following sense:

$$||v_\varepsilon - \tilde{v}_\varepsilon||_{L^1(\Omega)} \to 0 \qquad \text{as } \varepsilon \to 0.$$

The main point is to get rid of the possible small oscillations of v_ε on the set where it takes values close to $+1$ and -1. This is achieved by a composition with a suitable projection on $\Sigma = \{-1, 1\}$.

The fact that $v_* \in BV(\Omega)$ and $|v_*| = 1$ a.e. in Ω yields some important properties for the jump set. In order to get some insight for this type of result, let us first consider the one dimensional case, which captures already some of the essential features of the problem.

2.1.1 The case $N = 1$

Let $\Omega = I$ be a bounded interval of $\mathbb{R}$. We have

Proposition 2.2. *Let $v \in BV(I)$, $|v| = 1$ a.e.. Then v has only a finite number ℓ of jumps $a_1, \ldots, a_\ell$, and there exists $\chi \in \{-1, 1\}$ such that*

$$v(x) = \chi \prod_{i=1}^{\ell} \left(\frac{x - a_i}{|x - a_i|} \right). \tag{8}$$

Proof. The result follows immediately from the definition of the BV norm in dimension one: it is the sum of the L^1 norm and the total variation V_I, defined by $V_I(v) = \sup\{\sum |v(x_{i+1}) - v(x_i)|\,,\ \{x_i\} \text{ partition of } I\}$. □

Remark 2.2. Note that if v_* is given by (8), then $J_* = \nabla v_* = 2\chi \sum_{i=1}^{\ell} (-1)^i \delta_{a_i}$, and in particular $||J_*|| = 2\ell$.

Next we show inequality (1) in dimension one, that is

Proposition 2.3. *i) Let v_* given by (8). Then for any sequence $(v_\varepsilon)_{0<\varepsilon<1}$ such that $v_\varepsilon \to v_*$ in L^1 as $\varepsilon \to 0$, we have*

$$\liminf_{\varepsilon\to 0} \varepsilon\mathcal{E}_\varepsilon(v_\varepsilon) \geq \frac{2\sqrt{2}}{3}\ell\,. \tag{9}$$

ii) The bound (9) is sharp, i.e. there exists a sequence $(u_\varepsilon)_{0<\varepsilon<1}$ such that $u_\varepsilon \to v_*$ in L^1, as $\varepsilon \to 0$, and

$$\lim_{\varepsilon\to 0} \varepsilon\mathcal{E}_\varepsilon(u_\varepsilon) = \frac{2\sqrt{2}}{3}\ell\,. \tag{10}$$

Proof. i) Going back to the first inequality in (5), we have

$$\int_\Omega |\zeta(v_\varepsilon)| \leq \sqrt{2}\varepsilon\mathcal{E}_\varepsilon(v_\varepsilon)\,. \tag{11}$$

On the other hand, $\zeta(v_\varepsilon) \to \zeta(v_*)$ in L^1, and lower semicontinuity of the total variation gives

$$\liminf_{\varepsilon\to 0} \int_\Omega |\nabla\zeta(v_\varepsilon)| \geq \int_\Omega |\nabla\zeta(v_*)|\,. \tag{12}$$

Since $\zeta(v_*) = \frac{2}{3}v_*$, we have $\int_\Omega |\nabla\zeta(v_*)| = \frac{4}{3}\ell$, and (9) follows.

ii) The main idea is to construct an optimal profile (on the whole of $\mathbb{R}$) for the transition from -1 to $+1$. Indeed, consider the problem

$$-\ddot{v} = v(1-v^2)\,, \qquad v(-\infty) = -1\,, \quad v(+\infty) = 1\,. \tag{13}$$

Actually, it is elementary to show that the solution is the unique minimizer (up to translations) of $\mathcal{E}_1$ subject to the above boundary conditions. It is explicitely given by the formula $v(x) = \tanh(\frac{x}{\sqrt{2}})$.

Next set

$$u_\varepsilon(x) = \chi \prod_{i=1}^{\ell} v\left(\frac{x-a_i}{\varepsilon}\right)\,. \tag{14}$$

A few computations show that $u_\varepsilon \to v_*$ in L^1, and

$$\varepsilon\mathcal{E}_\varepsilon(u_\varepsilon) = \frac{\sqrt{2}}{3}\ell + O(\exp(-\frac{K}{\varepsilon}))\,, \tag{15}$$

for some constant $K > 0$. □

Remark 2.3. Multiplying equation (13) by $\dot{v}$ we obtain the pointwise equality

$$2\dot{v}^2 = (1-v^2)^2\,. \tag{16}$$

This yields the equipartition of energy for u_ε

$$\frac{1}{2}\dot{u}_\varepsilon^2 = \frac{1}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2 + O(\exp(-\frac{K}{\varepsilon}))\,. \tag{17}$$

More generally, for any sequence w_ε verifying statement ii), it is elementary to prove equipartition of the energies

$$\int_\Omega \frac{|\nabla w_\varepsilon|^2}{2} = \int_\Omega \frac{(1-|w_\varepsilon|^2)^2}{4\varepsilon^2} + o(1)\,. \tag{18}$$

This equality holds also in higher dimensions (see Proposition 2.5).

We would like to draw the attention of the reader that in the scalar case considered here the **exact** form of the optimal profile plays a central role in the analysis. We will see that in the complex case the exact form of the optimal profile does not really enter in the corresponding theory.

Remark 2.4. In view of (15), we see that the interaction between jumps is exponentially weak.

2.1.2 The case $N \geq 2$

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$. As in dimension one, the fact that $v_* \in BV(\Omega)$ and $|v_*| = 1$ a.e. in Ω allows to deduce regularity properties for the jump set of v_*, which are best expressed in the language of Geometric Measure Theory. More precisely, we have

Proposition 2.4. *Let $v_* \in BV(\Omega)$, $|v_*| = 1$ a.e.. There exists a set $E \subset \Omega$ of finite perimeter in Ω, such that $v_* = 2\chi_E - 1$, where χ_E is the characteristic of E. In particular, the jump set of v_* is $(N-1)$-rectifiable, and $2Per_\Omega(E) = \int_\Omega |\nabla v_*| = ||J_*||$.*

Comment. i) We recall that a set $E \subset \mathbb{R}^N$ is k-rectifiable, for $1 \leq k \leq N$, if it has locally finite k-dimensional Hausdorff measure $\mathcal{H}^k$, and is contained, up to an $\mathcal{H}^k$-negligible set, in a countable union of k-dimensional surfaces of class $\mathcal{C}^1$. For such sets, the tangent space $\mathrm{Tan}(E,x)$ is well-defined in a measure theoretic sense for $\mathcal{H}^k$ a. e. $x \in E$. An important aspect of rectifiable sets is that they are limits of finite unions of k-dimensional polyhedral sets in a suitable weak norm.

ii) The proof of Proposition 2.4 is far from being elementary, and relies on De Giorgi's theory of finite perimeter sets. More precisely, let $w_* \in BV(\Omega)$ (so that Dw_* is a measure), and $|w_*| = 1$ a.e.. Let $\Omega_*^\pm = \{x \in \Omega\,,\ w_*(x) = \pm 1\}$. Then Dw_* is supported on the $(N-1)$-rectifiable set $\partial^*\Omega_*^\pm$, the reduced boundary of $\Omega_*^\pm$. [For the definition of reduced boundary, see e.g. [30]; the reduced boundary is included of the usual topological boundary. In the smooth case they actually coincide, but in general they may be different].

The N-dimensional analog of Proposition 2.3 is the following

Proposition 2.5. *i) Let $v_* \in BV(\Omega)$, $|v_*| = 1$ a.e.. Then for any sequence $(v_\varepsilon)_{0<\varepsilon<1}$ such that $v_\varepsilon \to v_*$ in L^1 as $\varepsilon \to 0$, we have*

$$\liminf_{\varepsilon \to 0} \varepsilon \mathcal{E}_\varepsilon(v_\varepsilon) \geq \frac{\sqrt{2}}{3}\int_\Omega |\nabla v_*| = \frac{\sqrt{2}}{3}||J_*|| = \frac{\sqrt{2}}{3}Per_\Omega(E)\,. \tag{19}$$

ii) The bound (19) is sharp, i.e. there exists a sequence $(u_\varepsilon)_{0<\varepsilon<1}$ such that $u_\varepsilon \to v_*$ in L^1, as $\varepsilon \to 0$, and

$$\lim_{\varepsilon\to 0} \varepsilon\mathcal{E}_\varepsilon(u_\varepsilon) = \frac{\sqrt{2}}{3}\int_\Omega |\nabla v_*| = \frac{\sqrt{2}}{3}||J_*|| = \frac{\sqrt{2}}{3} Per_\Omega(E)\,. \tag{20}$$

Comment. The previous proposition is a classical example of Γ-convergence (see [25])

Sketch of the proof. The proof of i) is identical to the proof of i) in Proposition 2.3.

The easiest way to prove ii) is to use an approximation of E by a set with a polyhedral boundary in Ω. Then the u_ε are constructed using essentially the optimal profile (rescaled at the level ε) in the orthogonal direction to the approximating boundary.

2.2 The complex case

Here we will consider $u : \Omega \to \mathbb{C} \simeq \mathbb{R}^2$, so that $\Sigma = \{y \in \mathbb{C},\ V(y) = 0\} = \{y \in \mathbb{C},\ |y| = 1\} = S^1$. A new type of singularity can appear here, due to the fact that $\pi_1(S^1) = \mathbb{Z} \neq 0$. Interesting new cases of topological defects appear therefore for planar Ω, i.e. for $N = 2$ (this is somewhat similar to the one dimensional case for scalar problems).

2.2.1 Vortices

We start the discussion here with a minimization problem which, in a vague sense, corresponds to the selection of optimal profiles. For that purpose, let $\Omega = D^2 = \{z \in \mathbb{C} \simeq \mathbb{R}^2\,,\ |z| = 1\}$, and consider a regular function

$$g : \partial\Omega = S^1 \to S^1$$

with winding number $d \neq 0$. In contrast to the scalar case, there is of course a large choice of boundary conditions verifying $|g| = 1$. Let us consider next the minimization problem

$$I_\varepsilon = \inf\{\mathcal{E}_\varepsilon(v)\,,\ v \in H^1_g(D^2;\mathbb{C})\}\,.$$

If $d \neq 0$, any minimizer for $\mathcal{E}_\varepsilon$ has to vanish at some points. Moreover, it can be proved that $H^1_g(D^2, S^1) = \emptyset$, and therefore I_ε diverges as $\varepsilon \to 0$. The asymptotic analysis here is of course more involved, since we have PDE's instead of ODE's. It was initiated in [7], where the following was established.

Proposition 2.6. *Assume $d > 0$, and let u_ε be a minimizer for $\mathcal{E}_\varepsilon$. Then we have*

$$\mathcal{E}_\varepsilon(u_\varepsilon) = \pi d|\log\varepsilon| + O(1)\,, \qquad \frac{1}{4\varepsilon^2}\int_\Omega (1 - |u_\varepsilon|^2)^2 = O(1)\,. \tag{21}$$

Moreover, there exists d points $a_1, \ldots, a_d$ in Ω, and a harmonic function $\varphi : \Omega \to \mathbb{R}$ such that $u_\varepsilon \to u_*$ as $\varepsilon \to 0$ in $W^{1,p}(\Omega)$ for any $p < 2$, and in $C^k_{loc}(\Omega \setminus \{a_1, \ldots, a_d\})$, where

$$u_*(z) = \exp(i\varphi(z)) \prod_{i=1}^{d} \frac{z - a_i}{|z - a_i|}\,.$$

The points a_i are usually called "vortices" (in analogy with the terminology of fluid dynamics). Since φ is harmonic, it is completely determined by the boundary condition and the location of the points a_d. As a matter of fact it can be proved that the configuration $(a_1, \dots, a_d)$ is not arbitrary, but minimizes a suitable renormalized energy (i.e. independent of ε). Again, the boundary condition enters in an essential way in the definition of this energy.

Remark 2.5. As the reader might already have noticed, there are strong analogies between the 1-dimensional scalar case and the planar complex case: clearly vortices and jumps play a somewhat similar role. Let us stress however a few differences:
i) the typical energy necessary to the formation of a vortex is of order $|\log \varepsilon|$, whereas for jumps it is ε^{-1};
ii) from (21) one sees that there is no energy balance in the complex case, and the diverging part of the energy is concentrated in the gradient term;
iii) in a (vague) sense, jumps do not "interact", whereas vortices do. Their interaction is governed by the renormalized energy.

Another striking difference concerns the way the theory has been developed in both cases. Indeed, PDE techniques have played an important role in the starting development for the complex case, while the emphasis was put first, for the scalar case, on variational methods (e.g. compactness, Γ-convergence...).

Remark 2.6. Consider the boundary condition $g(z) = z = Id_{S^1}$, which is the simplest possible with non-zero winding number.

It is natural, due to the symmetries in the problem, to seek solutions of the form

$$w_\varepsilon(z) = f_\varepsilon(r) \exp(i\theta) = f_\varepsilon(r) \frac{z}{|z|}$$

where $z = r\exp(i\theta)$ (in polar coordinates), and $f_\varepsilon : \mathbb{R}^+ \to \mathbb{R}$ is smooth and such that

$$f_\varepsilon(0) = 0, \qquad f_\varepsilon(r) = 1 \quad \text{for } r \geq \varepsilon, \qquad |f'| \leq 2\varepsilon^{-1}, \tag{22}$$

a simple computation shows that

$$I_\varepsilon \leq \mathcal{E}_\varepsilon(w_\varepsilon) \leq \pi |\log \varepsilon| + K\,,$$

which establishes the upper bound for I_ε. Actually, it has been proved that the minimizers u_ε, for small ε, do have radial symmetry [23, 26]. Moreover, as in the scalar case, we may define an optimal profile (although it is not given by an explicit formula). More precisely, there exists a unique function $f : \mathbb{R}^+ \to \mathbb{R}$ satisfying

$$\begin{cases} -f'' - \frac{1}{r}f' + \frac{1}{r^2}f = f(1-f^2) & \text{on } [0, +\infty) \\ f(0) = 0 & . \end{cases} \tag{23}$$

Then we have

$$u_\varepsilon(z) \simeq f(\frac{|z|}{\varepsilon}) \exp(i\theta)$$

and

$$u_\varepsilon(z) \to u_*(z) = \frac{z}{|z|} \qquad \text{as } \varepsilon \to 0, \qquad \text{in } W^{1,p}\,,\ p<2\,,$$

and in $C^k_{loc}(D^2 \setminus \{0\})$. The map $u_*(z) = z/|z|$ realizes thus the prototypical singularity that can appear in the asymptotics for minimization problems.

2.2.2 The quest of compactness

As in the scalar case, the energy bound $\mathcal{E}_\varepsilon(v_\varepsilon) \le M_0|\log\varepsilon|$ enables to derive some compactness for the sequence $(v_\varepsilon)_{0<\varepsilon<1}$. However the discussion is a little more involved. Indeed, a simple example shows that no general compactness result for reasonable norms can be derived, due to possible divergences in the phase. Take, for instance

$$w_\varepsilon(z) = \exp(i\varphi(z)\sqrt{|\log\varepsilon|})\,,$$

with $\varphi : \Omega \to \mathbb{R}$ a non-constant smooth function. We have $|w_\varepsilon| = 1$, hence

$$\mathcal{E}_\varepsilon(w_\varepsilon) = \frac{1}{2}\int_\Omega |\nabla w_\varepsilon|^2 = \frac{|\log\varepsilon|}{2}\int_\Omega |\nabla\varphi|^2 \le K|\log\varepsilon|\,.$$

On the other hand, $|\nabla w_\varepsilon| = O(|\log\varepsilon|^{1/2})$, so that any norm of the gradient will diverge as $\varepsilon \to 0$. Actually, even for solutions of the stationary Ginzburg-Landau equation, no compactness has to be expected even in L^1 (see [14]).

However, one may split the contribution of the "topological" part from the rest of the phase to assert, in analogy with Remark 2.1, ii), (see [2])

Proposition 2.7. *Let $M_0 > 0$ and $(v_\varepsilon)_{\varepsilon>0}$, $v_\varepsilon : \Omega \to \mathbb{C}$ such that*

$$\mathcal{E}_\varepsilon(v_\varepsilon) \le M_0|\log\varepsilon|\,.$$

Let $G \subset\subset \Omega$ be a smooth open simply connected set. Then, there exists a subsequence $\varepsilon_n \to 0$, ℓ points $a_1, \dots, a_\ell \in G$, integers $d_1, \dots, d_\ell \neq 0$, with $\sum_1^\ell |d_i| \le K'$, for some constant K' depending only on M_0, and functions $\varphi_{\varepsilon_n} : G \to \mathbb{R}$ such that

$$\int_G |\nabla\varphi_{\varepsilon_n}|^2 \le M_0|\log\varepsilon|$$

and

$$v_{\varepsilon_n}\cdot\exp(-i\varphi_{\varepsilon_n}) \to \prod_{i=1}^{\ell}\left(\frac{z-a_i}{|z-a_i|}\right)^{d_i} \qquad \text{in } H^s(G)\,,\ s<1.$$

Notice that in the previous example, $\ell = 0$ (i.e. there are no vortices) and taking $\varphi_\varepsilon = \varphi\cdot\sqrt{|\log\varepsilon|}$, one may write, as above,

$$w_{\varepsilon_n}\cdot\exp(-i\varphi_{\varepsilon_n}) \equiv 1\,.$$

Sketch of proof: the idea is to introduce a regularization of v_ε in order to get rid of possible "small dipoles" (i.e. pairs of vortices having opposite multiplicities and whose distance is say $o(\varepsilon^{1/2})$), and to keep only the "relevant" part of the vorticity of v_ε.

Assume for simplicity that $|v_\varepsilon| \leq 2$, and consider a minimizer w_ε of

$$F_\varepsilon(u) = \frac{1}{2}\int_\Omega \frac{|u - v_\varepsilon|^2}{\varepsilon} + \mathcal{E}_\varepsilon(u)\,, \qquad u \in H^1(\Omega;\mathbb{R}^2)\,.$$

Then w_ε verifies the perturbed Ginzburg-Landau equation

$$\frac{w_\varepsilon - v_\varepsilon}{\varepsilon} = \Delta w_\varepsilon + \frac{1}{\varepsilon^2} w_\varepsilon(1 - |w_\varepsilon|^2)\,. \tag{24}$$

One can easily show that $\mathcal{E}_\varepsilon(w_\varepsilon) \leq \mathcal{E}_\varepsilon(v_\varepsilon) \leq M_0|\log\varepsilon|$, and

$$\int_\Omega |w_\varepsilon - v_\varepsilon|^2 \leq 2M_0\varepsilon|\log\varepsilon|\,.$$

Performing a change of scale, and denoting

$$\begin{aligned} \tilde{\varepsilon} &= \varepsilon^{1/2} \\ \tilde{w}(x) &= w(\tilde{\varepsilon}x)\,, \end{aligned} \tag{25}$$

we are then led to the equation

$$\tilde{w}_\varepsilon - \tilde{v}_\varepsilon = \Delta\tilde{w}_\varepsilon + \frac{1}{\tilde{\varepsilon}^2}\tilde{w}_\varepsilon(1 - |\tilde{w}_\varepsilon|^2)\,, \tag{26}$$

and the left hand side in (26) is bounded in L^∞. Many techniques developed in the context of the stationary Ginzburg-Landau equation (see [7, 31, 11]) apply to (26). In particular, on G, the maps w_ε will have a finite number of vortices, bounded independently of ε. More precisely, for any $1/2 \leq \delta < 1$, there exists points $a_1^\varepsilon, \ldots, a_\ell^\varepsilon$, integers $d_1^\varepsilon, \ldots, d_\ell^\varepsilon$, and a constant $\lambda > 0$ such that $|w_\varepsilon| \geq \delta$ on $G \setminus \cup_1^\ell B(a_i^\varepsilon, \lambda\varepsilon)$, and

$$\frac{w_\varepsilon(z)}{|w_\varepsilon(z)|} = \exp(i\varphi_\varepsilon(z))\prod_{i=1}^{\ell}\left(\frac{z - a_i^\varepsilon}{|z - a_i^\varepsilon|}\right)^{d_i^\varepsilon} \qquad \text{on } G \setminus \cup_{i=1}^\ell B(a_i^\varepsilon, \lambda\varepsilon)\,, \tag{27}$$

where $\varphi_\varepsilon : G \to \mathbb{R}$ are suitable functions. Moreover, we have

$$\begin{aligned} ||w_\varepsilon - v_\varepsilon||_{L^2} &\leq C\varepsilon^{1/2}|\log\varepsilon|^{1/2} \\ ||\nabla(w_\varepsilon - v_\varepsilon)||_{L^2} &\leq C|\log\varepsilon|^{1/2}\,, \end{aligned} \tag{28}$$

so that, for $s < 1$, $||w_\varepsilon - v_\varepsilon||_{H^s} \leq C\varepsilon^\alpha$, for some $0 < \alpha < 1$, and after a few simple computations the conclusion follows. □

Comment. i) Theorem 2 shows that the possible lack of compactness is merely due to the phase (which is a real-valued function). On the other hand, the "topological" contribution due to the vortices is essentially compact.

ii) In view of the previous remark, some topological properties of the level sets of $\mathcal{E}_\varepsilon$ can be reduced to the properties of the level sets of the renormalized energy on the space of configurations of vortices (which is finite dimensional). This fact has been used in [2, 29, 33, 12] in order to find solutions to the stationary equation by variational methods (mountain pass, Ljusternik-Schnirelman theory, etc...).

2.2.3 Compactness for Jacobians

A related but conceptually different approach for locating the vorticity for maps v_ε satisfying the bound $\mathcal{E}_\varepsilon(v_\varepsilon) \le M_0|\log\varepsilon|$ has been proposed first in [20] and, independently, in [1].

The main idea here is to look at the Jacobians of v_ε, which allows to characterize its topological part. More precisely, for $v = (v^1, v^2) : \Omega \to \mathbb{R}^2$ a smooth map, its Jacobian Jv is the 2-form defined by

$$Jv = dv^1 \wedge dv^2 = \frac{1}{2} d(v^1 dv^2 - v^2 dv^1).$$

In two dimensions, it may be identified with a scalar function, namely

$$Jv = \det(\nabla v) = v_x \times v_y,$$

where, for $a, b \in \mathbb{R}^2$, $a \times b = a^1 b^2 - a^2 b^1$. Note that $v_x \times v_y = 0$ whenever v_x and v_y are colinear. Hence, when $|v| = 1$, we have $Jv \equiv 0$. In particular, oscillations in the phase of v are not "seen" by its Jacobian Jv.

It is then proved that

Proposition 2.8. *Let $v_\varepsilon : \Omega \to \mathbb{R}^2$ such that $\mathcal{E}_\varepsilon(v_\varepsilon) \le M_0|\log\varepsilon|$. Then there exists a subsequence $\varepsilon_n \to 0$, ℓ points $a_1, \dots, a_\ell \in \Omega$, and integers $d_1, \dots, d_\ell \ne 0$, with $\sum_1^\ell |d_i| \le K'$, for some constant K' depending only on M_0, such that*

$$Jv_{\varepsilon_n} \rightharpoonup J_* = \pi \sum_{i=1}^{\ell} d_i \delta_{a_i} \qquad \text{in } [\mathcal{C}_c^{0,\alpha}(\Omega)]^*, \text{ for any } \alpha > 0. \tag{29}$$

Remark 2.7. i) Recall that the corresponding result in the one dimensional scalar case would be $\dot{v}_{\varepsilon_n} \to 2\chi \sum_{i=1}^{\ell} (-1)^i \delta_{a_i}$ (see Remark 2.1).

ii) Proposition 2.8 could also be derived using Proposition 2.7. However, the approaches in [20, 1] are more complete and give also interesting results for higher energy levels than the ones considered here.

2.2.4 Γ-convergence

The following result, stated in [20, 1], has to be compared with Proposition 2.3.

Proposition 2.9. *i) Let J_* be as in (29). Then for any sequence $(v_\varepsilon)_{0<\varepsilon<1}$ such that $Jv_\varepsilon \rightharpoonup J_*$ in $[\mathcal{C}_c^{0,\alpha}(\Omega)]^*$ as $\varepsilon \to 0$, we have*

$$\liminf_{\varepsilon \to 0} \frac{\mathcal{E}_\varepsilon(v_\varepsilon)}{|\log\varepsilon|} \ge ||J_*|| = \pi \sum_{i=1}^{\ell} |d_i|. \tag{30}$$

ii) The bound (30) is sharp, i.e. there exists a sequence $(u_\varepsilon)_{0<\varepsilon<1}$ such that $Ju_\varepsilon \rightharpoonup J_*$ in $[\mathcal{C}_c^{0,\alpha}(\Omega)]^*$ as $\varepsilon \to 0$, and

$$\lim_{\varepsilon \to 0} \frac{\mathcal{E}_\varepsilon(u_\varepsilon)}{|\log\varepsilon|} = ||J_*|| = \pi \sum_{i=1}^{\ell} |d_i|. \tag{31}$$

Proof. i) If the l.h.s. of (30) is equal to infinity there is nothing to prove. Therefore we may assume without loss of generality that $\mathcal{E}_\varepsilon(v_\varepsilon) \leq M_0|\log\varepsilon|$. Thus, going back to Proposition 2.7, we have $\mathcal{E}_\varepsilon(w_\varepsilon) \leq \mathcal{E}_\varepsilon(v_\varepsilon)$, and $Jw_\varepsilon \to J_*$ as $\varepsilon \to 0$ by (28), and we may work now on w_ε instead of v_ε. The main advantage is that the vorticity of w_ε is located, in view of (27), in a finite number of disjoint balls of size ε, and $|w_\varepsilon| > \delta$ outside the balls, where $1/2 \leq \delta < 1$ is fixed. Then, elementary computations (see [7], Chapter 1) show that

$$\mathcal{E}_\varepsilon(w_\varepsilon) \geq |\log\varepsilon|\delta^2\pi\sum_{i=1}^{\ell}|d_i| - K\,, \tag{32}$$

for a constant $K > 0$ independent of ε. The conclusion follows by letting $\varepsilon \to 0$ and then $\delta \to 1$.

ii) For $j = 1,\ldots,d_i$, let $b^\varepsilon_{i,j} = a_i + |\log\varepsilon|^{-1}\exp(i2\pi j/d_i)$. Consider the map

$$u_\varepsilon(x) = \prod_{i=1}^{\ell}\prod_{j=1}^{d_i} f_\varepsilon(x - b^\varepsilon_{i,j})\frac{x - b^\varepsilon_{i,j}}{|x - b^\varepsilon_{i,j}|}\,, \tag{33}$$

where f_ε is defined as in (22). Elementary computations show that the sequence u_ε enjoys the desired properties.

□

2.2.5 The case $N \geq 3$

Since in dimension two vortices are points, and therefore codimension two defects, one expects, likewise, that in higher dimensions defects for the complex Ginzburg-Landau functional will concentrate on sets of codimension two. The following result, first proved in [20] gives a precise formulation of that.

Proposition 2.10. *Let $(v_\varepsilon)_{0<\varepsilon<1}$ be a sequence such that $\mathcal{E}_\varepsilon(v_\varepsilon) \leq M_0|\log\varepsilon|$. Then, for a subsequence $\varepsilon_n \to 0$,*

$$Jv_{\varepsilon_n} \rightharpoonup J_* \qquad in\ [\mathcal{C}^{0,\alpha}_c(\Omega)]^*\,, \tag{34}$$

where $\frac{1}{\pi}J_*$ is an $(N-2)$- (integer multiplicity) rectifiable current without boundary.

Comment. i) We recall some terminology from Geometric Measure Theory. A k-dimensional current on Ω is an element of the dual of the space of smooth k-forms with compact support in Ω. A k-current is called rectifiable if it can be represented by integration over a k-rectifiable set, with an integer valued density function.

ii) The proof of Proposition 2.10 in [20] relies on reduction to the two dimensional case by slicing arguments.

A different proof has been derived independently in [1]: the strategy is to approximate the Jacobian of v_ε by polyhedral currents with uniformly bounded mass, and then apply the classical Federer-Fleming compactness theorem.

The corresponding Γ-convergence result (i.e. the generalization of Proposition 2.9 to higher dimensions) is proved in [1].

To conclude Section 2, we emphasize once more that, for maps v_ε verifying the energy bound

$$\mathcal{E}_\varepsilon(v_\varepsilon) \leq M_0 k_\varepsilon ,$$

the topological defects concentrate on $N-d$-dimensional sets with some regularity (i.e. they are rectifiable). In view of inequalities (19), (30), the concentration set for defects is also a concentration set for the energy (however, for arbitrary maps, energy might concentrate outside J_*).

Finally, we also would like to point out that, even though J_* is rectifiable, its geometrical support might not be closed, so that in particular, the distributional support could be the whole domain.

3 Properties of $(\mathrm{GL})_\varepsilon$ in the scalar case

We consider first the case $\Omega = I$ is an interval of $\mathbb{R}$. Although this case is extremely simple, and although most of the questions reduce to ODE, we believe it gives some insight in the general picture.

3.1 The case $\Omega = I$ is an interval of $\mathbb{R}$

Let $\Omega = I$ be an interval of $\mathbb{R}$. Equation $(\mathrm{GL})_\varepsilon$ becomes the ODE

$$-\ddot{v}_\varepsilon = \frac{1}{\varepsilon^2} v_\varepsilon (1 - |v_\varepsilon|^2) \quad \text{in I} . \tag{35}$$

Multiplying (35) by $\dot{v}_\varepsilon$ we obtain the conservation law

$$\frac{1}{2}\dot{v}_\varepsilon^2 - \frac{1}{4\varepsilon^2}(1 - |v_\varepsilon|^2)^2 = C . \tag{36}$$

On the other hand, we have

$$-\frac{d^2}{dx^2}(v_\varepsilon^2 - 1) + \frac{2}{\varepsilon^2} v_\varepsilon^2 (v_\varepsilon^2 - 1) = -2\left|\frac{dv_\varepsilon}{dx}\right|^2 \leq 0 . \tag{37}$$

Concerning boundary conditions we are going to consider two special cases

A) $I = (-1, 1)\,, \quad v_\varepsilon(-1) = -1\,, \qquad v_\varepsilon(1) = 1$

B) $I = (0, 1)\,, \quad v_\varepsilon(0) = 0\,, \qquad v_\varepsilon(1) = 0$ (homogeneous case) .

In both cases, the maximum principle applies to (37), and yields

$$|v_\varepsilon| \leq 1 .$$

Boundary conditions A. In this case, we have the following result.

Proposition 3.1. *There exists a unique solution u_ε to (35) satisfying boundary conditions A. This solution is minimizing, increasing and odd (i.e. $u_\varepsilon(x) = -u_\varepsilon(-x)$ for each $x \in [-1,1]$). Moreover, $|\dot{u}_\varepsilon(x)| \leq \frac{C}{\varepsilon}$ for each $x \in [-1,1]$, where C is a constant independent of ε.*

Proof. One easily obtains a solution u_ε by minimization. Next let v_ε be an arbitrary solution of (35) verifying A. The main point is to establish the following claim

$$v_\varepsilon(0) = 0, \qquad \text{and } v_\varepsilon(x) \leq 0 \text{ on } [-1,0]. \tag{38}$$

To this aim, consider the point $x_0 = \inf\{x \in [-1,1] \; v_\varepsilon(x) = 0\}$. Suppose first that $x_0 < 0$: we will show that this leads to a contradiction. Set

$$w_\varepsilon(x) = v_\varepsilon(x) \quad \text{on } [-1, x_0] \quad \text{and } w_\varepsilon(x) = -v_\varepsilon(2x_0 - x) \quad \text{on } [x_0, 2x_0+1].$$

One verifies that w_ε is of class C^1, and since $t(1-t^2)$ is odd, w_ε solves the equation on $[-1, 2x_0+1]$. It follows by Cauchy-Lipschitz theorem, that $w_\varepsilon = v_\varepsilon$ on $[-1, 2x_0+1]$. In particular, $v_\varepsilon(2x_0+1) = w_\varepsilon(2x_0+1) = -v_\varepsilon(-1) = 1$.

On the other hand, by the maximum principle, $v_\varepsilon \leq 1$, hence $\dot{v}_\varepsilon(2x_0+1) = 0$. By unique continuation we have $v_\varepsilon \equiv 1$, a contradiction. We show likewise that the case $x_0 > 0$ is impossible, whence $x_0 = 0$, i.e. $v_\varepsilon(0) = 0$, and $v_\varepsilon(x) \leq 0$ on $[-1,0]$. The claim is proved.

The next step is to establish the uniqueness on $[-1,0]$ of a negative solution verifying $v_\varepsilon(-1) = -1$, $v_\varepsilon(0) = 0$. The other properties follow easily. □

As already mentioned, the parameter ε corresponds to a characteristic lenght. In order to study the asymptotic behavior of u_ε as $\varepsilon \to 0$, it is convenient to introduce the change of variables $x \to \frac{x}{\varepsilon}$, and the function $\tilde{u}_\varepsilon(x) = u_\varepsilon(\varepsilon x)$, in such a way that $\tilde{u}_\varepsilon$ verifies

$$-\ddot{\tilde{u}}_\varepsilon = \tilde{u}_\varepsilon(1 - |\tilde{u}_\varepsilon|^2) \qquad \text{on } I_\varepsilon = [-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}], \; \tilde{u}_\varepsilon(\pm\frac{1}{\varepsilon}) = \pm 1\,. \tag{39}$$

Proposition 3.2. i) *$\tilde{u}_\varepsilon(x) \to \tanh(\frac{x}{\sqrt{2}})$ uniformly on $\mathbb{R}$ as $\varepsilon \to 0$. In particular, $u_\varepsilon(x) \to u_*(x)$, where $u_*(x) = -1$ if $x < 0$, $u_*(x) = 1$ if $x > 0$.*

ii) *We have the energy balance* $$\frac{1}{2}\dot{u}_\varepsilon^2 = \frac{1}{4\varepsilon^2}(1 - u_\varepsilon^2)^2 + o(1)\,. \tag{40}$$

iii) *The energy diverges as ε^{-1}, more precisely* $$\mathcal{E}_\varepsilon(u_\varepsilon) = \frac{2\sqrt{2}}{3\varepsilon} + o(1)\,. \tag{41}$$

Proof. By classical arguments one easily shows that $\tilde{u}_\varepsilon$ has a uniform limit v verifying the optimal profile condition:

$$v \text{ is increasing}, \quad v(-\infty) = -1, \quad v(+\infty) = 1, \quad v(0) = 0, \quad -\ddot{v} = v(1-v^2) \quad \text{on } \mathbb{R}.$$

By the conservation law (36), we have $2\dot{v}^2 = (1-v^2)^2$. Since v is increasing, we deduce

$$\sqrt{2}\dot{v} = 1 - v^2\,, \tag{42}$$

and therefore, since $v(0) = 0$, we may integrate (42) to deduce that

$$\int_0^{v(x)} \frac{dv}{1-v^2} = \frac{x}{\sqrt{2}}.$$

This proves i). For ii) remark that, by i), $\dot{u}_\varepsilon(\pm 1) \to 0$, and (40) follows integrating the conservation law (36).

Finally, for iii) we write, by Cauchy-Schwarz inequality,

$$\sqrt{2}\mathcal{E}_\varepsilon(v) \geq \int_{-1}^{1} \dot{v}(1-v^2) = \int_{-1}^{1} \frac{d}{dx}(v - \frac{v^3}{3}) = \frac{4}{3}. \tag{43}$$

Since by (40) the two terms in (43) are "almost" equal, (41) follows. □

Homogeneous conditions B. Here v_ε verifies the equation

$$-\ddot{v}_\varepsilon = \frac{1}{\varepsilon^2} v_\varepsilon(1 - v_\varepsilon^2), \qquad v_\varepsilon(0) = v_\varepsilon(1) = 0. \tag{44}$$

It is straightforward to establish the existence of a minimizing solution u_ε. Note that $-u_\varepsilon$ is also a minimizing solution (since $\mathcal{E}_\varepsilon$ is an even functional).

Proposition 3.3. i) *If $\varepsilon \geq \pi^{-1}$, the only minimizing solution is zero. If $\varepsilon < \pi^{-1}$, then the set of minimizing solutions is given by $\{u_\varepsilon, -u_\varepsilon\}$, where u_ε is strictly positive on $(0,1)$.*
ii) *Every solution which does not change sign is minimizing.*

Proof. i) First notice that $\mathcal{E}_\varepsilon(0) = \frac{1}{4\varepsilon^2}$. For $v \in H_0^1([0,1])$, $v \neq 0$, we have the expansion

$$\mathcal{E}_\varepsilon(v) = \frac{1}{2}\int_0^1 (\dot{v}^2 - \frac{1}{\varepsilon^2}v^2) + \frac{1}{4\varepsilon^2}\int_0^1 (v^4 + 1) > \frac{1}{2}\int_0^1 (\dot{v}^2 - \frac{1}{\varepsilon^2}v^2) + \mathcal{E}_\varepsilon(0).$$

If $\varepsilon^{-2} \leq \pi^2$ (which is the smallest eigenvalue of the operator $-\ddot{v}$ on $H_0^1([0,1])$), we have, for any $v \in H_0^1([0,1])$,

$$\frac{1}{2}\int_0^1 (\dot{v}^2 - \frac{1}{\varepsilon^2}v^2) \geq 0,$$

and hence

$$\mathcal{E}_\varepsilon(v) > \mathcal{E}_\varepsilon(0) \qquad \text{for any } v \neq 0.$$

If $\varepsilon < \pi$, taking $v_\lambda = \lambda \sin(\pi x)$, we have

$$\mathcal{E}_\varepsilon(v_\lambda) = -(\frac{1}{\varepsilon^2} - \pi^2)\lambda^2 \int_0^1 \sin^2(\pi x) + \lambda^4 \int_0^1 \sin^4(\pi x) + \mathcal{E}_\varepsilon(0).$$

Choosing λ sufficiently small, we obtain $\mathcal{E}_\varepsilon(v_\lambda) < \mathcal{E}_\varepsilon(0)$.

For the last statement in i), it suffices to show that if u_ε is minimizing, then u_ε does not vanish: the conclusion will then follow from assertion ii), which we will prove later.

To see that u_ε does not vanish, notice that $|u_\varepsilon|$ is also minimizing, hence a solution of (44). In particular, if $|u_\varepsilon(x_0)| = 0$, then $\dot{u}_\varepsilon(x_0) = 0$. By unique continuation we obtain $u_\varepsilon \equiv 0$, a contradiction.

ii) Observe first that every positive solution has to be convex and symmetric with respect to $\frac{1}{2}$ (i.e. $v(x) = v(1-x)$). Next we claim:
if v_1 and v_2 are two distinct positive solutions such that

$$\dot{v}_1(0) > \dot{v}_2(0) \geq 0,$$

then we have $v_1 > v_2$.

To prove the claim, we argue by contradiction. Otherwise there would exists $x_0 \in (0, \frac{1}{2})$ such that $v_1(x_0) = v_2(x_0)$, $\dot{v}_1(x_0) \leq \dot{v}_2(x_0)$. By (36),

$$0 < [\dot{v}_1(0)]^2 - [\dot{v}_2(0)]^2 = [\dot{v}_1(x_0)]^2 - [\dot{v}_2(x_0)]^2,$$

a contradiction. To conclude, we multiply the equation for v_1 by v_2 and proceed also *vice versa*, then we substract and we obtain, after integration by parts,

$$\int_0^1 v_1 v_2 (v_1^2 - v_2^2) = 0.$$

This implies that either one of the solutions is identically zero, or they coincide. □

The previous proposition leads to a complete classification of **all** solutions. We denote by u_ε, pour $\varepsilon < \pi^{-1}$, the positive solution of problem B. We verify that $u_\varepsilon \to 1$ as $\varepsilon \to 0$, that $u_\varepsilon(\varepsilon x) \to \tanh(\frac{x}{\sqrt{2}})$, and that outside the boundary layer in 0 and in 1,

$$|u_\varepsilon(x) - 1| = O(\exp(\frac{-x}{\sqrt{2}\varepsilon}))$$

(i.e. convergence to 1 is exponentially fast). Moreover,

$$\mathcal{E}_\varepsilon(u_\varepsilon) = \frac{2\sqrt{2}}{3\varepsilon} + o(\exp(\frac{-c}{\varepsilon})),$$

for some constant $c > 0$.

For $k \in \mathbb{N}^*$ we consider next the interval $I_k = [0, k^{-1}]$, and the function

$$U_\varepsilon^k(x) = u_{k\varepsilon}(kx) \qquad \text{for } x \in I_k\,,$$

so that if $k\varepsilon < \pi^{-1}$, i.e. $\varepsilon < (k\pi)^{-1}$, U_ε^k is well defined. Then, we may construct a solution $u_\varepsilon^k(x)$ to (44), for $k < [(\pi\varepsilon)^{-1}]$, by setting

$$u_\varepsilon^k(x) = (-1)^i U_\varepsilon^k(x - \frac{i}{k}) \qquad \forall\, x \in [\frac{i}{k}, \frac{i+1}{k}]\,, \quad i = 1, \ldots, k\,. \tag{45}$$

Classical arguments then show

Proposition 3.4. *Let $N = [(\pi\varepsilon)^{-1}]$. Then problem (44) has* **exactly** *N pairs of non-zero solutions $\{u_\varepsilon^k\,,\,-u_\varepsilon^k\}$, given, for $k = 1,\ldots,N$, by (45).*

Remark that, for k fixed,

$$u_\varepsilon^k \to u_*^k \equiv \prod_{i=1}^{k-1}\left(\frac{x-a_i}{|x-a_i|}\right)\,, \quad \text{where } a_i = \frac{i}{k}\,, \qquad \text{as } \varepsilon \to 0\,. \tag{46}$$

This shows that the limit verifies $|u_*^k| = 1$ and has exactly $k-1$ equidistant jumps. In particular, the configuration of the jumps is not arbitrary as in general setting of Section 2, but is strongly imposed by the equation.

3.2 The case $\Omega \subset \mathbb{R}^N$, $N \geq 2$

A minimization problem for $\mathcal{E}_\varepsilon$. Here the equation becomes an elliptic PDE with Dirichlet boundary conditions g_ε.

We first consider a simple example where $\Omega = D^2$, and the boundary data $g_\varepsilon : \partial D^2 = S^1 \to \mathbb{R}$ has essentially two jumps, and $|g_\varepsilon| = 1$ away from the jumps. Take for instance, for $-\pi < \theta_1 < \theta_2 \leq \pi$ given,

$$g_\varepsilon(\exp(i\theta)) = f(\varepsilon^{-1}(\theta-\theta_1)) \qquad \text{if } \theta_1 - \varepsilon \leq \theta \leq \theta_1 + \varepsilon,$$

$$g_\varepsilon(\exp(i\theta)) = -f(\varepsilon^{-1}(\theta-\theta_2)) \qquad \text{if } \theta_2 - \varepsilon \leq \theta \leq \theta_2 + \varepsilon,$$

and $|g_\varepsilon(\exp(i\theta))| = 1$ otherwise, in such a way that $g_\varepsilon \in \mathcal{C}^\infty(S^1)$. Here $f : \mathbb{R} \to \mathbb{R}$ represents a $\mathcal{C}^\infty$ function such that $f(t) = 1$ if $t > 1$, $f(t) = -1$ if $t < -1$. We see that $g_\varepsilon \to g_*$ as $\varepsilon \to 0$, where $g_* = 1$ if $\theta \leq \theta \leq \theta_2$, $g_* = -1$ otherwise.

Let u_ε be a minimizer for $\mathcal{E}_\varepsilon$ on $H^1_{g_\varepsilon}(D^2)$. Following the Γ-convergence analysis of Section 2, suitably modified in order to take into account the boundary conditions (see for instance [25, 31]), one proves that u_ε converges in L^1 to u_*, where $|u_*| = 1$ in D^2, $u_* = 1$ in Ω_1, $u_* = -1$ in $D_1 \setminus \Omega$. Here $\Omega_1 = D^2 \cap P$, where P is the half-plane whose boundary is the line passing through $\exp(i\theta_1)$ and $\exp(i\theta_2)$ containing the segment $S = \{\exp(i\theta)\,,\ \theta_1 \leq \theta \leq \theta_2\}$. In other words, u_* is of modulus 1 and has a jump on the interface given by the segment S. Moreover,

$$\mathcal{E}_\varepsilon(u_\varepsilon) = \frac{2\sqrt{2}}{3\varepsilon}|\exp(i\theta_1) - \exp(i\theta_2)| + o(1) \qquad \text{as } \varepsilon \to 0\,,$$

i.e. the cost of the interface is the product of its lenght times the cost of a single jump in dimension one.

The minimality of u_ε is expressed at the level of the interface S by the fact that it represents the curve of minimal lenght joining the singularities of g_* (and is therefore a line segment).

In dimension ($N \geq 3$) a similar phenomenon appears: the interface S is a minimal surface bounded by the singular set of g_*. Moreover,

$$\mathcal{E}_\varepsilon(u_\varepsilon) \simeq \frac{2\sqrt{2}}{3\varepsilon}\mathcal{H}^{N-1}(S) \qquad \text{as } \varepsilon \to 0\,.$$

One thus sees that this problem is closely connected to a classical geometric problem, that is the Plateau problem for minimal surfaces.

Critical points of $\mathcal{E}_\varepsilon$. One may ask if the results obtained in dimension one through ODE techniques can be extended to higher dimensions. Concerning the homogeneous Dirichlet problem (i.e. $g_\varepsilon = 0$ on $\partial\Omega$), the following results can be established:

i) The number of pairs of solutions of opposite sign is at least L, the number of eigenvalues of the Laplacian strictly less than ε^{-2}. This partially extends the result in Proposition 3.4. The proof is more involved and relies on the fact that the functional is even, the analysis of the linearized operator near the special solution $u = 0$, and then on the use of genus theory.

It is however not clear that the number of pairs of solutions is exactly L. In particular, for a given $K > 0$, no lower bound on the number of solutions with energy smaller than $K\varepsilon^{-1}$ is available.

ii) The minimizing solution u_ε is positive and different from zero for $\varepsilon^2 < \lambda_1^{-1}$, where λ_1 represents the first eigenvalue of the Laplacian. This solution u_ε converges to 1, and we observe a boundary layer effect on $\partial\Omega$. Moreover,

$$\mathcal{E}_\varepsilon(u_\varepsilon) \simeq \frac{\sqrt{2}}{3\varepsilon}|\partial\Omega|.$$

Uniqueness of positive solutions for small ε has been established by Angenent [5]. The proof relies on Leray-Schauder degree theory.

iii) Consider next a sequence v_ε of solutions to $(\mathrm{GL})_\varepsilon$ verifying the energy bound (H_0). In view of the analysis of Section 2 it can be proved that, for a asubsequence ε_n, $v_{\varepsilon_n} \to v_*$, where $v_* \in BV(\Omega)$, $|v_*| = 1$. Moreover, the limiting jump set is a minimal surface in a generalized sens (more precisely, a stationary varifold), see [17].

4 Properties of $(\mathrm{GL})_\varepsilon$ in the complex case

4.1 The critical case $N = 2$

In contrast to the scalar case, there is of course a large choice of boundary conditions verifying $|g| = 1$. The general situation where Ω is a smooth bounded starshaped domain in $\mathbb{R}^2$ was analyzed in [7].

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded starshaped domain, $g : \partial\Omega \to S^1$ and $w_\varepsilon \in H^1_g(\Omega;\mathbb{R}^2)$ a critical point of the Ginzburg-Landau functional $\mathcal{E}_\varepsilon$. Then there exists $K > 0$ such that*

$$\mathcal{E}_\varepsilon(w_\varepsilon) \le K|\log\varepsilon|\,, \qquad \frac{1}{4\varepsilon^2}\int_\Omega (1 - |w_\varepsilon|^2)^2 \le K\,.$$

Moreover, there exists points $a_1, \dots, a_\ell$ in Ω, integers $d_1, \dots, d_\ell \neq 0$, with $\ell \leq K$, and a harmonic function $\varphi : \Omega \to \mathbb{R}$ such that $w_\varepsilon \to w_*$ in $W^{1,p}(\Omega)$ for any $p < 2$ and in $C^k_{loc}(\Omega \setminus \{a_1, \dots, a_\ell\})$, where

$$w_*(z) = \exp(i\varphi(z)) \prod_{i=1}^{\ell} \left(\frac{z - a_i}{|z - a_i|} \right)^{d_i} .$$

The integers d_i are the multiplicities of the vortices. Since φ is harmonic, it is completely determined by the boundary condition, the location of the points a_i and their multiplicities. As a matter of fact it can be proved that the configuration $(a_1, d_1), \dots, (a_\ell, d_\ell)$ is not arbitrary, but has to be critical for a suitable renormalized energy (i.e. independent of ε). Again, the boundary condition enters in an essential way in the definition of this energy.

4.2 The case $N \geq 3$

Since in dimension two vortices are points, and therefore codimension two defects, one expects, likewise, that in higher dimensions defects for the complex Ginzburg-Landau functional will concentrate on sets of codimension two.

Let Ω be a smooth bounded, simply connected domain in $\mathbb{R}^N$, $N \geq 3$. For $\varepsilon > 0$ a small parameter, consider solutions $w_\varepsilon : \Omega \to \mathbb{C}$ of the Ginzburg-Landau equation with Dirichlet data g_ε in $H^{1/2}(\partial\Omega; \mathbb{C})$. We assume moreover that there exist positive constants M_0, M_1, M_2 such that w_ε and g_ε verify conditions (H1), (H2), (H3) or (H3bis) below.

$$\mathcal{E}_\varepsilon(w_\varepsilon) \leq M_0 |\log \varepsilon| \,, \tag{H1}$$

$$||g_\varepsilon||^2_{H^{1/2}(\partial\Omega)} \leq M_1 \,, \tag{H2}$$

$$|g_\varepsilon| = 1 \qquad \text{a.e. in } \partial\Omega \,, \tag{H3}$$

$$\frac{1}{2} \int_{\partial\Omega} |\nabla g_\varepsilon|^2 + \frac{1}{4\varepsilon^2} \int_{\partial\Omega} (1 - |g_\varepsilon|^2)^2 \leq M_2 |\log \varepsilon| \,. \tag{H3bis}$$

Theorem 4.2. *Let w_ε be a solution of (GL_ε) satisfying $(H1)$, $(H2)$, $(H3)$ or $(H3bis)$. Then, for a subsequence $\varepsilon_n \to 0$, there exist a map $w_* \in W^{1,p}(\Omega)$, $\forall\, 1 \leq p < \frac{N}{N-1}$, and a map $g_* \in H^{1/2}(\partial\Omega)$ such that*

i) $|w_*| = 1$ *on* Ω, $|g_*| = 1\,, w_* = g_*$ *on* $\partial\Omega$*;*

ii) $w_{\varepsilon_n} \to w_*$ *in* $W^{1,p}(\Omega)$, $g_{\varepsilon_n} \rightharpoonup g_*$ *in* $H^{1/2}(\partial\Omega)$*;*

iii) $\mathrm{div}(w_* \times \nabla w_*) = 0$ *in* Ω*;*

iv) $\dfrac{e_{\varepsilon_n}(w_{\varepsilon_n})}{|\log \varepsilon_n|} \rightharpoonup \mu_*$ *as measures, where* μ_* *is a bounded measure on* $\bar{\Omega}$*.*

Set $\Sigma_\mu = supp(\mu_*)$*.*

v) Σ_μ *is a closed subset of* $\bar{\Omega}$ *with* $\mathcal{H}^{N-2}(\Sigma_\mu) < +\infty$*;*

vi) $w_* \in C^\infty(\Omega \backslash \Sigma_\mu)$ *and for any ball* $B(x_0, r)$ *included in* $\Omega \backslash \Sigma_\mu$ *there exists a function* $\varphi_* \in C^\infty(B(x_0, r))$, *such that* $\Delta \varphi_* = 0$, $w_* = \exp(i\varphi_*)$;

vii) $w_{\varepsilon_n} \to w_*$ *in* $C^k(K)$, *for any compact subset* K *of* $\Omega \setminus \Sigma_\mu$;

viii) Σ_μ *is* $\mathcal{H}^{N-2}$*-rectifiable;* μ_* *is a stationary varifold.*

Remark 4.1. i) Statement viii) uses crucially the analysis of Ambrosio and Soner [4].

ii) In case w_ε is minimizing, then Σ_μ is an area-minimizing integer multiplicity rectifiable current (see [21, 28, 20, 1]).

iii) More generally, the following equation has been considered:

$$i\,|\log \varepsilon| \vec{c}(x) \cdot \nabla w = \Delta w + \frac{1}{\varepsilon^2} w(1 - |w|^2) + |\log \varepsilon|^2 d(x) w\,, \tag{47}$$

where $\vec{c}$ and d are smooth functions on $\bar{\Omega}$. The Ginzburg-Landau equation for superconductivity, as well as the travelling wave equation for $(\mathrm{NLS})_\varepsilon$ are included as particular cases of this equation. In [9], an analysis similar to the analysis in Theorem 4.2 was carried out for equation (47). The main difference is that the varifold associated to μ_* is no longer stationary, but satisfies the curvature equation

$$\vec{H}(x) = \star \left(\vec{c}(x) \wedge \star \frac{dJ_*}{d\mu_*} \right), \qquad \text{for } \mu_*\text{-a.e. } x \in \Sigma_\mu\,,$$

where $\vec{H}(x)$ denotes the generalized mean curvature of the varifold associated to μ_* at x, the measure $J_* = d(w_* \times dw_*)$ is the Jacobian of w_*, and $\frac{dJ_*}{d\mu_*}$ is the Radon-Nykodim derivative of J_* with respect to μ_*.

References

[1] G. Alberti, S. Baldo and G. Orlandi, *Variational convergence for functionals of Ginzburg-Landau type*, preprint 2002.

[2] L. Almeida and F. Bethuel, *Topological methods for the Ginzburg-Landau equation*, J. Math Pures Appl. **11** (1998), 1–49.

[3] L. Almeida, F. Bethuel and G. Orlandi, *Interfaces et condensation de vorticité dans les modèles de Ginzburg-Landau*, Actes du Colloque SMT, (2002).

[4] L. Ambrosio and M. Soner, *A measure theoretic approach to higher codimension mean curvature flow*, Ann. Sc. Norm. Sup. Pisa, Cl. Sci. **25** (1997), 27-49.

[5] S. Angenent, *Uniqueness of the solution of a semilinear boundary value problem,* Math. Ann. **272** (1985), 129–138.

[6] F. Bethuel, J. Bourgain, H. Brezis and G. Orlandi, $W^{1,p}$ *estimates for solutions to the Ginzburg-Landau functional with boundary data in* $H^{\frac{1}{2}}$, C.R. Acad. Sci. Paris (I) **333** (2001), 1-8.

[7] F. Bethuel, H. Brezis and F. Hélein, *Ginzburg-Landau vortices*, Birkhäuser, Boston, 1994.

[8] F. Bethuel, H. Brezis and G. Orlandi, *Asymptotics for the Ginzburg-Landau equation in arbitrary dimensions*, J. Funct. Anal. **186** (2001), 432-520. Erratum **188** (2002), 548-549.

[9] F. Bethuel, G. Orlandi and D. Smets, *Vortex rings for the Gross-Pitaevskii equation*, J. Eur. Math. Soc., to appear.

[10] F. Bethuel, G. Orlandi and D. Smets, *Convergence of the parabolic Ginzburg-Landau equation to motion by mean curvature*, C. R. Acad. Sci. Paris (I) **336**, (2003) and preprint Lab. J.L. Lions (2003).

[11] F. Bethuel and T. Rivière, *A minimization problem related to superconductivity,* Annales IHP, Analyse Non Linéaire **12** (1995), 243-303.

[12] F. Bethuel and J.C. Saut, *Travelling waves for the Gross-Pitaevskii equation,* Annales IHP, Phys. Théor. **70**, (1999).

[13] J. Bourgain, H. Brezis and P. Mironescu, *On the structure of the Sobolev space $H^{\frac{1}{2}}$ with values into the circle,* C.R. Acad. Sci. Paris (I) **331** (2000), 119-124, and detailed paper in preparation.

[14] H. Brezis and P. Mironescu, *Sur une conjecture de E. De Giorgi relative à l'énergie de Ginzburg-Landau,* C. R. Acad. Sci. Paris Sér. I Math. **319** (1994), 167-170.

[15] H. Federer, *Geometric Measure Theory,* Springer, Berlin, 1969.

[16] M. Gurtin, *On a theory of phase transitions with interfacial energy,* Arch. Rat. Mech. Anal. **87** (1985), 187–212.

[17] J. Hutchinson and Y. Tonegawa, *Convergence of phase interfaces in the van der Walls-Cahn-Hilliard theory,* Calc. Var. and P.D.E. **10** (2000), 49–84.

[18] T. Ilmanen, *Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature,* J. Diff. Geom. **38** (1993), 417-461.

[19] R.L. Jerrard and H.M. Soner, *Scaling limits and regularity results for a class of Ginzburg-Landau systems,* Ann. Inst. H. Poincaré Anal. Non Linéaire **16** (1999), 423-466.

[20] R.L. Jerrard and H.M. Soner, *The Jacobian and the Ginzburg-Landau energy,* Calc. Var. PDE **14** (2002), 151-191.

[21] F.H. Lin and T. Rivière, *Complex Ginzburg-Landau equation in high dimension and codimension 2 area minimizing currents,* J. Eur. Math. Soc. **1** (1999), 237-311. Erratum, Ibid.

[22] F.H. Lin and T. Rivière, *A quantization property for static Ginzburg-Landau vortices,* Comm. Pure Appl. Math. **54** (2001), 206-228.

[23] P. Mironescu, *Les minimiseurs locaux pour l'équation de Ginzburg-Landau sont à symétrie radiale,* C.R. Acad. Sci. Paris (I) **323** (1996), 593-598.

[24] L. Modica, *The gradient theory of phase transitions and the minimal interface criterion,* Arch. Rat. Mech. Anal. **98** (1987), 123–142.

[25] L. Modica and S. Mortola, Un esempio di Γ-convergenza, Boll. Un. Mat. Ital. B **14** (1977), 285–299.

[26] F. Pacard and T. Rivière, *Linear and non linear aspects of vortices,* Birkhäuser, (2001).

[27] T. Rivière, *Line vortices in the $U(1)$ Higgs model,* ESAIM, C.O.C.V. **1** (1996), 77-167.

[28] E. Sandier, *Ginzburg-Landau minimizers from $\mathbb{R}^{N+1}$ to $\mathbb{R}^N$ and minimal connections,* Indiana Univ. Math. J. **50** (2001), 1807–1844.

[29] S. Serfaty, *Local minimizers for the Ginzburg-Landau energy near critical magnetic fields, I and II,* Comm. Contemp. Math. **1** (1999), 213-254 and 295-333.

[30] L. Simon, *Lectures on Geometric Measure Theory,* Proc. of the Centre for Math. Anal., Austr. Nat. Univ., 1983.

[31] P. Sternberg, *The effect of a singular perturbation on nonconvex variational problems,* Arch. Rat. Mech. Anal. **101** (1988), 209–260.

[32] M. Struwe, *On the asymptotic behavior of the Ginzburg-Landau model in 2 dimensions,* J. Diff. Equ. **7** (1994), 1613-1624, Erratum **8** (1995), 224.

[33] F. Zhou and Q. Zhou, *A remark on the multiplicity of solutions for the Ginzburg-Landau equation,* Ann. IHP, Analyse Nonlinéaire **16** (1999), 255-267.

Address

Fabrice Bethuel, Laboratoire Jacques-Louis Lions, Université de Paris 6, 4 place Jussieu BC 187, 75252 Paris, France & Institut Universitaire de France.
E-mail : `bethuel@ann.jussieu.fr`