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Topology and Sobolev Spaces - an Introduction

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1 Introduction

Degree theory is an important basic topic in topology which has many applications in different domains. Classically, the Brouwer degree was used in the study of continuous maps between manifolds or domains in finite dimension. A generalization to a certain class of continuous operators in infinite dimension, the Leray-Schauder degree, is very useful in the theory of nonlinear partial differential equations. In recent years degree theory was extended to a class of maps in finite dimensional spaces, which are possibly *discontinuous*. This was first done for some Sobolev spaces, which are limiting cases of the Sobolev imbedding, such as $W^{1,n}(S^n, S^n)$ and $H^{1/2}(S^1, S^1)$. Brezis and Nirenberg [9, 10] unified and generalized the previous results by defining a degree for VMO-maps, (following a suggestion of L. Boutet de Monvel and O. Gabber, see [5]). This is a class of maps that includes continuous maps as well as the limiting cases of Sobolev spaces.

In these notes we shall describe, for simplicity, only degree theory for maps from S^1 to S^1, although the theory extends to maps $u : X \to Y$ where X and Y are arbitrary smooth n-dimensional oriented manifolds without boundary. These notes are based mainly on [7]. A good survey on the subject is [8] and relevant research papers containing more advanced material are [9, 10, 4, 5].

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2 Degree for maps in $C(S^1, S^1)$

Consider a continuous map g from the unit circle S^1 to itself. The function $f : [0, 2\pi] \to S^1$ which is defined by $f(\theta) = g(e^{i\theta})$, $\forall \theta \in [0, 2\pi]$ is also continuous. Therefore, there exists a continuous *scalar* function $\phi : [0, 2\pi] \to \mathbb{R}$ such that

$$f(\theta) = e^{i\phi(\theta)}, \ \forall \theta \in [0, 2\pi].$$

Since $f(0) = f(2\pi)$ we must have $\phi(2\pi) - \phi(0) = 2\pi k$ for some $k \in \mathbb{Z}$. We then define the *degree* of g on S^1 by

$$\deg(g, S^1) = \deg g = k. \tag{2.1}$$

An immediate consequence of this definition is:

$$\deg g \neq 0 \implies \text{image}(g) = S^1. \tag{2.2}$$

Note that if $\deg g = 0$ then the associated ϕ above satisfies $\phi(0) = \phi(2\pi)$. We can then define a *single valued* and continuous function, $\phi_0 : S^1 \to \mathbb{R}$, by $\phi_0(e^{i\theta}) = \phi(\theta)$, which satisfies

$$g(x) = e^{i\phi_0(x)}, \ \forall x \in S^1. \tag{2.3}$$

A function ϕ_0 satisfying (2.3) is called a *lifting* of g. Actually it is easy to see that g has a lifting if and only if $\deg g = 0$.

An important property of the degree is:

$$\deg(g_1 g_2) = \deg g_1 + \deg g_2 \ , \ \ \deg(g_1/g_2) = \deg g_1 - \deg g_2, \quad \forall g_1, g_2 \in C(S^1, S^1). \tag{2.4}$$

Another important property is the stability of the degree with respect to small perturbations. If $u_1, u_2 \in C(S^1, S^1)$ satisfy $\|u_2 - u_1\|_\infty < 2$ then $\deg u_1 = \deg u_2$. Indeed, the map $v = u_2/u_1 \in C(S^1, S^1)$ satisfies $\|v - 1\|_\infty < 2$, so its image *does not* cover all of S^1. We must have then $\deg v = 0$, so that by (2.4), $\deg u_1 = \deg u_2$. This implies, in particular, that the degree remains constant under continuous homotopy, namely

$$H \in C(S^1 \times [0, 1], S^1) \implies \deg H(\cdot, 0) = \deg H(\cdot, 1). \tag{2.5}$$

Let us present two classical results which demonstrate how degree theory can help in proving existence of solutions for some equations.

Proposition 2.1. *Let D denote the unit disc in $\mathbb{R}^2$ and consider a continuous map $u \in C(\overline{D}, \mathbb{R}^2)$ such that its restriction to $S^1 = \partial D$ is an S^1-valued map g of degree $d \neq 0$. Then, there exists a point $x \in D$ with $u(x) = 0$.*

Proof. Arguing by contradiction, assume that there exists such a map u with $u \neq 0$ on D. Then, the map $v := u/|u| \in C(\overline{D}, S^1)$ and satisfies $v|_{\partial D} = g$. We can now define an homotopy $H \in C(S^1 \times [0,1], S^1)$ by

$$H(e^{i\theta}, r) = v(re^{i\theta})\,.$$

By (2.5) we have $\deg H(\cdot, 0) = \deg H(\cdot, 1)$. But this is a contradiction since $H(\cdot, 0) \equiv v(0)$ implies that $\deg H(\cdot, 0) = 0$ while $\deg H(\cdot, 1) = \deg g = d \neq 0$. □

We can now deduce a special case of Brouwer fixed point theorem.

Corollary 2.1. *Let T be a continuous self-map of $\overline{D}$. Then, T has a fixed point, i.e. there exists some $x_0 \in \overline{D}$ such that $Tx_0 = x_0$.*

Proof. We argue again by contradiction and assume that there exists a continuous $T : \overline{D} \to \overline{D}$ with no fixed point. Then,

$$\alpha := \min_{x \in \overline{D}} |x - Tx| > 0\,. \tag{2.6}$$

We define a new mapping $R : \overline{D} \to \partial D$ by setting $R(x)$ to be the point on ∂D hit by the ray emanating from Tx and passing through x. Using (2.6) it is easy to see that R is well defined and continuous. Moreover, $Rx = x$, $\forall x \in \partial D$, so that $\deg R|_{\partial D} = 1 \neq 0$. By Proposition 2.1 it follows that $Rx = 0$ for some $x \in D$. Contradiction. □

When $g \in C^1(S^1, S^1)$ the following explicit convenient formula for the degree is available:

$$\deg g = \frac{1}{2\pi} \int_{S^1} (g \wedge g_\tau)\, ds, \tag{2.7}$$

where $a \wedge b = a_1 b_2 - a_2 b_1$ (i.e. $a \wedge b$ is the $z-$component of the vectorial product $a \times b$ of the two planner vectors a and b) and g_τ denotes tangential derivative (in the positive sense). To prove (2.7), let $\phi : [0, 2\pi] \to \mathbb{R}$ be the function associated with g as above, i.e. $g(e^{i\theta}) = e^{i\phi(\theta)}$, $\forall \theta \in [0, 2\pi]$, so that $\deg g = (\phi(2\pi) - \phi(0))/(2\pi)$. Under our current assumption ϕ is a C^1-function, and we compute for all $\theta \in [0, 2\pi]$:

$$\begin{aligned} g(e^{i\theta}) \wedge \frac{d}{d\theta}\big(g(e^{i\theta})\big) &= e^{i\phi(\theta)} \wedge \frac{d}{d\theta}\big(e^{i\phi(\theta)}\big) \\ &= (\cos\phi(\theta), \sin\phi(\theta)) \wedge (-\phi'(\theta)\sin\phi(\theta), \phi'(\theta)\cos\phi(\theta)) = \phi'(\theta). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{2\pi}\int_{S^1} (g \wedge g_\tau)\, ds &= \frac{1}{2\pi}\int_0^{2\pi} g(e^{i\theta}) \wedge \frac{d}{d\theta}\big(g(e^{i\theta})\big)\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \phi'(\theta)\, d\theta = \frac{\phi(2\pi) - \phi(0)}{2\pi} = \deg g, \end{aligned}$$

as claimed.

We close this section with two other formulas for the degree. The first is the index formula (change of argument) for a function $g \in C^1(S^1, \mathbb{C} \setminus \{0\})$:

$$\deg g = \frac{1}{2\pi i}\int_{S^1} \frac{g_\tau}{g}\, ds. \tag{2.8}$$

To see that (2.8) coincides with the degree that we defined above, it suffices to note that when g is S^1-valued, $1/g = \bar{g}$, hence,

$$\begin{aligned} \frac{1}{2\pi i}\int_{S^1} \frac{g_\tau}{g}\, ds &= \frac{1}{2\pi i}\int_{S^1} \bar{g} g_\tau\, ds = \frac{1}{2\pi i}\int_0^{2\pi} e^{-i\phi(\theta)} \frac{d}{d\theta}\big(e^{i\phi(\theta)}\big)\, d\theta \\ &= \frac{1}{2\pi i}\int_0^{2\pi} e^{-i\phi(\theta)}(i\phi'(\theta))e^{i\phi(\theta)}\, d\theta = \frac{1}{2\pi}(\phi(2\pi) - \phi(0)). \end{aligned}$$

Next we present another formula for the degree, assuming that $g \in C^2(S^1, S^1)$, although it is valid for $g \in C^1(S^1, S^1)$, and actually, as we shall see below, also for more general classes of functions. For such g we consider *any* extension $u \in C^2(\overline{D}, \mathbb{R}^2)$ such that $u = g$ on $\partial D = S^1$ and claim that

$$\deg g = \frac{1}{\pi}\int_D u_x \wedge u_y\, dx\, dy\,. \tag{2.9}$$

To prove (2.9) we compute,

$$\begin{aligned}\frac{1}{\pi}\int_D u_x\wedge u_y &= \frac{1}{2\pi}\int_D \left[(u\wedge u_y)_x+(u_x\wedge u)_y\right]dx\,dy = \frac{1}{2\pi}\int_D \operatorname{div}(u\wedge u_y, u_x\wedge u)\,dx\,dy\\ &= \frac{1}{2\pi}\int_{\partial D}(u\wedge u_y, u_x\wedge u)\cdot\mathbf{n}\,ds = \frac{1}{2\pi}\int_{\partial D}(u\wedge u_y)\mathbf{n}_x+(u_x\wedge u)\mathbf{n}_y\,ds\\ &= \frac{1}{2\pi}\int_{\partial D}(u\wedge u_y)\boldsymbol{\tau}_y-(u_x\wedge u)\boldsymbol{\tau}_x\,ds = \frac{1}{2\pi}\int_{\partial D}u\wedge(u_x\boldsymbol{\tau}_x+u_y\boldsymbol{\tau}_y)\,ds\\ &= \frac{1}{2\pi}\int_{\partial D} g\wedge g_\tau\,ds,\end{aligned}$$

and the result follows from (2.7).

3 Degree for VMO maps

As pointed out by L. Boutet de Monvel and O. Gabber (see the appendix in [5]), a notion of degree makes sense for self-maps of S^1 which belong to the class VMO (= vanishing mean oscillation), although such maps are not necessarily continuous. This notion was later considerably developed and generalized to Riemannian manifolds in dimension N by Brezis and Nirenberg in [7, 8]. Here we restrict ourselves to degree theory in $VMO(S^1, S^1)$.

We start with the definition of the space $VMO(S^1,\mathbb{C})$. Consider a function $f\in L^1(S^1,\mathbb{C})$. For any $x\in S^1$ and $\varepsilon>0$ put

$$A_\varepsilon(x)=\{y\in S^1\ :\ |y-x|<\varepsilon\},$$

so that if $x=e^{i\phi(x)}$ then $A_\varepsilon(x)=\{e^{it}\ :\ t\in(\phi(x)-\alpha,\phi(x)+\alpha)\}$ with $\alpha=2\sin^{-1}(\varepsilon/2)$. Next define

$$\bar{f}_\varepsilon(x)=\fint_{A_\varepsilon(x)} f(y)\,dy := \frac{1}{|A_\varepsilon(x)|}\int_{A_\varepsilon(x)} f(y)\,dy,\quad \forall x\in S^1. \tag{3.1}$$

The function f belongs to $VMO(S^1,\mathbb{C})$ if

$$\lim_{\varepsilon\to 0}\fint_{A_\varepsilon(x)}|f(y)-\bar{f}_\varepsilon(x)|\,dy=0 \quad \text{uniformly in } x\in S^1. \tag{3.2}$$

It will be also convenient to use an equivalent condition to (3.2), namely:

$$\lim_{\varepsilon\to 0} ⨍_{A_\varepsilon(x)} ⨍_{A_\varepsilon(x)} |f(y)-f(z)|\,dy\,dz = 0 \quad \text{uniformly in } x\in S^1. \tag{3.3}$$

The equivalence follows from the inequalities:

$$\begin{aligned} ⨍_{A_\varepsilon(x)} |f(y)-\bar f_\varepsilon(x)|\,dy &\le ⨍_{A_\varepsilon(x)} ⨍_{A_\varepsilon(x)} |f(y)-f(z)|\,dy\,dz \\ &\le ⨍_{A_\varepsilon(x)} ⨍_{A_\varepsilon(x)} |f(y)-\bar f_\varepsilon(x)+\bar f_\varepsilon(x)-f(z)|\,dy\,dz \\ &\le 2 ⨍_{A_\varepsilon(x)} |f(y)-\bar f_\varepsilon(x)|\,dy\,. \end{aligned} \tag{3.4}$$

The basic property of S^1-valued VMO-maps which enables us to define the degree is given by the following lemma.

Lemma 3.1. *If $g\in VMO(S^1,S^1)$ then*

$$\lim_{\varepsilon\to 0} |\bar g_\varepsilon(x)| = 1, \quad \textit{uniformly in } x\in S^1. \tag{3.5}$$

Proof. For every $x\in S^1$ we have,

$$0 \le 1-|\bar g_\varepsilon(x)| = \operatorname{dist}(\bar g_\varepsilon(x),S^1) \le ⨍_{A_\varepsilon(x)} |g(y)-\bar g_\varepsilon(x)|\,dy,$$

and the conclusion follows from (3.2). □

By the above, for $g\in VMO(S^1,S^1)$ there exists $\varepsilon_0>0$ such that

$$|\bar g_\varepsilon(x)| \ge 1/2,\ \forall x\in S^1,\ \forall \varepsilon\le\varepsilon_0.$$

Therefore,

$$g_\varepsilon(x) = \frac{\bar g_\varepsilon(x)}{|\bar g_\varepsilon(x)|},\ \forall x\in S^1,\ \forall\varepsilon\le\varepsilon_0, \tag{3.6}$$

is well defined and belongs to $C(S^1,S^1)$. We define

$$\deg g = \deg g_\varepsilon \text{ for any } \varepsilon\le\varepsilon_0. \tag{3.7}$$

Note that $\deg g_\varepsilon = \deg g_{\varepsilon_0}$, $\forall\varepsilon<\varepsilon_0$, since g_{ε_0} can be connected to g_ε by the continuous homotopy $H(t,x) = g_{t\varepsilon+(1-t)\varepsilon_0}(x)$ (see (2.5)). Therefore, the definition (3.7) makes sense. In particular, for $g \in C(S^1, S^1)$ the definition in (3.7) coincides with the one given in Section 2 (since $\lim_{\varepsilon \to 0} \|g_\varepsilon - g\|_\infty = 0$). It can be shown (see [7]) that the VMO-degree is stable with respect to small perturbations in the VMO-metric:

$$d(f,g) = \sup_{\substack{x \in S^1 \\ \delta > 0}} \fint_{A_\delta(x)} |(f-g)(y) - (\overline{f-g})_\delta(x)| \, dy \,.$$

More precisely, for every $g \in VMO(S^1, S^1)$ there exists $\eta > 0$ such that for every $f \in VMO(S^1, S^1)$ with $d(f,g) < \eta$ we have $\deg f = \deg g$. Note however a significant difference with respect to the continuous case: η really depends on the map g and cannot be taken uniformly in the map (see [7]).

Examples of discontinuous maps in $VMO(S^1, S^1)$ will be given in the next section, once we show that $H^{1/2}(S^1, S^1) \subset VMO(S^1, S^1)$. We mention in passing that the map $g(x) = \exp(i|\log|x-1||^\alpha)$ is in $VMO(S^1, S^1)$ for every $\alpha \in (0,1)$, and its degree is zero. We close this section with a simple result characterizing $\mathbb{Z}$-valued VMO-maps. We shall use it in the study of lifting for $H^{1/2}$-maps.

Proposition 3.1. *Every map u in $VMO(S^1, \mathbb{Z})$ is constant.*

Proof. For $u \in VMO(S^1, \mathbb{Z})$ define $\bar{u}_\varepsilon$ as in (3.1). By the VMO-property (3.2),

$$\operatorname{dist}(\bar{u}_\varepsilon(x), \mathbb{Z}) \leq \fint_{A_\varepsilon(x)} |\bar{u}_\varepsilon(x) - u(y)| \, dy \to 0, \quad \text{as } \varepsilon \to 0, \text{ uniformly in } x \in S^1. \tag{3.8}$$

Since $\bar{u}_\varepsilon$ is continuous, it follows from (3.8) that there exist integers $\{m_\varepsilon\}_{\{\varepsilon > 0\}}$ such that $\|\bar{u}_\varepsilon - m_\varepsilon\|_\infty \to 0$. Using the convergence $\bar{u}_\varepsilon \to u$ in $L^1(S^1)$, it follows that $\{m_\varepsilon\}$ remains bounded as $\varepsilon \to 0$, hence $m_\varepsilon = m$ for $\varepsilon \leq \varepsilon_0$. Therefore, $u = m$ a.e. on S^1. □

4 The space $H^{1/2}$

An important space, which contains discontinuous functions, on which degree theory can be applied is $H^{1/2}(S^1, S^1)$. We first define,

$$H^{1/2}(S^1, \mathbb{C}) = \{f \in L^2(S^1, \mathbb{C}) \,:\, \int_{S^1}\int_{S^1} \frac{|f(x)-f(y)|^2}{|x-y|^2}\,dx\,dy < \infty\}\,, \tag{4.1}$$

and then set,

$$H^{1/2}(S^1, S^1) = \{g \in H^{1/2}(S^1, \mathbb{C}) \,:\, |g| = 1 \text{ a.e. on } S^1\}\,. \tag{4.2}$$

The standard semi-norm on $H^{1/2}(S^1, \mathbb{C})$ is given by

$$\|f\|^2_{H^{1/2}} = \int_{S^1}\int_{S^1} \frac{|f(x)-f(y)|^2}{|x-y|^2}\,dx\,dy\,. \tag{4.3}$$

As a norm on this space one can use

$$|||f|||^2 = \|f\|^2_{H^{1/2}} + \|f\|^2_{L^2}\,.$$

The $H^{1/2}$ space is only a special case ($s = 1/2$, $p = 2$) of the fractional Sobolev spaces $W^{s,p}$, see [1]. The next theorem gives a characterization of $H^{1/2}(S^1, \mathbb{C})$ in terms of Fourier series and harmonic extensions.

Theorem 1. *(i)For $f \in L^2(S^1, \mathbb{C})$ let $f(e^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ be the associated Fourier series and $u(re^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{in\theta} = \frac{1}{2\pi}(f * P_r)(e^{i\theta})$ its harmonic extension to the unit disc D ($P_r(e^{i\theta}) = \sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta}$ is the Poisson kernel). Then,*

$$f \in H^{1/2}(S^1, \mathbb{C}) \iff \sum_{n=-\infty}^{\infty} |n||a_n|^2 < \infty \iff u \in H^1(D, \mathbb{C}). \tag{4.4}$$

Moreover, when $f \in H^{1/2}(S^1, \mathbb{C})$ we have,

$$\|f\|^2_{H^{1/2}} = (2\pi)^2 \sum_{n=-\infty}^{\infty} |n||a_n|^2 = 2\pi \int_D |\nabla u|^2\,. \tag{4.5}$$

(ii) The space $H^{1/2}(S^1, \mathbb{C})$ consists exactly of all the functions f which are the trace of some harmonic function u in $H^1(D, \mathbb{C})$, in the following sense:

$$\lim_{r \nearrow 1} u_r(e^{i\theta}) = f(e^{i\theta}) \text{ in } L^2(S^1), \quad \text{where } u_r(e^{i\theta}) = u(re^{i\theta})\,. \tag{4.6}$$

Proof. Note first that,

$$\|f\|^2_{H^{1/2}} = \int_{S^1}\int_{S^1} \frac{|f(x)-f(y)|^2}{|x-y|^2}\,dx\,dy = \int_0^{2\pi}\int_0^{2\pi} \frac{|f(e^{i\tau})-f(e^{i\theta})|^2}{|e^{i\tau}-e^{i\theta}|^2}\,d\tau\,d\theta$$
$$= \int_0^{2\pi}\int_0^{2\pi} \frac{|f(e^{i\tau})-f(e^{i\theta})|^2}{|1-e^{i(\tau-\theta)}|^2}\,d\tau\,d\theta = \int_{-\pi}^{\pi}\int_0^{2\pi} \frac{|f(e^{i(\theta+h)})-f(e^{i\theta})|^2}{|1-e^{ih}|^2}\,d\theta\,dh\,. \tag{4.7}$$

Using

$$f(e^{i(\theta+h)}) - f(e^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}(e^{inh}-1)$$

we get,

$$\int_{-\pi}^{\pi}\int_0^{2\pi} \frac{|f(e^{i(\theta+h)})-f(e^{i\theta})|^2}{|1-e^{ih}|^2}\,d\theta\,dh = 2\pi\int_{-\pi}^{\pi}\sum_{n=-\infty}^{\infty}|a_n|^2\frac{|e^{inh}-1|^2}{|e^{ih}-1|^2}\,dh$$
$$= 2\pi\sum_{n=-\infty}^{\infty}|a_n|^2\cdot\int_{-\pi}^{\pi}\left(\frac{\sin\frac{nh}{2}}{\sin\frac{h}{2}}\right)^2 dh = (2\pi)^2\sum_{n=-\infty}^{\infty}|n||a_n|^2\,, \tag{4.8}$$

where we used the fact that for $m \geq 1$,

$$\frac{1}{m}\left(\frac{\sin\frac{mh}{2}}{\sin\frac{h}{2}}\right)^2 = K_m(h)$$

is the Fejér kernel, hence $\int_{-\pi}^{\pi} K_m(h)\,dh = 2\pi$. The first equivalence in (4.4) follows by combining (4.7) with (4.8).

Next we turn to the second equivalence in (4.4). By assumption, $\sum_{n=-\infty}^{\infty}|a_n|^2 < \infty$. This easily implies that $u \in L^2(D,\mathbb{C})$. Now $u \in H^1(D,\mathbb{C})$ if and only if

$$\sup_{R<1}\int_{\{|x|<R\}} |\nabla u|^2 < \infty\,. \tag{4.9}$$

But a direct computation yields

$$\int_{\{|x|<R\}} |\nabla u|^2 = 2\pi\sum_{-\infty}^{\infty}|n||a_n|^2 R^{2|n|}\,.$$

Hence, (4.9) is equivalent to $\sum_{n=-\infty}^{\infty}|n||a_n|^2 < \infty$, and the result follows.

(ii) Let $f \in H^{1/2}(S^1,\mathbb{C})$. By (i) we have $f(e^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ with $\sum_{-\infty}^{\infty}|n||a_n|^2 <$

∞. Therefore, the function $u(re^{i\theta}) = \frac{1}{2\pi}(f * P_r)(e^{i\theta})$ is harmonic in D and belongs to $H^1(D, \mathbb{C})$ by (i). Further, (4.6) holds since the Poisson kernel is a summability kernel, see [12].

Conversely, if u is harmonic in D, we may write $u(re^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{in\theta}$ for some $\{a_n\}_{n=-\infty}^{\infty}$ (one can use the Taylor expansion of $F + iG$ where F and G are analytic functions in D such that Re F = Re u and Re G = Im u). The assumption that $u \in H^1(D, \mathbb{C})$ implies that $\sum_{-\infty}^{\infty} |n||a_n|^2 < \infty$ i.e. $f(e^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta} \in H^{1/2}(S^1, \mathbb{C})$, and we conclude as above. □

Remark 4.1. Theorem 1 actually establishes the existence of a bounded linear operator $Tr : H^1(D) \to H^{1/2}(\partial D)$, called the *trace operator*, that satisfies $Tr\, u = u|_{\partial D}$ for $u \in H^1(D) \cap C(\overline{D})$. The result extends to smooth domains in $\mathbb{R}^N$, see [1].

Remark 4.2. From the definition (4.1) it can be easily seen that if Φ is globally Lipschitz and $f \in H^{1/2}(S^1, \mathbb{C})$ then $\Phi(f) \in H^{1/2}(S^1, \mathbb{C})$. Further, if $f_n \to f$ in $H^{1/2}(S^1, \mathbb{C})$, then $\Phi(f_n) \to \Phi(f)$ in $H^{1/2}(S^1, \mathbb{C})$ (this follows by the dominated convergence theorem, see Step 4 of the proof of Theorem 4 below).

Remark 4.3. We give some examples of discontinuous functions in $H^{1/2}$. We claim that $f(t) = \log|\log|t|| \in H^{1/2}((-R, R), \mathbb{R})$, $\forall R > 0$. Indeed, consider the extension of f to the disc $B_R(0) = \{|x| < R\}$ by $F(x) = f(|x|)$. Since

$$|\nabla F(x)| = \frac{1}{|x||\log|x||} \in L^2(B_R(0)),$$

we deduce that $F \in H^1(B_R(0))$. Consequently, its restriction to $(-R, R)$, f, belongs to $H^{1/2}((-R, R))$ as claimed. Using Remark 4.2 we get that $g(x) = e^{i \log|\log|x||} \in H^{1/2}(S^1, S^1)$. By a similar argument, $g_\alpha(x) = e^{i|\log|x||^\alpha} \in H^{1/2}(S^1, S^1)$ for $0 < \alpha < 1/2$.

Next we prove

Theorem 2. $H^{1/2}(S^1, \mathbb{C}) \subset VMO(S^1, \mathbb{C})$.

Proof. For $f \in H^{1/2}(S^1, \mathbb{C})$ and $\varepsilon > 0$ we have by the Cauchy-Schwarz inequality,

$$\begin{aligned} \fint_{A_\varepsilon(x)} \fint_{A_\varepsilon(x)} |f(y) - f(z)|\, dy\, dz &\leq \fint_{A_\varepsilon(x)} \fint_{A_\varepsilon(x)} \frac{|f(y) - f(z)|}{|y - z|} \cdot (2\varepsilon)\, dy\, dz \\ &\leq C \Big[\int_{A_\varepsilon(x)} \int_{A_\varepsilon(x)} \frac{|f(y) - f(z)|^2}{|y - z|^2}\, dy\, dz \Big]^{1/2}. \end{aligned} \tag{4.10}$$

Since the r.h.s. of (4.10) tends to zero with ε, uniformly in x, we deduce that condition (3.3) holds, hence $f \in VMO(S^1, \mathbb{C})$ as claimed. □

5 Lifting for H^1-maps

In the next section we will define a notion of degree for $H^{1/2}$-maps which is based on lifting. Since $H^{1/2}(\partial\Omega)$ is the trace space of $H^1(\Omega)$, we shall first investigate the lifting problem in $H^1(\Omega)$ where Ω is a domain in $\mathbb{R}^N$ for some $N \geq 2$ (although we will later use only the case $N = 2$). The next theorem is due to Bethuel and Zheng [2]. We give a proof based on Carbou [11] and Bourgain, Brezis and Mironescu [4].

Theorem 3. *Let Ω be a smooth, bounded and simply connected domain in $\mathbb{R}^N$ and $u \in H^1(\Omega, S^1)$. Then, there exists some $\phi \in H^1(\Omega, \mathbb{R})$ such that $u = e^{i\phi}$.*

The proof is based on the following generalized form of Poincaré's lemma.

Lemma 5.1. *Let Ω be as in Theorem 3 and let $f \in L^2(\Omega, \mathbb{R}^N)$. Then, the following properties are equivalent:*

(i) There is some $\phi \in H^1(\Omega, \mathbb{R})$ such that $f = \nabla\phi$.

(ii) One has,

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i} \quad \forall i, j,\ 1 \leq i, j \leq N, \tag{5.1}$$

in the sense of distributions, i.e.,

$$\int_\Omega f_i \frac{\partial \psi}{\partial x_j} = \int_\Omega f_j \frac{\partial \psi}{\partial x_i} \quad \forall \psi \in C_c^\infty(\Omega). \tag{5.2}$$

Proof. The implication (i)⇒(ii) is obvious, so we turn to the proof of (ii)⇒(i). Extend f to $\mathbb{R}^N$ by setting,

$$\bar{f}(x) = \begin{cases} f(x) & x \in \Omega, \\ 0 & x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

and let $\bar{f}_\varepsilon = \rho_\varepsilon * \bar{f} \in C^\infty(\mathbb{R}^N, \mathbb{R}^N)$ where $\{\rho_\varepsilon\}_{\varepsilon>0}$ is a family of mollifiers, i.e. $\rho_\varepsilon(x) = \frac{1}{\varepsilon^N}\rho(\frac{x}{\varepsilon})$ for some $\rho \in C_c^\infty(\mathbb{R}^N)$ satisfying $\rho \geq 0$ and $\int_{\mathbb{R}^N} \rho = 1$. For any $\omega \subset\subset \Omega$, smooth and simply connected domain, we have

$$\frac{\partial (f_\varepsilon)_i}{\partial x_j} = \rho_\varepsilon * \frac{\partial f_i}{\partial x_j} = \rho_\varepsilon * \frac{\partial f_j}{\partial x_i} = \frac{\partial (f_\varepsilon)_j}{\partial x_i} \quad \text{in } \omega, \ \forall i, j,$$

provided that $\varepsilon < \operatorname{dist}(\omega, \partial\Omega)$. Therefore, by the classical Poincaré's lemma there exists a function $\psi_\varepsilon \in C^\infty(\omega)$ such that $f_\varepsilon = \nabla\psi_\varepsilon$ in ω. Moreover, we can choose ψ_ε such that $\int_\omega \psi_\varepsilon = 0$. Since $\lim_{\varepsilon\to 0} f_\varepsilon = f$ in $L^2(\omega, \mathbb{R}^N)$, it follows that $\psi = \lim_{\varepsilon \to 0}\psi_\varepsilon$ exists in $H^1(\omega)$ and satisfies $\nabla\psi = f$.

Next we choose a sequence of such domains, $\omega^{(m)} \nearrow \Omega$. For each m there exists by the above, a function $\psi^{(m)} \in H^1(\omega^{(m)})$ such that $\nabla\psi^{(m)} = f$ in $\omega^{(m)}$. Setting for every m, $\bar{\psi}^{(m)} = c_m + \psi^{(m)}$ for an appropriate constant c_m, we get a new sequence $\{\bar{\psi}^{(m)}\}$ satisfying,

$$\bar{\psi}^{(n)} = \bar{\psi}^{(m)} \text{ on } \omega^{(m)} \quad \text{whenever } n > m.$$

Passing to the limit, we obtain a function $\phi \in H^1_{\text{loc}}(\Omega)$ satisfying $\nabla\phi = f$. By a theorem in Sobolev spaces, see Maz'ja [13], it follows that $\phi \in H^1(\Omega)$. □

Proof of Theorem 3. The basic idea behind the proof is the following. Assume a lifting ϕ does exists. Then we have, $u = u_1 + iu_2 = \cos\phi + i\sin\phi$. It follows that,

$$\nabla u_1 = -(\sin\phi)\nabla\phi = -u_2\nabla\phi \quad \text{and} \quad \nabla u_2 = (\cos\phi)\nabla\phi = u_1\nabla\phi\,,$$

i.e.,

$$\nabla\phi = u_1\nabla u_2 - u_2\nabla u_1\,. \tag{5.3}$$

Our strategy will be to find ϕ by solving (5.3), using Lemma 5.1.

Setting $f = u_1\nabla u_2 - u_2\nabla u_1 = (u_1\frac{\partial u_2}{\partial x_i} - u_2\frac{\partial u_1}{\partial x_i})_{i=1}^N$, we need to verify condition (5.1). Consider first the case of a smooth u. Then,

$$\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} = 2\Big(\frac{\partial u_1}{\partial x_j}\frac{\partial u_2}{\partial x_i} - \frac{\partial u_1}{\partial x_i}\frac{\partial u_2}{\partial x_j}\Big) = -2\det\begin{pmatrix} \frac{\partial u_1}{\partial x_i} & \frac{\partial u_2}{\partial x_i} \\ \frac{\partial u_1}{\partial x_j} & \frac{\partial u_2}{\partial x_j} \end{pmatrix}. \tag{5.4}$$

But differentiating the relation $|u|^2 = u_1^2 + u_2^2 = 1$ yields,

$$u_1\frac{\partial u_1}{\partial x_i} + u_2\frac{\partial u_2}{\partial x_i} = (u_1, u_2)\cdot(\frac{\partial u_1}{\partial x_i}, \frac{\partial u_2}{\partial x_i}) = 0, \quad \forall i = 1, \dots, N, \tag{5.5}$$

i.e. the vector (u_1, u_2) is orthogonal to $(\frac{\partial u_1}{\partial x_i}, \frac{\partial u_2}{\partial x_i})$ for all i. It follows that the vectors $\{(\frac{\partial u_1}{\partial x_i}, \frac{\partial u_2}{\partial x_i})\}_{i=1}^N$ are all collinear. Therefore, the r.h.s. of (5.4) is zero and (5.1) is satisfied.

The case of a general $u \in H^1(\Omega, S^1)$ will follow by approximation. By the density of smooth functions in $H^1(\Omega)$, see [1], there exist sequences $\{u_{1,n}\}, \{u_{2,n}\}$ in $C^\infty(\bar{\Omega}, \mathbb{R})$ such that $u_{1,n} \to u_1$ and $u_{2,n} \to u_2$ in $H^1(\Omega, \mathbb{R})$ and $\|u_{1,n}\|_\infty \le 1, \|u_{2,n}\|_\infty \le 1$ (note that we do not claim that $u_n = (u_{1,n}, u_{2,n})$ takes its values in S^1). Put $f_n = u_{1,n}\nabla u_{2,n} - u_{2,n}\nabla u_{1,n}$ so that $f_n \to f$ in $L^2(\Omega, \mathbb{R}^N)$. Indeed, to prove for example that $u_{1,n}\nabla u_{2,n} \to u_1\nabla u_2$ in L^2 write,

$$u_{1,n}\nabla u_{2,n} - u_1\nabla u_2 = u_{1,n}(\nabla u_{2,n} - \nabla u_2) + (u_{1,n} - u_1)\nabla u_2,$$

and apply the dominated convergence theorem. Next we claim that for all $1 \le i, j \le N$,

$$\frac{\partial (f_n)_i}{\partial x_j} - \frac{\partial (f_n)_j}{\partial x_i} = 2\Big[\frac{\partial u_{1,n}}{\partial x_j}\frac{\partial u_{2,n}}{\partial x_i} - \frac{\partial u_{1,n}}{\partial x_i}\frac{\partial u_{2,n}}{\partial x_j}\Big] \to 2\Big[\frac{\partial u_1}{\partial x_j}\frac{\partial u_2}{\partial x_i} - \frac{\partial u_1}{\partial x_i}\frac{\partial u_2}{\partial x_j}\Big] \text{ in } L^1(\Omega). \tag{5.6}$$

Indeed, to prove, for example, that $\frac{\partial u_{1,n}}{\partial x_j}\frac{\partial u_{2,n}}{\partial x_i} \to \frac{\partial u_1}{\partial x_j}\frac{\partial u_2}{\partial x_i}$ in L^1, write first,

$$\frac{\partial u_{1,n}}{\partial x_j}\frac{\partial u_{2,n}}{\partial x_i} - \frac{\partial u_1}{\partial x_j}\frac{\partial u_2}{\partial x_i} = \frac{\partial u_{1,n}}{\partial x_j}(\frac{\partial u_{2,n}}{\partial x_i} - \frac{\partial u_2}{\partial x_i}) + (\frac{\partial u_{1,n}}{\partial x_j} - \frac{\partial u_1}{\partial x_j})\frac{\partial u_2}{\partial x_i}. \tag{5.7}$$

Next use the Cauchy-Schwarz inequality to get,

$$\begin{aligned}\int_\Omega |\frac{\partial u_{1,n}}{\partial x_j}|\cdot|\frac{\partial u_{2,n}}{\partial x_i} - \frac{\partial u_2}{\partial x_i}| &\le \Big(\int_\Omega |\frac{\partial u_{1,n}}{\partial x_j}|^2\Big)^{1/2}\Big(\int_\Omega |\frac{\partial u_{2,n}}{\partial x_i} - \frac{\partial u_2}{\partial x_i}|^2\Big)^{1/2} \\ &\le C\Big(\int_\Omega |\frac{\partial u_{2,n}}{\partial x_i} - \frac{\partial u_2}{\partial x_i}|^2\Big)^{1/2} \to 0.\end{aligned}$$

The second term on the r.h.s. of (5.7) is treated similarly, and (5.6) follows.

Testing the equality in (5.6) against $\psi \in C_c^\infty(\Omega)$ yields,

$$-\int_\Omega \left((f_n)_i \frac{\partial \psi}{\partial x_j} - (f_n)_j \frac{\partial \psi}{\partial x_i}\right) = \int_\Omega 2\Big[\frac{\partial u_{1,n}}{\partial x_j}\frac{\partial u_{2,n}}{\partial x_i} - \frac{\partial u_{1,n}}{\partial x_i}\frac{\partial u_{2,n}}{\partial x_j}\Big]\psi\,.$$

Passing to the limit $n \to \infty$, using (5.6), yields

$$-\int_\Omega \left(f_i \frac{\partial \psi}{\partial x_j} - f_j \frac{\partial \psi}{\partial x_i}\right) = \int_\Omega 2\Big[\frac{\partial u_1}{\partial x_j}\frac{\partial u_2}{\partial x_i} - \frac{\partial u_1}{\partial x_i}\frac{\partial u_2}{\partial x_j}\Big]\psi\,. \tag{5.8}$$

But for any $u \in H^1(\Omega, S^1)$ we have, by the same argument as for smooth u,

$$\det \begin{pmatrix} \frac{\partial u_1}{\partial x_i} & \frac{\partial u_2}{\partial x_i} \\ \frac{\partial u_1}{\partial x_j} & \frac{\partial u_2}{\partial x_j} \end{pmatrix} = 0 \text{ a.e. on } \Omega. \tag{5.9}$$

Therefore, the r.h.s. of (5.8) vanishes and (5.2) follows. Applying Lemma 5.1 we get that there exists some $\phi \in H^1(\Omega, \mathbb{R})$ such that $f = \nabla\phi$. We will now prove that, up to an additive constant, ϕ is the required lifting.

Recall that if $g, h \in H^1(\Omega) \cap L^\infty(\Omega)$ then $gh \in H^1(\Omega) \cap L^\infty(\Omega)$ and

$$\frac{\partial}{\partial x_i}(gh) = g\frac{\partial h}{\partial x_i} + h\frac{\partial g}{\partial x_i}\,.$$

Therefore, $v = ue^{-i\phi} \in H^1(\Omega, S^1)$ satisfies

$$\begin{aligned}\nabla v &= e^{-i\phi}(\nabla u - iu\nabla\phi) = ue^{-i\phi}(\bar{u}\nabla u - i\nabla\phi)\\ &= ue^{-i\phi}(\bar{u}\nabla u - if) = ue^{-i\phi}(u_1\nabla u_1 + u_2\nabla u_2) = 0\,.\end{aligned}$$

Therefore, $v \equiv \text{const}$, and since $|v| = 1$ we have $v = e^{ic}$ for some constant $c \in \mathbb{R}$. Thus $u = e^{i(\phi+c)}$, i.e. $\phi + c$ is the required lifting. □

Remark 5.1. An analog to Theorem 3 holds when we replace $H^1(\Omega, S^1) = W^{1,2}(\Omega, S^1)$ by $W^{1,p}(\Omega, S^1)$ with $p \ge 2$. But the conclusion fails for every $1 \le p < 2$ (although an analog of Lemma 5.1 holds for any $1 \le p < \infty$). Assume for simplicity that $N = 2$ and assume w.l.o.g. that $0 \in \Omega$. It is easy to see that $u(x) = x/|x| \in W^{1,p}(\Omega, S^1)$ for every $1 \le p < 2$. We claim that there exists no $\phi \in W^{1,p}(\Omega, \mathbb{R})$ such that

$u = e^{i\phi}$. Assume by contradiction that such ϕ exists. Since $u \in C^\infty(\Omega \setminus \{0\})$ it follows that also $\phi \in C^\infty(\Omega \setminus \{0\})$. Fix $r > 0$ small such that $\{|x| \le r\} \subset \Omega$. Then, on $S_r = \{|x| = r\}$ we have $e^{i\phi(r,\theta)} = e^{i\theta}$. Therefore for some constant $k \in \mathbb{Z}$ we must have $\phi(r,\theta) = \theta + 2\pi k$. But this is impossible since ϕ is single-valued and continuous.

6 Existence of local lifting in $H^{1/2}$

Since by Theorem 2, $H^{1/2}(S^1, S^1) \subset VMO(S^1, S^1)$, every $g \in H^{1/2}(S^1, S^1)$ has a degree as defined in Section 3. We denote this degree temporarily by $\deg_V g$ and investigate in the sequel other definitions of degree for $H^{1/2}$-maps (we shall show later that they all coincide).

In the next section we shall define another notion of degree which is based on a local lifting. The rest of this section is devoted to the lifting problem. Consider a map $f \in H^{1/2}(I, S^1)$, where $I \subset \mathbb{R}$ is a bounded interval, i.e. we require that,

$$\int_I \int_I \frac{|f(x) - f(y)|^2}{|x-y|^2}\, dx\, dy < \infty\,.$$

Applying extension via reflections we may assume that $f \in H^{1/2}_{\text{loc}}(\mathbb{R}, S^1)$. The next lemma shows that such f can be extended to an H^1-map on a rectangle. We present the proof from [5].

Lemma 6.1. *For every $f \in H^{1/2}(I, S^1)$ there exists an $\varepsilon_0 > 0$ such that,*

$$F(x,\varepsilon) = f_\varepsilon(x) = \frac{\bar f_\varepsilon(x)}{|\bar f_\varepsilon(x)|} \in H^1(I \times (-\varepsilon_0, \varepsilon_0), S^1),$$

where for $\varepsilon \neq 0$ we define (analogously to (3.1)):

$$\bar f_\varepsilon(x) = \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} f(t)\, dt.$$

Proof. We first prove that $\bar F(x,\varepsilon) = \bar f_\varepsilon(x)$ belongs to $H^1(I \times (-\varepsilon_0, \varepsilon_0), \mathbb{C})$ for every $\varepsilon_0 > 0$. Note that $\bar F(x,\varepsilon) = \frac{1}{2\varepsilon}(\int_{x-\varepsilon}^{x} f(t)\, dt + \int_x^{x+\varepsilon} f(t)\, dt)$. Hence it suffices to show that $G(x,\varepsilon) := \frac{1}{\varepsilon}\int_x^{x+\varepsilon} f(t)\,dt \in H^1(I\times(-\varepsilon_0,\varepsilon_0),\mathbb{C})$. Clearly

$$\frac{\partial G}{\partial x} = \frac{1}{\varepsilon}(f(x+\varepsilon)-f(x)) \in L^2(I\times(-\varepsilon_0,\varepsilon_0),\mathbb{C}),$$

since $f \in H^{1/2}(I,\mathbb{C})$ implies that,

$$\int_{-\varepsilon_0}^{\varepsilon_0}\int_I \frac{|f(x+\varepsilon)-f(x)|^2}{\varepsilon^2}\,dx\,d\varepsilon < \infty\,.$$

Next we compute,

$$\begin{aligned}\frac{\partial G}{\partial x}(x,\varepsilon) - \frac{\partial G}{\partial \varepsilon}(x,\varepsilon) &= \frac{1}{\varepsilon^2}\int_x^{x+\varepsilon} f(t)\,dt - \frac{f(x)}{\varepsilon} = \frac{1}{\varepsilon^2}\int_0^{\varepsilon}(f(x+t)-f(x))\,dt \\ &= \frac{1}{\varepsilon^2}\int_0^{\varepsilon}\big(t\frac{\partial G}{\partial x}(x,t)\big)\,dt = \int_0^1 s\frac{\partial G}{\partial x}(x,\varepsilon s)\,ds.\end{aligned}$$

Therefore,

$$\Big\|\frac{\partial G}{\partial x} - \frac{\partial G}{\partial \varepsilon}\Big\|_{L^2} \le \int_0^1 s\Big\|\frac{\partial G}{\partial x}(x,s\varepsilon)\Big\|_{L^2}\,ds \le \frac{2}{3}\Big\|\frac{\partial G}{\partial x}\Big\|_{L^2},$$

where we used

$$\begin{aligned}\Big\|\frac{\partial G}{\partial x}(x,s\varepsilon)\Big\|_{L^2}^2 &= \int_{-\varepsilon_0}^{\varepsilon_0}\int_I \Big|\frac{\partial G}{\partial x}(x,s\varepsilon)\Big|^2\,dx\,d\varepsilon = \frac{1}{s}\int_{-s\varepsilon_0}^{s\varepsilon_0}\int_I \Big|\frac{\partial G}{\partial x}(x,\sigma)\Big|^2\,dx\,d\sigma \\ &\le \frac{1}{s}\int_{-\varepsilon_0}^{\varepsilon_0}\int_I \Big|\frac{\partial G}{\partial x}(x,\sigma)\Big|^2\,dx\,d\sigma = \frac{1}{s}\Big\|\frac{\partial G}{\partial x}\Big\|_{L^2}^2.\end{aligned}$$

Our claim that $\bar{F} \in H^1(I\times(-\varepsilon_0,\varepsilon_0),\mathbb{C})$ is established.

Next, $f \in H^{1/2}(I,S^1)$ implies that $f \in VMO(I,S^1)$ by the same argument as in Theorem 2 (the space $VMO(I,S^1)$ is defined analogously to the definition in (3.2)). Therefore,

$$\lim_{\varepsilon\to 0}|\bar{F}(x,\varepsilon)| = 1, \text{ uniformly in } x\in I.$$

In particular, there exists some $\varepsilon_0 > 0$ such that,

$$|\bar{F}(x,\varepsilon)| \ge \frac{1}{2}, \quad \forall x\in I,\ \forall\varepsilon\in(-\varepsilon_0,\varepsilon_0)\,.$$

It follows that $F = \bar{F}/|\bar{F}| \in H^1(I\times(-\varepsilon_0,\varepsilon_0),S^1)$ as required. □

An easy consequence is the existence of a lifting for maps in $f \in H^{1/2}(I,S^1)$.

Proposition 6.1. *For every $f \in H^{1/2}(I, S^1)$ there exists a function $\phi \in H^{1/2}(I, \mathbb{R})$ such that $f = e^{i\phi}$.*

Proof. Fix any bounded open interval J such that $I \subset\subset J$ and recall that we may assume that $f \in H^{1/2}(J, S^1)$. Applying Lemma 6.1 we obtain, for some $\varepsilon_0 > 0$, an extension $F \in H^1(J \times (-\varepsilon_0, \varepsilon_0), S^1)$. Let Ω be a smooth, simply connected domain such that,

$$I \times \{0\} \subset \Omega \subset J \times (-\varepsilon_0, \varepsilon_0)\,.$$

By Theorem 3 we may write $F = e^{i\psi}$ for some $\psi \in H^1(\Omega, \mathbb{R})$. The restriction $\phi = \psi|_I$ then belong to $H^{1/2}(I, \mathbb{R})$ (see Remark 4.1, it can also be deduced from Theorem 1) and satisfies

$$f = e^{i\phi} \text{ on } I. \tag{6.1}$$

□

7 Degree theory on $H^{1/2}(S^1, S^1)$

In this section we shall define different notions of degree for maps in $H^{1/2}(S^1, S^1)$ (in addition to the VMO-degree $\deg_V$ that was defined in the beginning of Section 6) and show that they all coincide.

We begin by showing how the results about local lifting from Section 6 can be used to define a degree for a map $g \in H^{1/2}(S^1, S^1)$. Consider the function $f(t) = g(e^{it})$, $\forall t \in \mathbb{R}$. Clearly $f \in H^{1/2}_{\text{loc}}(\mathbb{R}, S^1)$ and $f(t + 2\pi) = f(t)$ a.e.. By Proposition 6.1 there exists some $\phi \in H^{1/2}_{\text{loc}}(\mathbb{R}, \mathbb{R})$ such that $f = e^{i\phi}$, which is unique up to an additive constant in $2\pi\mathbb{Z}$, by Proposition 3.1, or rather by a variant of it for functions defined on an interval (since $H^{1/2} \subset VMO$ by Theorem 2). Clearly $\frac{1}{2\pi}(\phi(t+2\pi) - \phi(t)) \in \mathbb{Z}$ a.e. on $\mathbb{R}$. By Proposition 3.1 there exists a constant $k \in \mathbb{Z}$ such that,

$$\phi(t + 2\pi) - \phi(t) = 2\pi k \quad \text{a.e. on } \mathbb{R}. \tag{7.1}$$

Since ϕ is unique up to an additive constant it follows that the integer k is independent of the choice of ϕ, and it makes sense to define:

$$\deg_L g = k. \tag{7.2}$$

Next we define two additional notions of degree for a map $g \in H^{1/2}(S^1, S^1)$. The first one is a generalization of (2.9). Take any $u \in H^1(D, \mathbb{R}^2)$ such that $u = g$ on $\partial D = S^1$ in the sense of traces (one can take for example the harmonic extension, see Theorem 1). We set

$$\deg_E g = \frac{1}{\pi} \int_D u_x \wedge u_y \, dx \, dy. \tag{7.3}$$

To see why $\deg_E g$ is well defined, we need to prove that the r.h.s. of (7.3) is independent of the choice of u. Consider another $v \in H^1(D, \mathbb{R}^2)$ such that $v = g$ on ∂D, i.e. $w = v - u \in H_0^1(D, \mathbb{R}^2)$. Now,

$$\int_D (v_x \wedge v_y - u_x \wedge u_y) = \int_D w_x \wedge u_y + \int_D u_x \wedge w_y + \int_D w_x \wedge w_y \, . \tag{7.4}$$

It suffices to show that

$$\int_D w_x \wedge f_y = \int_D w_y \wedge f_x \quad \forall w \in H_0^1(D, \mathbb{R}^2), \ \forall f \in H^1(D, \mathbb{R}^2). \tag{7.5}$$

Indeed, applying (7.5) with $f = u$ and $f = w$ in (7.4) yields $\int_D v_x \wedge v_y = \int_D u_x \wedge u_y$. Consider first $w \in C_c^\infty(D, \mathbb{R}^2)$. Then,

$$\int_D w_x \wedge f_y = \int_D (w_x \wedge f)_y - \int_D w_{xy} \wedge f = -\int_D w_{xy} \wedge f$$

and

$$\int_D w_y \wedge f_x = \int_D (w_y \wedge f)_x - \int_D w_{xy} \wedge f = -\int_D w_{xy} \wedge f,$$

so that (7.5) holds. The general case follows by approximation.

Finally we give a definition generalizing (2.8). Consider $g \in C^\infty(S^1, S^1)$ with its Fourier series expansion

$$g(e^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta} \, .$$

Then, by (2.8)

$$\deg g = \frac{1}{2\pi i}\int_{S^1} g_\tau \bar g\, ds = \frac{1}{2\pi i}\int_0^{2\pi} \Big(\sum_{n=-\infty}^{\infty} ina_n e^{in\theta}\Big)\Big(\sum_{n=-\infty}^{\infty} \bar a_n e^{-in\theta}\Big)\, d\theta = \sum_{n=-\infty}^{\infty} n|a_n|^2 .$$

Motivated by the above we define for $g = \sum_{n=-\infty}^{\infty} a_n e^{in\theta} \in H^{1/2}(S^1, S^1)$:

$$\deg_F g = \sum_{n=-\infty}^{\infty} n|a_n|^2 . \tag{7.6}$$

Note that the series on the r.h.s. of (7.6) converges absolutely thanks to (4.4). The next theorem shows that all this four notions of degree coincide for $H^{1/2}$ maps.

Theorem 4. *For a map $g \in H^{1/2}(S^1, S^1)$ we have*

$$\deg_V g = \deg_L g = \deg_E g = \deg_F g := \deg g$$

and when $g \in H^{1/2} \cap C(S^1, S^1)$ we obtain the usual degree. Moreover, the map $g \mapsto \deg g$ is continuous from $H^{1/2}$ to $\mathbb{Z}$.

For the proof we need the following density result.

Lemma 7.1. *For every $g \in H^{1/2}(S^1, S^1)$, g_ε (as defined in (3.6))tends to g in $H^{1/2}$. Moreover, $C^\infty(S^1, S^1)$ is dense in $H^{1/2}(S^1, S^1)$.*

Proof. Put $\bar g_\varepsilon(x) = ⨍_{A_\varepsilon(x)} g(y)\, dy$ as in (3.1). Then,

$$\bar g_\varepsilon(e^{i\theta}) = \frac{1}{2\alpha_\varepsilon}\int_{-\alpha_\varepsilon}^{\alpha_\varepsilon} \sum_{n=-\infty}^{\infty} a_n e^{in(\theta+t)}\, dt = a_0 + \sum_{n\neq 0} a_n e^{in\theta}\Big(\frac{e^{in\alpha_\varepsilon} - e^{-in\alpha_\varepsilon}}{2in\alpha_\varepsilon}\Big),$$

with $\alpha_\varepsilon = 2\sin^{-1}(\frac{\varepsilon}{2})$. Therefore, by Theorem 1

$$\|\bar g_\varepsilon - g\|^2_{H^{1/2}} \le C\sum_{n\neq 0} |na_n^2|\Big|1 - \frac{e^{in\alpha_\varepsilon} - e^{-in\alpha_\varepsilon}}{2in\alpha_\varepsilon}\Big|^2 \to 0 \text{ as } \varepsilon \to 0.$$

Since $|\bar g_\varepsilon(x)| \to 1$ uniformly in x and the function $\Phi(\zeta) = \zeta/|\zeta|$ is Lipschitz on $\{|\zeta| > 1/2\}$ we deduce from Remark 4.2 that $g_\varepsilon \to g$ in $H^{1/2}$. Finally, for the assertion about density, given $\delta > 0$ we can find by the above $\varepsilon_1 > 0$ such that $\tilde g := g_{\varepsilon_1}$ satisfies $\|\tilde g - g\|_{H^{1/2}} < \delta/2$. Since $\tilde g \in H^{1/2} \cap C(S^1, S^1)$ we can find some $h \in C^\infty(S^1, S^1)$ such that $\|h - \tilde g\|_{H^{1/2}} < \delta/2$ by convolution with a mollifier and normalization. ∎

Proof of Theorem 4. The proof is carried out in 5 steps.

Step 1: The maps $g \mapsto \deg_E g$ and $g \mapsto \deg_F g$ are continuous from $H^{1/2}$ to $\mathbb{Z}$.

For $g, h \in H^{1/2}(S^1, S^1)$ let $u, v \in H^1(D, \mathbb{R}^2)$ be their respective harmonic extensions (see Theorem 1). Then,

$$\begin{aligned}\Big|\int_D (u_x \wedge u_y - v_x \wedge v_y)\Big| &\leq \Big|\int_D u_x \wedge (u_y - v_y)\Big| + \Big|\int_D (u_x - v_x) \wedge v_y\Big| \\ &\leq \|u - v\|_{H^1}(\|u\|_{H^1} + \|v\|_{H^1}) \\ &\leq C\|g - h\|_{H^{1/2}}(\|g\|_{H^{1/2}} + \|h\|_{H^{1/2}}),\end{aligned}$$

and the continuity of $\deg_E$ on $H^{1/2}$ follows.

Let $g(e^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ and $h(e^{i\theta}) = \sum_{n=-\infty}^{\infty} b_n e^{in\theta}$. Then,

$$\begin{aligned}|\deg_F g - \deg_F h| = |\sum_{n=-\infty}^{\infty} n(|a_n|^2 - |b_n|^2)| &\leq \sum_{n=-\infty}^{\infty} |n||a_n - b_n|(|a_n| + |b_n|) \\ &\leq C\|g - h\|_{H^{1/2}}(\|g\|_{H^{1/2}} + \|h\|_{H^{1/2}}),\end{aligned}$$

and the continuity of $\deg_F$ on $H^{1/2}$ follows as well. Using Lemma 7.1 we get that $\deg_F g, \deg_E g \in \mathbb{Z}$, $\forall g \in H^{1/2}(S^1, S^1)$.

Step 2: $\deg_E g = \deg_F g$, $\forall g \in H^{1/2}(S^1, S^1)$ and $\deg_E g = \deg g$, $\forall g \in H^{1/2}(S^1, S^1) \cap C(S^1, S^1)$.

The equality for smooth g extends to all of $H^{1/2}(S^1, S^1)$ by Step 1 and Lemma 7.1. The same argument, combined with (2.9), shows that $\deg_E g = \deg g$ for $g \in H^{1/2} \cap C(S^1, S^1)$.

Step 3: For $g \in H^{1/2}(S^1, S^1)$ with $\deg_L g = 0$ there exists $\psi \in H^{1/2}(S^1, \mathbb{R})$ such that $g = e^{i\psi}$.

By the definition of $\deg_L$ it follows that there exists some $\phi \in H^{1/2}_{\text{loc}}(\mathbb{R}, \mathbb{R})$ such that $f(t) = g(e^{it}) = e^{i\phi(t)}$ on $\mathbb{R}$, and in our case we have,

$$\phi(t + 2\pi) - \phi(t) = 2\pi \deg_L g = 0 \quad \text{a.e. on } \mathbb{R}.$$

We can then define on S^1, $\psi(e^{it}) = \phi(t)$, which satisfies $\psi \in H^{1/2}(S^1, \mathbb{R})$ and $g = e^{i\psi}$.

Step 4: $\deg_L g = \deg_E g = \deg_F g,\ \forall g \in H^{1/2}(S^1, S^1)$.

First note that $H^{1/2}(S^1, S^1)$ is an algebra since for $g, h \in H^{1/2}(S^1, S^1)$ we have

$$|g(x)h(x) - h(y)g(y)|^2 \leq 2\Big(|g(x) - g(y)|^2 + |h(x) - h(y)|^2\Big), \tag{7.7}$$

which leads to

$$\|gh\|^2_{H^{1/2}} \leq 2(\|g\|^2_{H^{1/2}} + \|h\|^2_{H^{1/2}}).$$

Moreover, if $g_n, h_n \in H^{1/2}(S^1, S^1)$ satisfy $g_n \to g$ and $h_n \to h$ in $H^{1/2}$, then $g_n h_n \to gh$ in $H^{1/2}$. Indeed, since we may assume after passing to a subsequence that $g_n \to g$ a.e. and $h_n \to h$ a.e., there exists a function $B(x,y) \in L^2(S^1 \times S^1)$ such that,

$$\frac{|g_n(x) - g_n(y)|}{|x-y|}, \frac{|h_n(x) - h_n(y)|}{|x-y|} \leq B(x,y), \text{ a.e. on } S^1 \times S^1,\ \forall n,$$

see [6, Théorème IV.9]. Therefore, by (7.7)

$$\frac{|g_n(x)h_n(x) - g_n(y)h_n(y)|}{|x-y|} \leq 2B(x,y), \text{ a.e. on } S^1 \times S^1,\ \forall n,$$

and we conclude by the dominated convergence theorem.

Next we claim that,

$$\deg_E(gh) = \deg_E g + \deg_E h \text{ and } \deg_L(gh) = \deg_L g + \deg_L h,\ \forall g, h \in H^{1/2}(S^1, S^1). \tag{7.8}$$

In fact, (7.8) is clear for smooth maps and the assertion concerning $\deg_E$ follow by density (see Lemma 7.1), Step 1 and the above discussion. On the other hand, the assertion in (7.8) for $\deg_L$ is clear from the definition (7.2).

Let $k = \deg_L g$, so that by (7.8), $\deg_L(gz^{-k}) = 0$. By Step 3 there exists $\psi \in H^{1/2}(S^1, \mathbb{R})$ such that $gz^{-k} = e^{i\psi}$. Therefore, by (7.8) we have,

$$\deg_E(gz^{-k}) = \deg_E\left((e^{i\psi/n})^n\right) = n \deg_E(e^{i\psi/n}), \quad \forall n \geq 1.$$

This implies that $\deg_E(gz^{-k}) = 0$ since otherwise we would get that $\deg_E(e^{i\psi/n}) \notin \mathbb{Z}$ for n large enough. Invoking once more (7.8) we finally obtain that $k = \deg_E g =$

$\deg_F g$.

Step 5: $\deg_E g = \deg_V g$, $\forall g \in H^{1/2}(S^1, S^1)$.

Fix any $g \in H^{1/2}(S^1, S^1)$. For $\varepsilon > 0$ small enough we have by definition, $\deg_V g = \deg g_\varepsilon$. Since $g_\varepsilon \in C \cap H^{1/2}(S^1, S^1)$, $\deg_E g_\varepsilon = \deg g_\varepsilon = \deg_V g$. Since $g_\varepsilon \to g$ in $H^{1/2}$ (by Lemma 7.1) and $\deg_E$ is continuous under $H^{1/2}$ convergence (see Step 1) we conclude that $\deg_V g = \deg_E g$. □

An immediate consequence is

Corollary 7.1. *Every $g \in H^{1/2}(S^1, S^1)$ of degree d can be written as*

$$g(z) = z^d e^{i\phi(z)}, \ \ z \in S^1,$$

for some $\phi \in H^{1/2}(S^1, \mathbb{R})$.

We conclude with an application to the Dirichlet problem for S^1-valued maps on the unit disc D. For $g \in H^{1/2}(S^1, S^1)$ we denote

$$H^1_g(D, \mathbb{R}^2) = \{u \in H^1(D, \mathbb{R}^2) \text{ s.t. } u = g \text{ on } \partial D\}$$

and

$$H^1_g(D, S^1) = \{u \in H^1_g(D, \mathbb{R}^2) \text{ s.t. } |u| = 1 \text{ a.e. in } D\}.$$

Theorem 5. *Let $g \in H^{1/2}(S^1, S^1)$. Then, $H^1_g(D, S^1) \neq \emptyset$ if and only if $\deg g = 0$.*

Proof. Assume first that $H^1_g(D, S^1) \neq \emptyset$ and take any $u \in H^1_g(D, S^1)$. Recall that $u_x \wedge u_y = 0$ a.e. on D (see (5.9)). Therefore, by (7.3)

$$\deg g = \deg_E g = \frac{1}{\pi} \int_D u_x \wedge u_y \, dx \, dy = 0.$$

Conversely, assume that $\deg g = 0$. By Step 3 in the proof Theorem 4 there exists $\psi \in H^{1/2}(S^1, \mathbb{R})$ such that $g = e^{i\psi}$. Let $v \in H^1_\psi(D, \mathbb{R})$ be the harmonic extension of ψ. Then clearly $u = e^{iv} \in H^1_g(D, S^1)$. □

Finally we present the solution to the minimization problem for the Dirichlet energy over $H^1_g(D,S^1)$ when $\deg g = 0$ (by Theorem 5 this is the only case for which the problem makes sense).

Theorem 6. *Let $g \in H^{1/2}(S^1,S^1)$ with $\deg g = 0$. Let $\psi \in H^{1/2}(S^1,\mathbb{R})$ be a lifting for g, i.e. $g = e^{i\psi}$, and let $v_0 \in H^1_\psi(D,\mathbb{R})$ denote its harmonic extension. Put $u_0 = e^{iv_0}$. Then,*

$$\int_D |\nabla u_0|^2 = \min\{\int_D |\nabla u|^2 \, : \, u \in H^1_g(D,S^1)\}. \tag{7.9}$$

Moreover, u_0 is the unique minimizer in (7.9).

Proof. For each $u \in H^1_g(D,S^1)$ there exists by Theorem 3 a $\phi \in H^1(D,\mathbb{R})$ such that $u = e^{i\phi}$. Let $\tilde{\phi} \in H^{1/2}(S^1,\mathbb{R})$ denote the trace of ϕ. Then,

$$g = e^{i\psi} = e^{i\tilde{\phi}} \text{ on } \partial D = S^1.$$

Hence by Theorem 2, $\frac{1}{2\pi}(\tilde{\phi} - \psi) \in H^{1/2}(S^1,\mathbb{Z}) \subset VMO(S^1,\mathbb{Z})$. By Theorem 3.1 it follows that for some $k \in \mathbb{Z}$ we have, $\tilde{\phi} = \psi + 2\pi k$, a.e.. Therefore, $u = e^{i\phi}$ with $\phi - 2\pi k \in H^1_\psi(D,\mathbb{R})$. Using $|\nabla u| = |\nabla \phi| = |\nabla(\phi - 2\pi k)|$ we deduce that,

$$\min\{\int_D |\nabla u|^2 \, : \, u \in H^1_g(D,S^1)\} = \min\{\int_D |\nabla v|^2 \, : \, v \in H^1_\psi(D,S^1)\}, \tag{7.10}$$

and any two minimizers, $\tilde{u}$ to the problem on the l.h.s. of (7.10) and $\tilde{v}$ to the problem on the r.h.s., are related via $\tilde{u} = e^{i\tilde{v}}$. But it is well-known that the problem on the r.h.s of (7.10) has a unique minimizer, namely v_0-the harmonic extension of ψ. Hence u_0 is the unique minimizer for the problem on the l.h.s. of (7.10), i.e. for (7.9). □

Remark 7.1. Consider a smooth $g : \partial D \to S^1$ of degree $d \neq 0$. Since $H^1_g(D,S^1)$ is empty the minimization problem (7.9) makes no sense. One may consider different ways to "get around" this topological obstruction by relaxation. In their fundamental work [3], Bethuel, Brezis and Hélein use the following approach. They denote for each $\varepsilon > 0$ by u_ε a minimizer for the Ginzburg-Landau type energy

$$E_\varepsilon(u) = \frac{1}{2}\int_D |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_D (1 - |u|^2)^2$$

over $H^1_g(D,\mathbb{C})$, and study the asymptotic behavior of $\{u_\varepsilon\}$ as ε goes to 0.

References

[1] R. Adams, Sobolev Spaces, Academic Press, 1975.

[2] F. Bethuel and X.M. Zheng, *Density of smooth functions between two manifolds in Sobolev spaces*, J. Funct. Anal. **80** (1988), 60–75.

[3] F. Bethuel, H. Brezis and F. Hélein, Ginzburg-Landau Vortices, Birkhäuser, 1994.

[4] J. Bourgain, H. Brezis and P. Mironescu, *Lifting in Sobolev spaces*, J. Anal. Math. **80** (2000), 37–86.

[5] A. Boutet de Monvel-Berthier, V. Georgescu and R. Purice, *A boundary value problem related to the Ginzburg-Landau model*, Comm. Math. Phys. **142** (1991), 1–23.

[6] H. Brezis, Analyse fonctionnelle, Masson, Paris, 1983.

[7] H. Brezis, Lectures on the Ginzburg-Landau Vortices, Scuola Normale Superiore, Pisa, 1996 (book in preparation).

[8] H. Brezis, *Degree theory: old and new*, Topological nonlinear analysis, II (Frascati, 1995), 87–108, Progr. Nonlinear Differential Equations Appl., 27, Birkhäuser Boston, Boston, MA, 1997.

[9] H. Brezis and L. Nirenberg, *Degree theory and BMO. I., Compact manifolds without boundaries*, Selecta Math. (N.S.) **1** (1995), 197–263.

[10] H. Brezis and L. Nirenberg, *Degree theory and BMO. II. Compact manifolds with boundaries, with an appendix by the authors and P. Mironescu*, Selecta Math. (N.S.) **2** (1996), 309–368.

[11] G. Carbou, *Applications harmoniques à valeurs dans un cercle* C. R. Acad. Sci. Paris Sér. I Math. **314** (1992), 359–362.

[12] Y. Katznelson, An introduction to harmonic analysis, Second corrected edition, Dover Publications, Inc., New York, 1976.

[13] V. Maz'ja, Sobolev Spaces, Springer Series in Soviet Mathematics, Springer-Verlag, Berlin, 1985.