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Vortex Analysis of the Ginzburg-Landau Model of Superconductivity

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Chapter 1

Introduction

These notes report on recent mathematical work [33, 34, 35, 36, 37] which aim at describing minimizers of the Ginzburg-Landau functional in the presence of an applied magnetic field in terms of *vortices*. For some part these results were already known to be true by physicists and applied mathematicians, but were only recently rigourously proved. Also the mathematical approach has made the knowledge more accurate, and has clarified the validity regime of certain formal calculations.

1.1 The Ginzburg-Landau model

The Ginzburg-Landau model was introduced in the 1950's as a semiphenomenological model describing superconductivity, before the phenomenon of superconductivity was actually explained from first principles by the Bardeen-Cooper-Schrieffer (or BCS) theory. In the Ginzburg-Landau (or GL) theory, superconductivity enters Maxwell's equations via an order parameter u which is a complex-valued function that can be interpreted in terms of the BCS theory as a condensed wave function of Cooper pairs. In a suitable normalization $|u|^2$ represents the local state of the material: $|u| = 0$ for a normal conductor and $|u| = 1$ for a superconductor.

For obvious reasons, much attention has been paid to the case where the superconducting material occupies an infinite cylinder and is surrounded by a perfect insulator (think of a wire). We may then assume translation invariance of the quantities involved along the axis of the cylinder and study the model in two dimensions. Letting Ω denote the section of the cylinder and using suitable units, the so-called Ginzburg-Landau energy assumes the form

$$J(u,A) = \frac{1}{2}\int_\Omega |(\nabla - iA)\,u|^2 + |\operatorname{curl} A - h_{\text{ext}}|^2 + \frac{\kappa^2}{2}\left(1 - |u|^2\right)^2. \qquad (1.1) \quad \boxed{\text{GL}}$$

Here $u : \Omega \to \mathbb{C}$ is the order parameter, $A : \Omega \to \mathbb{R}^2$ is the vector potential of the magnetic field (we are dealing with a static situation where there is no electric field), $h = \partial_1 A_2 - \partial_2 A_1$ is the induced magnetic field. The *covariant* gradient $(\nabla - iA)u$ that we also write $\nabla_A u$ is the vector with complex coordinates

$$\partial_1^A u = \partial_1 u - iA_1 u, \quad \partial_2^A u = \partial_2 u - iA_2 u.$$

There are two parameters present in the functional. The first is $h_{\text{ext}} \in \mathbb{R}$ which represents the intensity of the ambient magnetic field, assumed to be constant and oriented parallel to the axis of the cylinder. The second is the Ginzburg-Landau parameter $\kappa > 0$ which depends on the material considered.

Our interest will be in the phase diagram relative to those two parameters, i.e. in the description of the minimizers of (1.1) as h_{ext} and κ vary. We will concentrate on the description of the *mixed* state (neither superconducting or normal) predicted in 1959 by Abrikosov.

1.2 Gauge invariance

An infinite dimensional group leaves the functional $J(u, A)$ invariant, namely the group of gauge transformations.

Definition 1. *Two configurations (u, A), $(v, B) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)$ are said to be* gauge-equivalent *if there exists $f \in H^2(\Omega, \mathbb{R})$ such that $u = ve^{if}$ and $A = B + \nabla f$.*

Proposition 1. *If (u, A) and (v, B) are gauge-equivalent then $J(u, A) = J(v, B)$.*

Proof. Assume all the data is smooth, $u = ve^{if}, A = B + \nabla f$, with smooth f. Then $\nabla u = (\nabla v + i\nabla f v)e^{if}$ thus

$$\nabla_A u = (\nabla v + i\nabla f v)e^{if} - i(B + \nabla f)v = (\nabla v - iBv)e^{if}.$$

It follows that $|\nabla_A u|^2 = |\nabla_B v|^2$. It is clear that $|u| = |v|$ and that $\operatorname{curl} A = \operatorname{curl} B$. The proposition follows for smooth data. For the general result, density arguments pose no difficulty. □

1.3 Existence of minimizers.

Since J is a gauge invariant functional, a bound on $J(u, A)$ gives no estimate of the norm of u or A in a useful function space: a wild gauge transformation can allways be used to get a gauge-equivalent configuration with arbitrarily large H^1-norm, for instance. However we have

Proposition 2. *Let Ω be a bounded simply connected domain in $\mathbb{R}^2$. For any $v \in H^1(\Omega, \mathbb{C})$ and any $B \in H^1(\Omega, \mathbb{R}^2)$ there exists a gauge transformation $f \in H^2(\Omega, \mathbb{R})$ such that, letting $u(x) = v(x)\exp(if(x))$ and $A(x) = B(x) + \nabla f(x)$,*

$$div\, A = 0, \ \ in\ \Omega, \quad A.\nu = 0\ \ on\ \partial\Omega. \tag{1.2}$$ coul

This transformation is unique modulo a constant and the gauge thus chosen is called the Coulomb gauge.

Proof. Solve (see [10])

$$\begin{cases} \Delta f = -\operatorname{div} B & \text{in } \Omega \\ \partial_\nu f = -B.\nu & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$ coulbis

This is possible since $\int_\Omega \operatorname{div} B = \int_{\partial\Omega} B.\nu$ and the solution is unique modulo a constant. It is easy to see that $A = B + \nabla f$ will solve (1.2) and reciprocally that if $\nabla f = A - B$ and A solves (1.2) then f satisfies (1.3). □

The Coulomb gauge is interesting because it allows to bound the H^1 norm of the connection A in terms of the L^2-norm of its curl.

Proposition 3. *Let Ω be a bounded simply connected domain in $\mathbb{R}^2$. There exists $C(\Omega) > 0$ such that for any $A \in H^1(\Omega, \mathbb{R}^2)$ satisfying* (1.2)

$$\|A\|_{H^1} \le C \int_\Omega h^2,$$

where $h = \operatorname{curl} A$.

Proof. Solve $\Delta f = h$ in Ω and $f = 0$ on $\partial\Omega$. Then $\nabla^\perp f = (-\partial_2 f, \partial_1 f)$ satisfies $\operatorname{div} \nabla^\perp f = 0$ and $\operatorname{curl} \nabla^\perp f = h$ in Ω while $\nabla^\perp f.\nu = 0$ on $\partial\Omega$ therefore $A = \nabla^\perp f$ and since the H^2 norm of f is controlled by the L^2 norm of h from standard elliptic theory (see for instance [11]) the result follows. □

The above result is a simple instance of a general fact in the theory of connections on vector bundles ([44]). It makes the Coulomb gauge an indispensable tool and in our case yields the existence of minimizers for the Ginzburg-Landau functional.

existence **Proposition 4.** *Let Ω be a smooth bounded domain in $\mathbb{R}^2$. Then for any value of h_{ext}, κ there exists a minimizer of J over $H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)$.*

Proof. We only sketch the proof, see [10] for details. Take a minimizing sequence $(u_n, A_n)_{n\in\mathbb{N}}$, and assume the Coulomb gauge condition (1.2) is satisfied for each n. Then since $h_n = \operatorname{curl} A_n$ is bounded in L^2 by the energy, and from the previous proposition, the sequence $\{A_n\}_{n\in\mathbb{N}}$ is bounded in H^1. The energy bound also yields a bound of $|u_n|$ in L^4. Then the energy bounds $\nabla_{A_n} u_n$ in L^2 which implies — since A_n is bounded in L^4 by Sobolev imbedding, and u_n also — that ∇u_n is bounded in L^2. Finally u_n and A_n are bounded in H^1 and the result follows by checking that J is weakly lower semicontinuous in H^1. □

1.4 Euler-Lagrange equations

We will need the following identities which are the basis for calculus with gauge-invariant quantities. We denote by (v, w) the scalar product of two complex numbers, i.e. $2(u, v) = u\bar{v} + \bar{u}v$.

Proposition 5. *For any $u, v : \Omega \to \mathbb{C}$ and any $A : \Omega \to \mathbb{R}^2$, letting $h = \operatorname{curl} A$,*

$$\partial_i(u, v) = (\partial_i^A u, v) + (u, \partial_i^A v), \quad (\partial_1^A \partial_2^A - \partial_2^A \partial_1^A) u = ihu. \tag{1.4}$$ id

The proof is left as an exercise.

We will also use the following notation

$$(iu, \nabla_A u) = \left((iu, \partial_1^A u), (iu, \partial_2^A u)\right), \quad (\nabla_A)^2 u = (\partial_1^A \partial_1^A + \partial_2^A \partial_2^A) u, \quad \nabla^\perp = (-\partial_2, \partial_1). \tag{1.5}$$ not

Now we are ready to prove

Proposition 6. *A critical point* (u, A) *of* (1.1) *satisfies*

$$\begin{cases} -(\nabla_A)^2 u = \kappa^2 u(1-|u|^2) & \text{in } \Omega \\ -\nabla^\perp h = (iu, \nabla_A u) & \text{in } \Omega \\ h = h_{ext} & \text{on } \partial\Omega \\ \nu.\nabla_A u = 0 & \text{on } \partial\Omega. \end{cases} \qquad (1.6) \quad \boxed{\texttt{gl}}$$

Here we have written ∇_A *for the operator* $\nabla - iA$, $h = \operatorname{curl} A$ *and* ν *is the outward pointing unit normal to* $\partial\Omega$.

Proof. Let $(v, B) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)$ be a variation of (u, A) then

$$\frac{d}{dt}_{|t=0} J(u+tv, A+tB) = \int_\Omega (\nabla_A u, \nabla_A v - iBu) + (h - h_{\text{ext}}) \operatorname{curl} B - \kappa^2 (u,v)(1-|u|^2) = 0.$$

Using (1.4), we find

$$(\nabla_A)^2 u = \operatorname{div}(\nabla_A u, v) = \big((\nabla_A)^2 u, v\big) + (\nabla_A u, \nabla_A v),$$

hence integration by parts yields

$$-\int_\Omega \big((\nabla_A)^2 u + \kappa^2 u(1-|u|^2), v\big) - \int_\Omega \Big((\nabla_A u, iu) + \nabla^\perp h\Big).B + \int_{\partial\Omega} (h - h_{\text{ext}}) B.\tau + \int_{\partial\Omega} (\nu.\nabla_A u, v) = 0.$$

Since this is true for any v, B we get the result. □

1.5 Properties of critical points

Proposition 7 (regularity). *Let* Ω *be a smooth bounded domain in* $\mathbb{R}^2$ *and* $\kappa, h_{ext} > 0$. *If* (u, A) *is a critical point of* J *and if* A *satisfies the Coulomb gauge condition* (1.2), *then* u, A *are smooth.*

Sketch of the proof. Together with the Coulomb gauge condition, the Ginzburg-Landau equations (1.6) become

$$\begin{cases} -\Delta u = \kappa^2 u(1-|u|^2) - 2i(A.\nabla)u - |A|^2 u & \text{in } \Omega \\ -\Delta A = (iu, \nabla u - iAu) & \text{in } \Omega \\ h = h_{\text{ext}} & \text{on } \partial\Omega \\ \nu.\nabla_A u = 0 & \text{on } \partial\Omega. \end{cases} \qquad (1.7) \quad \boxed{\texttt{glcoul}}$$

The first equation is obtained by expanding $(\nabla_A)^2 u$. To obtain the second equation from (1.6), note that

$$-\nabla^\perp h = \left(\partial_2(\partial_1 A_2 - \partial_2 A_1), -\partial_1(\partial_1 A_2 - \partial_2 A_1)\right). \qquad (1.8) \quad \boxed{\texttt{hcoul}}$$

Differentiating $\partial_1 A_1 + \partial_2 A_2 = 0$ with respect to both variables we find $\partial_{12} A_2 = -\partial_{11} A_1$ and $\partial_{12} A_1 = -\partial_{22} A_2$. Replacing in (1.8) yields $-\nabla^\perp h = -\Delta A$ and thus (1.7).

But (1.7) are a couple of elliptic equation for which we easily derive regularity by bootstrapping arguments. Since (u, A) are both in H^1, hence in every L^q, the right-hand side of the equations (1.6) are in L^p for any $p < 2$ and therefore (u, A) are both in $W^{2,p}$ by standard elliptic theory, and therefore in every $W^{1,q}$, etc...

Note that the above argument yields interior regularity, boundary regularity requires a more careful inspection of the boundary conditions $h = h_{\text{ext}}$, $\nu.\nabla_A u = 0$ supplemented by $A.\nu = 0$. See [10] for more details. □

Proposition 8. *Let Ω be a smooth bounded domain in $\mathbb{R}^2$ and $\varepsilon, h_{ext} > 0$. If (u, A) is acritical point of J then $|u| \leq 1$ in Ω.*

Proof. This is a consequence of the maximum principle. Using (1.4) we find

$$\frac{1}{2}\Delta|u|^2 = 2\sum_{i=1}^{2}\partial_i(\nabla_i^A u, u) = \left((\nabla_A)^2 u, u\right) + |\nabla_A u|^2.$$

Using (1.7) we find

$$-\frac{1}{2}\Delta|u|^2 = \kappa^2|u|^2(1-|u|^2) - |\nabla_A u|^2.$$

and therefore

$$-\frac{1}{2}\Delta\left(1-|u|^2\right) + \kappa^2|u|^2(1-|u|^2) = |\nabla_A u|^2.$$

We multiply this equation by $(1-|u|^2)_- = min(1-|u|^2, 0)$ and integrate in Ω to find

$$-\frac{1}{2}\int_\Omega (1-|u|^2)_-\Delta\left(1-|u|^2\right) + \kappa^2|u|^2(1-|u|^2)(1-|u|^2)_- \leq 0.$$

Integrating by parts we get

$$-\frac{1}{2}\int_{\partial\Omega}(1-|u|^2)_-\partial_\nu\left(1-|u|^2\right) + \int_\Omega \nabla(1-|u|^2).\nabla(1-|u|^2)_- + \kappa^2|u|^2(1-|u|^2)(1-|u|^2)_- \leq 0.$$

It is a well-known property of Sobolev functions that $\nabla f = 0$ a.e. on any level set $\{f = a\}$ therefore the above inequality may be rewritten

$$-\frac{1}{2}\int_{\partial\Omega}(1-|u|^2)_-\partial_\nu\left(1-|u|^2\right) + \int_{|u|\geq 1} |\nabla(1-|u|^2)|^2 + \kappa^2|u|^2(1-|u|^2)^2 \leq 0.$$

The Neumann boundary condition $\nabla_A u.\nu = 0$ implies, taking the scalar product with u that $\partial_\nu|u| = 0$ thus the above inequality implies $\{x \in \Omega \mid |u| > 1\} = \varnothing$. □

Chapter 2

Introduction to critical fields in $\mathbb{R}^2$

We now return to our main interest, namely the phase diagram of the GL energy with respect to the parameters κ, h_{ext}. For simplicity we work in $\mathbb{R}^2$, and we summarize some of the consideration which can be found in physics book (e.g. [43]) on the subject of critical fields.

2.1 First critical line

There are two trivial solutions to (1.6) in $\mathbb{R}^2$ (the boundary conditions becoming of course irrelevant). The first is the *superconducting solution* or Meissner solution $u = 1$, $A = 0$. If h_{ext} is different from zero its energy is of course infinite, but its energy per unit area is

$$J_s = \frac{1}{2}h_{\text{ext}}^2.$$

The second solution is the *normal* solution $u = 0$, $A = h_{\text{ext}}A_n$, where A_n is any vector field satisfying $\operatorname{curl} A_n = 1$ in $\mathbb{R}^2$. It has infinite energy as soon as $\kappa > 0$ but its energy per unit area is

$$J_n = \frac{\kappa^2}{4}.$$

We have therefore a so-called critical line $h_{\text{ext}} = \kappa/\sqrt{2}$. If h_{ext} is higher the normal solution is favorable in terms of energy and if it is lower the superconducting solution is favorable.

Fortunately our problem is not solved by these trivial considerations. Indeed these two particular solutions need not be the only solutions, and the energy minimizer may be something completely different. This is indeed the discovery of Abrikosov, which was not very seriously considered until it was verified experimentally.

2.2 A Second critical line

If one looks for perturbations of the normal solution, one may in first approximation consider that for such a solution $|u|$ is small and try to solve the linearized form of (1.6) and try to evaluate wether these solutions are energetically more favorable than the normal and superconducting solutions, this was the approach of Abrikosov. It turns out that these linearized equations can be solved exactly and yield a family of doubly periodic solutions in $\mathbb{R}^2$. Recently M.Dutour ([19]) showed that to these linearized solutions correponded solutions of the original equations).

We will not describe these solutions further, let us just say that their existence implies the existence of a second critical half-line $\kappa > 1/\sqrt{2}$, $h_{\text{ext}} = \kappa^2$. When h_{ext} crosses this line (from above) the normal solutions ceases to be energetically favorable compared to these doubly peridic solutions. And this happens before h_{ext} decreases below the first critical line $h_{\text{ext}} = \kappa/\sqrt{2}$.

2.3 A Third critical line

If we believe the minimizer belongs to one of the solutions considered until now, the situation is the following.

If $\kappa < 1/\sqrt{2}$ then for $h_{\text{ext}} > \kappa/\sqrt{2}$ the normal solution is minimizing, if $h_{\text{ext}} < \kappa/\sqrt{2}$ the superconducting solutions is minimizing. We let

$$H_c(\kappa) = \frac{\kappa}{\sqrt{2}}.$$

If $\kappa > 1/\sqrt{2}$ then for $h_{\text{ext}} > \kappa^2$ the normal solution is minimizing, if $h_{\text{ext}} < \kappa^2$ then the Abrikosov-Dutour doubly periodic solutions are better than the normal solution. We let

$$H_{c_2}(\kappa) = \kappa^2.$$

It follows from the above that when the normal solution ceases to be minimizing, the new minimizer is *not* the normal solution. Therefore the superconducting solution ceases to be minimizing (as h_{ext} increases) for a value of h_{ext} different (and smaller) then H_{c_2}. Let us call $H_{c_1}(\kappa)$ this value. It seems that H_{c_1} should be computed by comparing the energy of the superconducting solution and that of the Abrikosov-Dutour solutions. But another type of solution exists that we have not considered yet which will yield the correct result, at least for κ large.

2.3.1 Another expression for $\nabla_A u$

It is good to keep in mind when studying the Ginzburg-Landau energy another expression for $\nabla_A u$. Wherever $u \neq 0$, we may write

$$u(x) = \rho(x)e^{i\varphi(x)}$$

where ρ is a positive function and φ is locally a well defined real valued function. Then we have

$$\nabla_A u = (\nabla\rho + i\rho(\nabla\varphi - A))\, e^{i\varphi}, \quad (iu, \nabla_A u) = \rho^2(\nabla\varphi - A). \tag{2.1}$$ polar

We leave this to check to the reader.

2.3.2 The vortex configuration

A natural way to find new solutions is to search for symmetric solutions. It is easy to convince oneself the strongest symmetry which does not yield trivial configurations while leaving the energy invariant is *radial symmetry*. A configuration is *radially symmetric* if, using polar coordinates in $\mathbb{R}^2$, it is of the form

$$u(r,\theta) = f(r)e^{i\theta}, \quad A(r,\theta) = g(r)(-\sin\theta, \cos\theta),$$

for some real-valued functions f and g. A radially symmetric solution to (1.6) exists but we will be satisfied with computing the energy of an approximate solution that we construct now.

if we assume κ to be large, then $|u|$ should be close to 1 outside a small area, so we take $f(r) = 1$ for $r > r_0$, where $r_0 > 0$ is to be chosen later. For $r < r_0$ we let $f(r) = r/r_0$.

To determine $g(r)$ we come back to (1.6). The best A should solve $-\nabla^\perp h = (iu, \nabla_A u)$. Let us translate this when $u(r,\theta) = f(r)e^{i\theta}$. We have

$$(iu, \nabla_A u) = (ife^{i\theta}, \nabla f e^{i\theta} + f\nabla\theta i e^{i\theta} - Afie^{i\theta}) = f^2(\nabla\theta - A).$$

The equation $-\nabla^\perp h = (iu, \nabla_A u)$ becomes $-\nabla^\perp h = f^2(\nabla\theta - A)$ and taking the curl we find that outside the ball $B_{r_0} = B(0, r_0)$

$$-\Delta h + h = 0. \tag{2.2}$$ e1

On the other hand, assuming $-\nabla^\perp h = (iu, \nabla_A u)$ in B_{r_0}

$$\int_{B_{r_0}} -\Delta h + h = \int_{\partial B_{r_0}} A.\tau - \nu.\nabla h = \int_{\partial B_{r_0}} \tau.\nabla\theta = 2\pi, \tag{2.3}$$ e2

where τ is the unit tangent vector to ∂B_{r_0} such that (τ, ν) is a direct frame. Grouping (2.2), (2.3), we see that h approximately solves $-\Delta h + h = 2\pi\delta$ in $\mathbb{R}^2$.

Then we define A to be the unique vector field of the form $g(r)(-\sin\theta, \cos\theta)$ such that $h = \operatorname{curl} A$ satisfies $-\Delta h + h = 2\pi\delta$ in $\mathbb{R}^2$. The solution to this equation is very well known it is a radial function $h(r)$ and its behaviour at 0 and ∞ are

$$h(r) \approx_{r\to 0} |\log r|, \quad h(r) = O(e^{-r}), \quad \text{as } r \text{ tends to } +\infty. \tag{2.4}$$ asymp

Moreover, it is easy to check that A satisfies

$$-\nabla^\perp h = \nabla\theta - A. \tag{2.5}$$ gl2

2.3.3 Energy of the vortex configuration

Since we wish to compare the vortex solution to the superconducting solution we will compute

$$\Delta = \lim_{R\to+\infty} J_{B_R}(u, A) - \frac{1}{2}h_{\text{ext}}^2|B_R| \tag{2.6}$$ delta

where $|B_R|$ is the area of the ball B_R and $\frac{1}{2}h_{\text{ext}}^2|B_R|$ is the energy density of the superconducting solution integrated in the ball.

We first compute the contribution outside B_{r_0}, that we call Δ_1. Outside B_{r_0} we have $-\nabla^\perp h = (iu, \nabla_A u)$ and $|u| = 1$ thus $|\nabla_A u|^2 = |\nabla h|^2$ and $1 - |u|^2 = 0$. Thus

$$\begin{aligned} \Delta_1 &= -\frac{h_{\text{ext}}^2|B_R \setminus B_{r_0}|}{2} + \frac{1}{2}\int_{B_R\setminus B_{r_0}} |\nabla_A u|^2 + |\operatorname{curl} A - h_{\text{ext}}|^2 + \frac{\kappa^2}{2}\left(1 - |u|^2\right)^2 \\ &= \frac{1}{2}\int_{B_R\setminus B_{r_0}} |\nabla h|^2 + h^2 - 2h_{\text{ext}}h. \end{aligned} \tag{2.7}$$

Integrating by parts and using the fact that $-\Delta h + h = 0$ in $B_R \setminus B_{r_0}$ yields

$$\Delta_1 = \frac{1}{2}\int_{\partial B_R} (h\nu.\nabla h - 2h_{\text{ext}}A.\tau) + \frac{1}{2}\int_{\partial B_{r_0}} (h\nu.\nabla h - 2h_{\text{ext}}A.\tau),$$

where ν is the outward pointing normal to $B_R \setminus B_{r_0}$ and (τ, ν) is a direct frame. Since $A = \nabla\theta + \nabla^\perp h$ and since

$$\int_{\partial B_R} \nabla\theta.\tau = -\int_{\partial B_{r_0}} \nabla\theta.\tau = -2\pi$$

thus

$$\Delta_1 = \frac{1}{2}\int_{\partial B_R} (h\nu.\nabla h - 2h_{\text{ext}}\nabla^\perp h.\tau) - \frac{1}{2}\int_{\partial B_{r_0}} (h\nu.\nabla h - 2h_{\text{ext}}\nabla^\perp.\tau).$$

The asymptotics (2.4) then yield

$$\Delta_1 \approx_{R\to+\infty, r_0\to 0} \pi \log\frac{1}{r_0} - 2\pi h_{\text{ext}}. \tag{2.8}$$ d1

We wish now to evaluate

$$\Delta_2 = -\frac{1}{2}h_{\text{ext}}^2|B_{r_0}| + |\frac{1}{2}\int_{B_{r_0}} |\nabla_A u|^2 + |\text{curl}\, A - h_{\text{ext}}|^2 + \frac{\kappa^2}{2}\left(1-|u|^2\right)^2.$$

We have $|\nabla_A u|^2 = (f')^2 + f^2|\nabla\theta - A|^2$ thus using (2.5) and the definition of $f(r)$ we find

$$|\nabla_A u|^2 = \frac{1}{{r_0}^2} + \frac{r}{r_0}^2 |\nabla h|^2.$$

in B_{r_0}. It follows easily that as $r_0 \to 0$

$$\Delta_2 = O\left(1 + {r_0}^2 h_{\text{ext}}^2 + \kappa^2 {r_0}^2\right). \tag{2.9}$$ d2

Choosing $r_0 = 1/\kappa$ and assuming κ is large, $h_{\text{ext}} \ll \kappa$ we find, combining (2.8),(2.9)

$$\Delta = \lim_{R\to+\infty} J_{B_R}(u, A) - \frac{1}{2}h_{\text{ext}}^2|B_R| \approx \pi\log\kappa - 2\pi h_{\text{ext}}. \tag{2.10}$$ dfin

2.3.4 conclusion

We conclude that there is a critical line

$$h_{\text{ext}}(\kappa) = H_{c_1}(\kappa) \approx_{\kappa\to+\infty} \frac{\log\kappa}{2} \tag{2.11}$$ hci

such that when $h_{\text{ext}} > H_{c_1}$ then the superconducting solution is no longer minimizing.

Chapter 3

Vortex analysis and H_{c_1}

Here we try to define H_{c_1} and to compute it in a more satisfactory way. We have until now computed by comparing the energy of two particular configurations. But the true energy minimizer could have been a third configuration, and in fact "energy-minimizer" has no clear meaning in $\mathbb{R}^2$ since for $h_{\text{ext}} > 0$ every configuration has infinite energy!

To obtain a mathematically well posed problem, we choose to work in a bounded smooth and simply connected domain Ω and try to describe the minimizers of $J(u, A)$ over the function space $H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)$ as $\kappa \to +\infty$. The estimate (2.11) suggests that to capture the transition from superconducting state to mixed state we should let h_{ext} tend to $+\infty$ with κ. From now on we choose to let

$$h_{\text{ext}} = \Lambda \log \kappa. \tag{3.1}$$ he

The first critical field is the field for which the creation of a vortex becomes energetically favorable. To determine its value it is therefore necessary to estimate the energy of a vortexless state (i.e. the energy of a solution to equations (1.6) that does not vanish). It is not our goal here to prove that such a state actually exists or is stable, we will only rely on energy comparisons and therefore we will be satisfied with a good guess on the energy of such a vortexless state, which bounds from above the minimal energy.

3.1 Estimate of the minimal energy

The first a-priori bound on the infimum of the energy is given by

Proposition 9 (and definition). *Assume Ω is simply connected. The infimum*

$$\inf_{A \in H^1(\Omega, \mathbb{R}^2)} J(1, A)$$

is achieved by a unique connection A, and moreover $A = h_{ext} A_0$, where A_0 does not depend on h_{ext}. The function $h_0 = \operatorname{curl} A_0$ solves the following problem

$$\begin{cases} -\Delta h_0 + h_0 = 0 & \text{in } \Omega \\ h_0 = 1 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$ h0

Proof. The proof of the existence of a minimizer follows the lines of Proposition 4. Take a minimizing sequence $(1, A_n)$. It is gauge-equivalent to (v_n, B_n) where B_n satisfies the

Coulomb gauge condition, which yields the weak H^1 convergence of (v_n, B_n) to a minimizer (v, B). Since (v_n, B_n) and $(1, A_n)$ are gauge-equivalent then $|v_n| = 1$ in Ω and therefore $|v| = 1$ in Ω. Moreover (v, B) are smooth. Since Ω is simply connected, there is a smooth function φ such that $v = e^{i\varphi}$ and thus (v, B) is gauge equivalent to a configuration $(1, A)$. Uniqueness follows from the fact that $A \to J(1, A)$ is convex.

Now since A minimizes $J(1, A)$ the configuration $(1, A)$ satisfies the Ginzburg-Landau equation and boundary conditions that express criticality *with respect to variations of* A, i.e. $-\nabla^\perp h = -A$ in Ω and $h = h_{\text{ext}}$ on $\partial\Omega$. Taking the curl of the first equation we find $-\Delta h + h = 0$, hence the result.

□

Definition 2. *We let*

$$J_0 = \frac{1}{2}\int_\Omega |\nabla h_0|^2 + (h_0 - 1)^2 \tag{3.3}$$ jo

We have

upper **Proposition 10.** *The following estimate holds*

$$\min_{(u,A)\in H^1(\Omega,\mathbb{C})\times H^1(\Omega,\mathbb{R}^2)} J(u, A) \le h_{ext}^2 J_0.$$

Proof. Defining A_0, h_0 as above we have $\nabla^\perp h_0 = A_0$ and therefore

$$J(1, h_{\text{ext}} A_0) = \frac{h_{\text{ext}}^2}{2}\int_\Omega |\nabla h_0|^2 + (h_0 - 1)^2.$$

□

3.2 Vortex balls

The heart of our analysis is to interpret the energy $J(u, A)$ in terms of *vortices*. One should think of a vortex of the configuration (u, A) as a point in Ω near which (u, A) behaves not too differently from the radial configuration we constructed above. The following proposition gives a precise meaning to this. we denote by $M(\kappa)$ any function such that for any $\alpha > 0$

$$\lim_{\kappa\to+\infty} \kappa^{-\alpha} M(\kappa) = 0, \quad \lim_{\kappa\to+\infty} \frac{\log\kappa}{M(\kappa)^\alpha} = 0, \quad \log M(\kappa) = o(\log\kappa) \text{ as } \kappa \to +\infty. \tag{3.4}$$ majb

For example $M(\kappa) = \exp(\sqrt{\log\kappa})$ satisfies this.

vortex **Proposition 11.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. We assume $\kappa > 1$ and that $J(u, A) < KM(\kappa)$. Then there exists disjoint balls $B_1, \dots, B_n$ with $B_i = B(a_i, r_i)$ such that letting $\tilde\Omega = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > 1/\kappa\}$*

1. $\sum_i r_i \le 1/M(\kappa)$.
2. *For any $x \in \tilde\Omega \setminus \cup_i B_i$, $||u(x)| - 1| \le 2/M(\kappa)$.*
3. *If $B_i \subset \tilde\Omega$,*

$$J_{B_i}(u,A) \geq \pi |d_i| \log \kappa (1 - o(1)), \tag{3.5}$$ minoboules

where $d_i = \deg(u, \partial B_i)$. The $o(1)$ appearing in the lower bound is a function of κ that goes to zero when $\kappa \to +\infty$ and which depends only on K.

The balls constructed above are called *vortex balls* associated to (u, A). They can be associated to (u, A) under the assumption that $J(u, A) < KM(\kappa)$. In our case this is a very mild assumption. Note that since h_{ext} is defined by (3.1) and from Proposition 10, a minimizer of J satisfies the apriori bound

$$J(u, A) \leq C(\log \kappa)^2.$$

Since from (3.4) the function $M(\kappa)$ dominates any power of $\log \kappa$, Proposition 11 may be applied to any minimizer of J.

We will not prove the above proposition, it relies on a construction introduced in [32] and independantly in [23].

The vortex balls constructed above are related to the configuration (u, A) in a more direct way. The proposition below appears in [24] and in a weaker form in [33].

jac **Proposition 12.** *Under the hypothesis of Proposition 11 and using the same notations*

$$\left\| \operatorname{curl}(iu, \nabla_A u) + h - 2\pi \sum_{\{i | a_i \in \tilde{\Omega}\}} d_i \delta_{a_i} \right\|_{(C_c^{0,1}(\Omega))'} \leq C \frac{J(u,A)}{M(\kappa)}, \tag{3.6}$$ jaco

where C depends only on the constant K in the bound $J(u, A) < KM(\kappa)$.

Here C_c^1 denotes the space of C_c^1 functions with compact support.

Sketch of the proof. To simplify the proof we assume that $|u| = 1$ outside the vortex balls. We define

$$\mu = 2\pi \sum_i d_i \delta_{a_i}, \quad ju = (iu, \nabla_A u), \quad Ju = \operatorname{curl}(ju) + h. \tag{3.7}$$ defs

Let ξ be a smooth compactly supported function. Since $|u| = 1$ outside of $\tilde{\Omega} \cap (\cup_i B_i)$ we have $Ju = 0$ there, using (2.1). Therefore

$$\int_{\Omega} \xi Ju = \int_{\Omega \setminus \tilde{\Omega}} \xi Ju + \sum_{B_i \not\subset \tilde{\Omega}} \int_{B_i \cap \tilde{\Omega}} \xi Ju + \sum_{B_i \subset \tilde{\Omega}} \int_{B_i} \xi Ju = I_1 + I_2 + I_3. \tag{3.8}$$ split

Since ξ vanishes on $\partial\Omega$ and from the definition of $\tilde{\Omega}$ we find $|\xi(x)| < \kappa^{-1} \|\xi\|_{C^{0,1}(\Omega)}$ for any $x \in \Omega \setminus \tilde{\Omega}$. It is clear that $|Ju| < (|\nabla_A u|^2 + h^2)$ thus

$$I_1 \leq C \frac{J(u,A) + h_{\text{ext}}^2}{\kappa} \|\xi\|_{C^{0,1}(\Omega)}. \tag{3.9}$$ I1

The second integral is taken care of in a similar way. From the definition of $\tilde{\Omega}$ and since the radius of any ball is less than $M(\kappa)^{-1}$ it follows that if $B_i \not\subset \tilde{\Omega}$ and $x \in \Omega \cap B_i$ then $|\xi(x)| < \|\xi\|_{C^{0,1}} / M(\kappa)$. It follows that

$$I_2 \leq C \lambda^2 \frac{J(u,A)}{M(\kappa)} \|\xi\|_{C^{0,1}(\Omega)}. \tag{3.10}$$ I2

To deal with the third integral we define $\bar{\xi}$ to be equal to $\xi(a_i)$ on B_i for any $B_i = B(a_i, r_i) \subset \tilde{\Omega}$ and $\bar{\xi} = 0$ elsewhere. Then letting A be the union of the B_i's which are included in $\tilde{\Omega}$, we have $|\xi - \bar{\xi}| \leq \|\xi\|_{C^{0,1}}/M(\kappa)$ on A while

$$\int_A \bar{\xi} Ju = \sum_{B_i \subset \tilde{\Omega}} \xi(a_i) \int_{B_i} Ju = \sum_{B_i \subset \tilde{\Omega}} 2\pi d_i \xi(a_i) = \int \xi \, d\mu,$$

where we have used the fact that $|u| = 1$ on ∂B_i. Therefore

$$\left| I_3 - \int \xi \, d\mu \right| \leq C \frac{J(u,A)}{M(\kappa)} \|\xi\|_{C^{0,1}(\Omega)}. \tag{3.11}$$ I3

It follows from (2.6), (3.8), (3.9), (3.10) and (3.11) that for any compactly supported smooth ξ

$$\left| \int_\Omega \xi Ju - \int \xi \, d\mu \right| \leq C \frac{J(u,A)}{M(\kappa)} \|\xi\|_{C^{0,1}}.$$

and the proposition is proved. □

This construction allows us to give the following definitions.

Definition 3. *If (u, A) is a configuration satisfying the hypothesis of Proposition 11, we call*

$$2\pi \sum_{\{i | a_i \in \tilde{\Omega}\}} d_i \delta_{a_i}$$

the vorticity measure associated to (u, A). We say (u, A) is vortex-less if this measure is 0.

3.3 Energy lower bound

We now prove an energy lower bound which results from the previous vortex construction

exp **Proposition 13.** *For any $K > 0$, there exist positive constants κ_0, C such that for any $\kappa > \kappa_0$ and any $h_{ext} < K \log \kappa$, if (u, A) is a critical point of (1.6) satisfying $J(u, A) < KM(\kappa)$ and $\{(a_i, d_i)\}$ is an associated family of vortices then*

$$J(u,A) \geq h_{ext}^2 J_0 + \pi \left(\sum_i |d_i| \right) (\log \kappa - o(1)) + 2\pi h_{ext} \sum_i d_i (h_0 - 1)(a_i) + \\ + \frac{1}{2} \int_{\Omega \setminus \cup_i B_i} |\nabla (h - h_{ext} h_0)|^2 + \frac{1}{2} \int_\Omega |h - h_{ext} h_0|^2 - o(1), \tag{3.12}$$ 6

where we have written $h = \operatorname{curl} A$ and J_0, h_0 are defined by (3.3), (3.2).

Proof. It follows from $-\nabla^\perp h = (iu, \nabla_A u)$ and $|u| \leq 1$ that $|\nabla_A u|^2 \geq |\nabla h|^2$. Thus,

$$J(u,A) \geq J_{\cup_i B_i}(u,A) + \frac{1}{2} \int_{\tilde{\Omega}} |\nabla h|^2 + |h - h_{\text{ext}}|^2, \tag{3.13}$$ 20

where $\tilde{\Omega} = \Omega \setminus \cup_i B_i$. Also, from Proposition 11,

$$J_{\cup_i B_i}(u, A) \geq \pi \sum_i |d_i| \left(\log \kappa - o(1)\right) \qquad (3.14)$$ 21

and, letting $h = h_{\text{ext}} h_0 + f$

$$\frac{1}{2} \int_{\tilde{\Omega}} |\nabla h|^2 + |h - h_{\text{ext}}|^2 \geq h_{\text{ext}}^2 J_0 + \frac{1}{2} \|f\|^2_{H^1(\tilde{\Omega})} + h_{\text{ext}} \int_{\tilde{\Omega}} \nabla f . \nabla (h_0 - 1) + f(h_0 - 1). \qquad (3.15)$$ 22

Since the measure of $\cup_i B_i$ is less than $CM(\kappa)^{-2}$,

$$h_{\text{ext}}^2 \int_{\cup_i B_i} (h_0 - 1)^2 = o(1). \qquad (3.16)$$ 23

Moreover, $f = h_{\text{ext}} h_0 - h$ and both h and $h_{\text{ext}} h_0$ are bounded in H^1 norm and therefore in L^4 norm by $C \log \kappa$. Then, by Hölder's inequality,

$$\left(\int_{\cup_i B_i} f^2 \right)^2 \leq \left(\sum_i |B_i| \right) \int_{\cup_i B_i} |f|^4 = o(1). \qquad (3.17)$$ 24

Also,

$$\int_{\cup_i B_i} \nabla f . \nabla (h_0 - 1) + f(h_0 - 1) \leq C h_{\text{ext}} \int_{\cup_i B_i} |\nabla f| + |f| = o(1). \qquad (3.18)$$ 25

From (3.13) – (3.18) we get

$$J(u, A) \geq \pi \sum_i |d_i| \left(\log \kappa - o(1)\right) + h_{\text{ext}}^2 J_0 + \frac{1}{2} \int_{\tilde{\Omega}} |\nabla f|^2 + \frac{1}{2} \int_{\Omega} f^2 + $$
$$+ h_{\text{ext}} \int_{\Omega} \nabla f . \nabla (h_0 - 1) + f(h_0 - 1). \qquad (3.19)$$ 26

Moreover $-\Delta f + f = -\Delta h + h = \text{curl}(iu, \nabla_A u) + h$ therefore using Proposition 3.6

$$\int_{\Omega} \nabla f . \nabla (h_0 - 1) + f(h_0 - 1) = 2\pi \sum_i d_i (h_0 - 1)(a_i) + o(1),$$

which, together with (3.19), proves (3.12). □

c1 **Corollary 1.** *Assume that* $h_{ext}(\kappa) = \Lambda \log \kappa$ *with*

$$\Lambda < \frac{1}{2 \max_{\Omega} |h_0 - 1|}.$$

Then for κ *large enough any minimizer of the functional* J *is vortexless.*

Indeed it is clear from (3.12) that in this case, if κ is large enough and the vorticity measure is not zero then $J(u,A)$ is strictly greater than $h_{\text{ext}}^2 J_0$ hence cannot be minimizing. This proves in a weak form that the first critical field is greater than

$$H(\kappa) = \frac{\log \kappa}{2\max_\Omega |h_0 - 1|}. \tag{3.20}$$

Note that Proposition 13 also implies

Corollary 2. *Assume that $h_{ext}(\kappa) = \Lambda \log \kappa$ for any $\Lambda > 0$. Then the energy of a vortexless configuration (u, A) satisfies*

$$J(u,A) \geq h_{ext}^2 J_0 - o(1).$$

3.4 Energy upper bound

To complete our estimate of the first critical filed, we need to construct an upper bound. We prove the following

Proposition 14. *Assume that $h_{ext}(\kappa) = \Lambda \log \kappa$ with*

$$\Lambda > \frac{1}{2\max_\Omega |h_0 - 1|}.$$

Then for κ large enough there exists a test configuration (u, A) with energy smaller than any vortexless configuration.

Together with Corollaries 1 and 2 this implies

Theorem 1. *The first critical field $H_{c_1}(\kappa)$, defined as the highest value value of h_{ext} for which minimizing configuration are vortexless satisfies*

$$\frac{H_{c_1}(\kappa)}{\log \kappa} \approx_{\kappa\to\infty} \frac{1}{2\max_\Omega |h_0 - 1|}$$

Note that the function h_0 is equal to 1 on $\partial\Omega$ but decays like $\exp(-\operatorname{dist}(x, \partial\Omega))$ inside Ω. Therefore if Ω is very large, the minimum of h_0 in Ω is close to 0 and therefore $\max_\Omega |h_0 - 1|$ is close to 1. We then recover the value $\log \kappa/2$ for the first critical field that we computed in $\mathbb{R}^2$.

The proof of Proposition (14) follows closely the construction we did in $\mathbb{R}^2$. The idea is to let x_0 be a point in Ω where h_0 achieves it minimum, then to solve

$$-\Delta h + h = 2\pi\delta_0$$

in Ω together with the boundary condition $h = h_{\text{ext}}$ on $\partial\Omega$. Then we solve $\operatorname{curl} A = h$, and then $-\nabla^\perp h = \nabla\varphi - A$. This last equation is solved in $\Omega \setminus \{x_0\}$, the function φ plays the role that θ had in the radial case, it is defined modulo 2π only. Finally we define ρ to be equal to 1 outside $B(x_0, 1/\kappa)$ and $\rho(x) = \kappa|x - x_0|$ otherwise. Then the energy of the configuration $(u = rhoe^{i\varphi}, A)$ can be estimated as in the case of $\mathbb{R}^2$ and is strictly less than that of a vortexless configuration if κ is large enough. In fact the computation yields

$$J(u,A) \leq h_{\text{ext}}^2 J_0 + \pi \log \kappa + 2\pi h_{\text{ext}}(h_0 - 1)(x_0) + o(\log \kappa),$$

which is to the order of $\log \kappa$ exactly the right-hand side of (3.12), for a single vortex of degree 1 located at x_0.

Remark 1. *Note that refining the techniques above, much sharper results can be deduced, we refer to [38, 39, 37].*

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