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Concentrating solutions in nonlinear elliptic problems

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These are preliminary lecture notes, intended only for distribution to participants

Concentrating solutions in nonlinear elliptic problems

Lecture Notes by Andrea Malchiodi[1]

Recent Trends in Nonlinear Variational Problems

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1 Introduction

In these notes we present the main ideas for the applications of finite-dimensional reductions to the study of some singularly perturbed elliptic equations. We will discuss the following two problems

$$(NLS_\varepsilon) \qquad -\varepsilon^2 \Delta u + V(x) u = u^p \qquad \text{in } \mathbb{R}^n;$$

and

$$(P_\varepsilon) \qquad \begin{cases} -\varepsilon^2 \Delta u + u = u^p & \text{in } \Omega \subseteq \mathbb{R}^n, \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \\ u > 0 \text{ in } \Omega, \end{cases}$$

where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the usual Laplace operator, $p > 1$, and ν is the outward unit normal to Ω.

Problem (NLS_ε) arises in the study of standing-waves in the semiclassical limit for the Nonlinear Schrödinger equation. The function $V(x)$ is the potential, while ε is a small positive parameter which represents the Planck constant $\hbar$.

On the other hand, problem (P_ε) models pattern-formation in some biological experiments. More precisely, it is obtained from the study of stationary solutions of the system

$$(GM) \qquad \begin{cases} \mathcal{U}_t = d_1 \Delta \mathcal{U} - \mathcal{U} + \frac{\mathcal{U}^p}{\mathcal{V}^q} & \text{in } \Omega \times (0, +\infty); \\ \mathcal{V}_t = d_2 \Delta \mathcal{V} - \mathcal{V} + \frac{\mathcal{U}^r}{\mathcal{V}^s} & \text{in } \Omega \times (0, +\infty); \\ \frac{\partial \mathcal{U}}{\partial \nu} = \frac{\partial \mathcal{V}}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, +\infty), \end{cases}$$

proposed by Gierer and Meinhardt in 1972. Here the functions $\mathcal{U}, \mathcal{V}$ represent the densities of some chemicals and d_1, d_2 are diffusion coefficients. Problem (P_ε) is obtained from (GM) letting $d_2 \to +\infty$. Then non-constant solutions appear, according to the so-called *Turing's instability.*

Problems (NLS_ε) and (P_ε) have many common features. They are both variational and can often be treated with the same abstract approaches. They exhibit solutions u_ε which scale roughly in the following way

$$u_\varepsilon(x) \sim \overline{u}\left(\frac{x}{\varepsilon}\right),$$

where $\overline{u}$ solves a suitable *limit problem* in $\mathbb{R}^n$ or in $\mathbb{R}^n_+$ (the half-space $\{x_n > 0\}$).

Much work has been devoted to the study of *spike-layer* solutions, which concentrate at single points of $\mathbb{R}^n$, or of $\overline{\Omega}$. In the case of (NLS_ε), this happens at critical points of V, which are equilibria of the classical Newtonian motion.

About problem (P_ε), is the geometry of Ω which determines the location of spike-layers. At the boundary, concentration occurs at critical points of the mean curvature, while at the interior it occurs, roughly, at critical points of the distance from $\partial\Omega$.

Let us mention that both problems (NLS_ε) and (P_ε) possess *multibump* solutions as well, namely solutions with multiple peaks. The techniques used to produce these solutions are min-max theorems, Lyapunov-Schmidt reductions, penalization methods and gluing procedures.

Very recently the phenomenon of concentration at curves or manifolds has been investigated. Here we consider (NLS_ε) and (P_ε) under the assumption of spherical symmetry. In this case the method of finite-dimensional reduction can still be applied, using suitable modifications. For both problems, new phenomena take place. Non-symmetric cases, instead, require completely different methods, which go beyond the purpose of these notes.

The main references are the papers [1], [2], [3], which contain the motivations of this study, together with complete proofs and detailed bibliography.

2 Notation and preliminary results

Consider the problem

$$(P_0) \qquad \begin{cases} -\Delta u + u = u^p & \text{in } \mathbb{R}^n; \\ u(x) \to 0 & \text{as } |x| \to +\infty; \\ u > 0, \end{cases}$$

where $n \geq 1$ and $p > 1$.

If $p \leq \frac{n+2}{n-2}$ (in the case $n \geq 3$), and if $u \in H^1(\mathbb{R}^n)$, solutions of (P_0) can be found as critical points of the functional $\overline{I} : H^1(\mathbb{R}^n) \to \mathbb{R}$ defined as

$$(1) \qquad \overline{I}(u) = \frac{1}{2} \int_{\mathbb{R}^n} \left(|\nabla u|^2 + u^2 \right) - \frac{1}{p+1} \int_{\mathbb{R}^n} |u|^{p+1}.$$

Note that, by the Sobolev embedding theorem, $\overline{I}$ is well-defined (and is actually of class C^2) on $H^1(\mathbb{R}^n)$. We have the following result.

Proposition 2.1 *(Berestycki-Lions, Gidas-Ni-Nirenberg, Kwong). Assume $p < \frac{n+2}{n-2}$ (in the case $n \geq 3$). Then there holds*

(a) *problem (P_0) admits a radial decreasing solution $\overline{u}(r)$;*

(b) *there exists a positive constant $\alpha_{n,p}$ such that*

$$(2) \qquad \lim_{r \to +\infty} r^{\frac{n-1}{2}} e^r \overline{u}(r) = \alpha_{n,p}; \qquad \lim_{r \to +\infty} \frac{\overline{u}'(r)}{\overline{u}(r)} = -1;$$

(c) *all the solutions of (P_0) are of the form $\overline{u}(|x - \xi|)$, for some $\xi \in \mathbb{R}^n$;*

(d) *$\overline{u}(|x|) \in H^1(\mathbb{R}^n)$, and is a mountain-pass critical point of $\overline{I}$.*

Remark 2.2 *From the Pohozahev's identity (testing the equation in (P_0) on $(x, \nabla u)$) and from some decay estimates, it follows that problem (P_0) admits no solution for $p \geq \frac{n+2}{n-2}$.*

It is essential to understand the spectral properties of the linearized equation at $\overline{u}$, or equivalently of the operator $\overline{I}''(\overline{u})$, which is given by

$$(3) \qquad \overline{I}''(\overline{u})[v_1, v_2] = \int_{\mathbb{R}^n} \left((\nabla v_1, \nabla v_2) + v_1 v_2 \right) - p \int_{\mathbb{R}^n} \overline{u}^{p-1} v_1 v_2; \qquad v_1, v_2 \in H^1(\mathbb{R}^n).$$

With some abuse of notation, we will indifferently consider it defined on $\mathbb{R}^n$ or on $\mathbb{R}^+$. We have the following well-known result.

Proposition 2.3 *Assume $p < \frac{n+2}{n-2}$ (in the case $n \geq 3$). Then $\overline{I}''(\overline{u})$ is of the form Identity − Compact, and has the following properties*

(a) *the first eigenvalue is* $-(p-1)$, *is simple, and the corresponding eigenfunction is* $\overline{u}$;

(b) *the second eigenvalue is* 0 *with multiplicity* n, *and the corresponding eigenfunctions are*

$$\alpha_1 \frac{\partial \overline{u}}{\partial x_1} + \cdots + \alpha_n \frac{\partial \overline{u}}{\partial x_n},$$

where $\alpha_1, \ldots, \alpha_n$ *are arbitrary real constants;*

(c) *the remaining eigenvalues are bounded below by a positive constant.*

We also need to consider the following variant of problem (P_0), namely

$$(P_\lambda) \qquad \begin{cases} -\Delta u + \lambda^2 u = u^p & \text{in } \mathbb{R}^n; \\ u(x) \to 0 & \text{as } |x| \to +\infty; \\ u > 0, \end{cases}$$

where $n \geq 1$, $\lambda > 0$, and $p > 1$. It is immediate to check from Proposition 2.1 and some scaling argument that all the solutions of (P_λ) are the translates of the function $\overline{u}_\lambda$, which is given by

$$\overline{u}_\lambda(x) = \lambda^{\frac{2}{p-1}} \overline{u}(\lambda x); \qquad x \in \mathbb{R}^n.$$

Of course, the complete analogues of Propositions 2.1 and 2.3 hold true for the function $\overline{u}_\lambda$ and the functional $\overline{I}_\lambda : H^1(\mathbb{R}^n) \to \mathbb{R}$ defined as

$$(4) \qquad \overline{I}_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^n} \left(|\nabla u|^2 + \lambda^2 u^2 \right) - \frac{1}{p+1} \int_{\mathbb{R}^n} |u|^{p+1}; \qquad u \in H^1(\mathbb{R}^n).$$

We conclude this section recalling the weak formulation of problems (NLS_ε) and (P_ε) in the case $p \leq \frac{n+2}{n-2}$. About the former, after the scaling $x \mapsto \varepsilon x$, solutions can be found as critical points of the functional $f_\varepsilon : H^1(\mathbb{R}^n) \to \mathbb{R}$ defined as

$$(5) \qquad f_\varepsilon(u) = \frac{1}{2} \int_{\mathbb{R}^n} \left(|\nabla u|^2 + V(\varepsilon x) u^2 \right) - \frac{1}{p+1} \int_{\mathbb{R}^n} |u|^{p+1}; \qquad u \in H^1(\mathbb{R}^n).$$

About the latter, and using the same scaling, one is reduced to find non-negative critical points of the functional $I_\varepsilon : H^1(\Omega_\varepsilon) \to \mathbb{R}$

$$(6) \qquad I_\varepsilon(u) = \frac{1}{2} \int_{\Omega_\varepsilon} \left(|\nabla u|^2 + u^2 \right) - \frac{1}{p+1} \int_{\Omega_\varepsilon} |u|^{p+1}; \qquad u \in H^1(\Omega_\varepsilon),$$

where $\Omega_\varepsilon = \frac{1}{\varepsilon}\Omega$. In the sequel, we will not distinguish between problems (NLS_ε), (P_ε) and their scaled versions.

On the potential V, we will make the following assumptions

(V1) $V \in C^2(\mathbb{R}^n)$, and $\|V\|_{C^2(\mathbb{R}^n)} < +\infty$;

(V2) V is bounded and $\lambda_0^2 = \inf_{\mathbb{R}^n} V > 0$.

Throughout the notes, the constant C is allowed to vary from one formula to another, also within the same line, assuming larger and larger values.
Below, we use the notation $a \sim b$ if the quantities a and b are of the same order when $\varepsilon \to 0$. In the same way, we write $a \lesssim b$ if the order of a is not larger that the order of b when $\varepsilon \to 0$.

3 Concentration at points for (NLS_ε)

In this section we study concentration at points for (NLS_ε), and we present the general procedure to reduce singularly perturbed problems to finite-dimensional ones. The idea, see Proposition 3.1, is to find an n-dimensional manifold Z^ε of *pseudo-critical points*, which can be perturbed to obtain a *natural constraint* $\tilde{Z}^\varepsilon$ for f_ε. This means that a critical point of f_ε restricted to $\tilde{Z}^\varepsilon$ is also a critical point for f_ε. Throughout this section, we always assume that p is subcritical, namely $p < \frac{n+2}{n-2}$.

3.1 A Lyapunov-Schmidt type reduction

We set

$$z^{\varepsilon\xi}(x) = \alpha(\varepsilon\xi)\overline{u}(\beta(\varepsilon\xi)x); \qquad \xi \in \mathbb{R}^n, \tag{7}$$

where

$$\beta(\varepsilon\xi) = \sqrt{V(\varepsilon\xi)}; \qquad \alpha(\varepsilon\xi) = \beta(\varepsilon\xi)^{\frac{2}{p-1}}.$$

Then we define

$$Z^\varepsilon = \{z^{\varepsilon\xi}(x-\xi) : \xi \in \mathbb{R}^n\}.$$

When there is no possible misunderstanding, we will write z, resp. Z, instead of $z^{\varepsilon\xi}$, resp Z^ε. We will also use the notation z_ξ to denote the function $z_\xi(x) := z^{\varepsilon\xi}(x-\xi)$. Obviously all the functions in $z_\xi \in Z$ are solutions of (P_λ) or, equivalently, critical points of $\overline{I}_\lambda$, with $\lambda = \beta(\varepsilon\xi)$. For future reference, let us point out some estimates. First of all, we evaluate

$$\begin{aligned} &\partial_\xi z^{\varepsilon\xi}(x-\xi) = \partial_\xi\left[\alpha(\varepsilon\xi)\overline{u}(\beta(\varepsilon\xi)(x-\xi))\right] \\ &= \varepsilon\alpha'(\varepsilon\xi)\overline{u}(\beta(\varepsilon\xi)(x-\xi)) + \varepsilon\alpha(\varepsilon\xi)\beta'(\varepsilon\xi)\overline{u}'(\beta(\varepsilon\xi)(x-\xi)) - \alpha(\varepsilon\xi)\overline{u}'(\beta(\varepsilon\xi)(x-\xi)). \end{aligned}$$

Recalling the definition of α, β one finds:

$$\partial_\xi z^{\varepsilon\xi}(x-\xi) = -\partial_x z^{\varepsilon\xi}(x-\xi) + O(\varepsilon|\nabla V(\varepsilon\xi)|). \tag{8}$$

The main result of this section is the following proposition.

Proposition 3.1 *Let V satisfy the assumptions $(V1)$, $(V2)$. Then for $\varepsilon > 0$ small and $|\xi| \le \overline{\xi}$ there exists a unique $w = w(\varepsilon,\xi) \in (T_{z_\xi}Z)^\perp$ such that $\nabla f_\varepsilon(z_\xi + w) \in T_{z_\xi}Z$. The function $w(\varepsilon,\xi)$ is of class C^2 (resp. $C^{1,p-1}$) with respect to ξ, provided that $p \ge 2$ (resp. $1 < p < 2$). Moreover, the functional $\Phi_\varepsilon(\xi) = f_\varepsilon(z_\xi + w(\varepsilon,\xi))$ has the same regularity of w and satisfies*

$$\nabla\Phi_\varepsilon(\xi_0) = 0 \quad \Longrightarrow \quad \nabla f_\varepsilon\left(z_{\xi_0} + w(\varepsilon,\xi_0)\right) = 0.$$

In order to prove this proposition, we need some gradient estimates, and the invertibility of f''_ε on the orthogonal complement of TZ.

3.1.1 Gradient estimates

The next Lemma shows that $\nabla f_\varepsilon(z_\xi)$ is close to zero when ε is small, namely z_ξ is an approximate solution of (NLS_ε).

Lemma 3.2 *Assume* $(V1)$, $(V2)$ *hold. then for all* $\xi \in \mathbb{R}^n$ *and all* $\varepsilon > 0$ *small, one has*

$$\|\nabla f_\varepsilon(z_\xi)\| \le C\left(\varepsilon|\nabla V(\varepsilon\xi)| + \varepsilon^2\right), \quad C > 0.$$

PROOF. Since

$$f_\varepsilon(u) = \overline{I}_\lambda(u) + \frac{1}{2}\int_{\mathbb{R}^n} [V(\varepsilon x) - V(\varepsilon\xi)]\, u^2 dx; \qquad \lambda^2 = V(\varepsilon\xi),$$

and since z_ξ is a critical point of $\overline{I}_\lambda$, one has

$$\begin{aligned}(\nabla f_\varepsilon(z_\xi)|v) &= (\nabla \overline{I}_\lambda(z_\xi)|v) + \int_{\mathbb{R}^n} [V(\varepsilon x) - V(\varepsilon\xi)]\, z_\xi v\, dx \\ &= \int_{\mathbb{R}^n} [V(\varepsilon x) - V(\varepsilon\xi)]\, z_\xi v\, dx.\end{aligned}$$

Using the Hölder inequality, one finds

$$|(\nabla f_\varepsilon(z_\xi)|v)|^2 \le \|v\|_{L^2}^2 \int_{\mathbb{R}^n} |V(\varepsilon x) - V(\varepsilon\xi)|^2 z_\xi^2 dx. \tag{9}$$

From the assumption that $|D^2V(x)| \le C$, one infers

$$|V(\varepsilon x) - V(\varepsilon\xi)| \le C\varepsilon|\nabla V(\varepsilon\xi)||x-\xi| + C\varepsilon^2|x-\xi|^2.$$

This implies

$$\begin{aligned}&\int_{\mathbb{R}^n} |V(\varepsilon x) - V(\varepsilon\xi)|^2 z_\xi^2 dx \le \\ &C\varepsilon^2|\nabla V(\varepsilon\xi)|^2 \int_{\mathbb{R}^n} |x-\xi|^2 z^2(x-\xi)dx + C\varepsilon^4 \int_{\mathbb{R}^n} |x-\xi|^4 z^2(x-\xi)dx.\end{aligned} \tag{10}$$

Recalling Proposition 2.1 and (7), a direct calculation yields

$$\begin{aligned}\int_{\mathbb{R}^n} |x-\xi|^2 z^2(x-\xi)dx &= \alpha^2(\varepsilon\xi)\int_{\mathbb{R}^n} |y|^2\overline{u}^2(\beta(\varepsilon\xi)y)dy \\ &= \alpha^2\beta^{-n-2}\int_{\mathbb{R}^n} |y'|^2\overline{u}^2(y')dy' \le C.\end{aligned}$$

From this (and a similar calculation for the last integral in the above formula) one derives

$$\int_{\mathbb{R}^n} |V(\varepsilon x) - V(\varepsilon\xi)|^2 z_\xi^2 dx \le C\varepsilon^2|\nabla V(\varepsilon\xi)|^2 + C\varepsilon^4. \tag{11}$$

Putting together (9) and (11), the lemma follows. ■

3.1.2 Invertibility of $D^2 f_\varepsilon$ on $TZ^\perp$

In this section we will show that $D^2 f_\varepsilon$ is invertible on $TZ^\perp$. This will be the main tool to perform the finite dimensional reduction.

Let $L_{\varepsilon,\xi} : (T_{z_\xi} Z^\varepsilon)^\perp \to (T_{z_\xi} Z^\varepsilon)^\perp$ denote the operator defined by duality as $(L_{\varepsilon,\xi} v|w) = D^2 f_\varepsilon(z_\xi)[v,w]$. We want to show the following.

Lemma 3.3 *Under the assumptions* $(V1)$, $(V2)$ *there exists* $C > 0$ *such that for* ε *small enough one has that*

$$|(L_{\varepsilon,\xi} v|v)| \geq C^{-1}\|v\|^2, \qquad \forall\, |\xi|,\ \forall\, v \in (z_\xi \oplus T_{z_\xi} Z^\varepsilon)^\perp. \tag{12}$$

PROOF. From (8) it follows that every element $\zeta \in T_{z_\xi} Z$ can be written in the form $\zeta = -\partial_x z^{\varepsilon\xi}(x-\xi) + O(\varepsilon)$. As a consequence,

$(*)$ it suffices to prove (12) for all $v \in \mathrm{span}\{z_\xi, \phi\}$, where $\phi \perp \mathrm{span}\{z_\xi, \partial_x z^{\varepsilon\xi}(x-\xi)\}$.

Precisely we shall prove that there exist $C > 0$ such that for all $\varepsilon > 0$ small one has

$$(L_{\varepsilon,\xi} z_\xi | z_\xi) \ \leq\ -C^{-1} < 0; \tag{13}$$

$$(L_{\varepsilon,\xi} \phi | \phi) \ \geq\ C^{-1}\|\phi\|^2. \tag{14}$$

It is clear that the Lemma immediately follows from $(*)$, (13) and (14).

Proof of (13). First let us recall that, by Proposition 2.3, one has

$$D^2 \overline{I}_\lambda(z_\xi)[z_\xi, z_\xi] = -(p-1) < -C^{-1} < 0. \tag{15}$$

There holds

$$(L_{\varepsilon,\xi} z_\xi | z_\xi) = D^2 \overline{I}_\lambda(z_\xi)[z_\xi, z_\xi] + \int_{\mathbb{R}^n} [V(\varepsilon x) - V(\varepsilon\xi)]\, z_\xi^2 dx.$$

The last integral can be estimated as in (11) yielding

$$(L_{\varepsilon,\xi} z_\xi | z_\xi) \leq D^2 \overline{I}_\lambda(z_\xi)[z_\xi, z_\xi] + C\varepsilon|\nabla V(\varepsilon\xi)| + C\varepsilon^2. \tag{16}$$

From (15) and (16) the inequality in (13) follows.

Proof of (14). As before, by Proposition 2.3, we have

$$D^2 \overline{I}_\lambda(z_\xi)[\phi, \phi] > C^{-1}\|\phi\|^2. \tag{17}$$

Let $R \gg 1$ and consider a radial smooth function $\chi_1 : \mathbb{R}^n \mapsto \mathbb{R}$ such that

$$\chi_1(x) = 1, \quad \text{for } |x| \leq R; \qquad \chi_1(x) = 0, \quad \text{for } |x| \geq 2R; \tag{χ'}$$

$$|\nabla \chi_1(x)| \leq \frac{2}{R}, \quad \text{for } R \leq |x| \leq 2R. \tag{χ''}$$

We also set $\chi_2(x) = 1 - \chi_1(x)$. Given ϕ let us consider the functions

$$\phi_i(x) = \chi_i(x-\xi)\phi(x), \quad i = 1, 2.$$

A straight computation yields

$$\int_{\mathbb{R}^n} \phi^2 = \int_{\mathbb{R}^n} \phi_1^2 + \int_{\mathbb{R}^n} \phi_2^2 + 2\int_{\mathbb{R}^n} \phi_1\, \phi_2,$$

$$\int_{\mathbb{R}^n} |\nabla\phi|^2 = \int_{\mathbb{R}^n} |\nabla\phi_1|^2 + \int_{\mathbb{R}^n} |\nabla\phi_2|^2 + 2\int_{\mathbb{R}^n} \nabla\phi_1 \cdot \nabla\phi_2,$$

and hence

$$\|\phi\|^2 = \|\phi_1\|^2 + \|\phi_2\|^2 + 2\int_{\mathbb{R}^n} [\phi_1\, \phi_2 + \nabla\phi_1 \cdot \nabla\phi_2].$$

Letting I denote the last integral, one immediately finds

$$I = \underbrace{\int_{\mathbb{R}^n} \chi_1\chi_2(\phi^2 + |\nabla\phi|^2)}_{I_\phi} + \underbrace{\int_{\mathbb{R}^n} \phi^2 \nabla\chi_1 \cdot \nabla\chi_2}_{I'} + \underbrace{\int_{\mathbb{R}^n} \phi_1 \nabla\phi \cdot \nabla\chi_2 + \phi_2 \nabla\phi \cdot \nabla\chi_1}_{I''}.$$

Due to the definition of χ, the two integrals I' and I'' reduce to integrals in $\{R \le |x| \le 2R\}$, and thus they are of order $o_R(1)\|\phi\|^2$. As a consequence we have

$$\|\phi\|^2 = \|\phi_1\|^2 + \|\phi_2\|^2 + 2I_\phi + o_R(1)\|\phi\|^2, \tag{18}$$

After these preliminaries, let us evaluate the three terms in the equation below

$$(L_{\varepsilon,\xi}\phi|\phi) = \underbrace{(L_{\varepsilon,\xi}\phi_1|\phi_1)}_{\sigma_1} + \underbrace{(L_{\varepsilon,\xi}\phi_2|\phi_2)}_{\sigma_2} + 2\underbrace{(L_{\varepsilon,\xi}\phi_1|\phi_2)}_{\sigma_3}.$$

One has

$$\sigma_1 = (L_{\varepsilon,\xi}\phi_1|\phi_1) = D^2\overline{I}_\lambda[\phi_1, \phi_1] + \int_{\mathbb{R}^n} [V(\varepsilon x) - V(\varepsilon\xi)]\, \phi_1^2. \tag{19}$$

In order to use (17), we introduce the function $\overline{\phi}_1 = \phi_1 - \psi$, where

$$\psi = (\phi_1|z_\xi)z_\xi + (\phi_1|\partial_x z_\xi)\partial_x z_\xi.$$

Then we have

$$D^2\overline{I}_\lambda[\phi_1, \phi_1] = D^2\overline{I}_\lambda[\overline{\phi}_1, \overline{\phi}_1] + D^2\overline{I}_\lambda[\psi, \psi] + 2D^2\overline{I}_\lambda[\overline{\phi}_1, \psi]. \tag{20}$$

Let us explicitly point out that $\overline{\phi}_1 \perp \text{span}\{z_\xi, \partial_x z^{\varepsilon\xi}(x-\xi)\}$ and hence (17) implies

$$D^2\overline{I}_\lambda[\overline{\phi}_1, \overline{\phi}_1] \ge C^{-1}\|\overline{\phi}_1\|^2. \tag{21}$$

On the other side, since $(\phi|z_\xi)=0$ it follows that

$$
\begin{aligned}
(\phi_1|z_\xi) &= (\phi|z_\xi)-(\phi_2|z_\xi)=-(\phi_2|z_\xi)\\
&= -\int_{\mathbb{R}^n}\phi_2 z_\xi dx-\int_{\mathbb{R}^n}\nabla z_\xi\cdot\nabla\phi_2 dx\\
&= -\int_{\mathbb{R}^n}\chi_2(y)z(y)\phi(y+\xi)dy-\int_{\mathbb{R}^n}\nabla z(y)\cdot\nabla\chi_2(y)\phi(y+\xi)dy.
\end{aligned}
$$

Since $\chi_2(x)=0$ for all $|x|<R$, and since $z(x)\to 0$ as $|x|=R\to\infty$, we infer $(\phi_1|z_\xi)=o_R(1)\|\phi\|$. Similarly one shows $(\phi_1|\partial_x z_\xi)=o_R(1)\|\phi\|$, and it follows that

$$\|\psi\|=o_R(1)\|\phi\|. \tag{22}$$

We are now in position to estimate the last two terms in (20). Actually, using (22) we get

$$D^2\overline{I}_\lambda[\psi,\psi]=\|\psi\|^2+V(\varepsilon\xi)\int_{\mathbb{R}^n}\psi^2-p\int_{\mathbb{R}^n}z_\xi^{p-1}\psi^2=o_R(1)\|\phi\|^2. \tag{23}$$

The same arguments readily imply

$$D^2\overline{I}_\lambda[\overline{\phi}_1,\psi]=(\overline{\phi}_1|\psi)+V(\varepsilon\xi)\int_{\mathbb{R}^n}\overline{\phi}_1\psi-p\int_{\mathbb{R}^n}z_\xi^{p-1}\overline{\phi}_1\psi=o_R(1)\|\phi\|^2. \tag{24}$$

Putting together (21), (23) and (24) we infer

$$D^2\overline{I}_\lambda[\phi_1,\phi_1]\geq\|\phi_1\|^2+o_R(1)\|\phi\|^2. \tag{25}$$

Using arguments already carried out before, one has

$$
\begin{aligned}
\int_{\mathbb{R}^n}|V(\varepsilon x)-V(\varepsilon\xi)|\phi_1^2 dx &\leq C\int_{\mathbb{R}^n}|x-\xi|\chi^2(x-\xi)\phi^2(x)dx\\
&\leq \varepsilon C\int_{\mathbb{R}^n}y\chi(y)\phi^2(y+\xi)dy\\
&\leq \varepsilon C\|\phi\|^2.
\end{aligned}
$$

This and (25) yield

$$\sigma_1=(L_{\varepsilon,\xi}\phi_1|\phi_1)\geq C^{-1}\|\phi_1\|^2-\varepsilon C\|\phi\|^2+o_R(1)\|\phi\|^2. \tag{26}$$

Let us now estimate σ_2. One finds

$$\alpha_2=(L_{\varepsilon,\xi}\phi_2|\phi_2)=\int_{\mathbb{R}^n}|\nabla\phi_2|^2+\int_{\mathbb{R}^n}V(\varepsilon x)\phi_2^2-p\int_{\mathbb{R}^n}z_\xi^{p-1}\phi_2^2$$

and therefore, using $(V2)$,

$$\sigma_2\geq C^{-1}\|\phi_2\|^2-p\int_{\mathbb{R}^n}z_\xi^{p-1}\phi_2^2.$$

As before, $\phi_2(x) = 0$ for all $|x| < R$ and $z(x) \to 0$ as $|x| = R \to \infty$ imply

$$\sigma_2 \geq C^{-1}\|\phi_2\|^2 + o_R(1)\|\phi\|^2. \tag{27}$$

In a similar way one shows that

$$\sigma_3 \geq C^{-1}I_\phi + o_R(1)\|\phi\|^2. \tag{28}$$

Finally, (26), (27), (28) and the fact that $I_\phi \geq 0$ yield

$$\begin{aligned}(L_{\varepsilon,\xi}\phi|\phi) &= \sigma_1 + \sigma_2 + 2\sigma_3 \\ &\geq C^{-1}\left[\|\phi_1\|^2 + \|\phi_2\|^2 + 2I_\phi\right] - C\varepsilon\|\phi\|^{2'} + o_R(1)\|\phi\|^2.\end{aligned}$$

Recalling (18) we infer that

$$(L_{\varepsilon,\xi}\phi|\phi) \geq C^{-1}\|\phi\|^2 - C\varepsilon\|\phi\|^{2'} + o_R(1)\|\phi\|^2.$$

Taking ε small and R large, equation (14) follows. This completes the proof of Lemma 3.3. ■

3.1.3 Proof of Proposition 3.1

Let $P = P_{\varepsilon\xi}$ denote the projection onto $(T_{z_\xi}Z)^\perp$. We want to find a solution $w \in (T_{z_\xi}Z)^\perp$ of the equation $P\nabla f_\varepsilon(z_\xi + w) = 0$. One has

$$\nabla f_\varepsilon(z + w) = \nabla f_\varepsilon(z) + D^2 f_\varepsilon(z)[w] + R(z, w),$$

with

$$\|R(z,w)\| = o(\|w\|), \quad \text{and} \quad \|R(z, w_1 - w_2)\| = o(\|w_1\| + \|w_2\|)\|w_1 - w_2\|, \tag{29}$$

uniformly with respect to $z = z_\xi$. Using the notation introduced in subsection 3.1.2, we are led to the equation

$$L_{\varepsilon,\xi}w + P\nabla f_\varepsilon(z) + PR(z,w) = 0.$$

According to Lemma 3.3, this is equivalent to

$$w = N_{\varepsilon,\xi}(w), \quad \text{where} \quad N_{\varepsilon,\xi}(w) = -L_{\varepsilon,\xi}^{-1}\left(P\nabla f_\varepsilon(z) + PR(z,w)\right).$$

From Lemma 3.2 and the first inequality in (29) it follows that

$$\|N_{\varepsilon,\xi}(w)\| \leq c_1(\varepsilon|\nabla V(\varepsilon\xi)| + \varepsilon^2) + o(\|w\|). \tag{30}$$

By the second inequality in (29) there holds

$$\|N_{\varepsilon,\xi}(w_1) - N_{\varepsilon,\xi}(w_1)\| = o(\|w_1\| + \|w_2\|)\|w_1 - w_2\|. \tag{31}$$

Then one readily checks that $N_{\varepsilon,\xi}$ is a contraction on some ball in $(T_{z_\xi}Z)^\perp$ provided that $\varepsilon > 0$ is small enough. Then there exists a unique w such that $w = N_{\varepsilon,\xi}(w)$. Let us point out that we cannot use the Implicit Function Theorem to find $w(\varepsilon,\xi)$, because the map $(\varepsilon,u)\mapsto P\nabla f_\varepsilon(u)$ fails to be C^2. However, fixed $\varepsilon>0$ small, we can apply the Implicit Function Theorem to the map $(\xi,w)\mapsto P\nabla f_\varepsilon(z_\xi+w)$. Then, in particular, the function $w(\varepsilon,\xi)$ turns out to be of class C^1 with respect to ξ.

The final assertion in the proposition is proved as follows. Studying the derivative of $w(\varepsilon,\xi)$ with respect to ξ, one can verify that

$$T_{z_\xi}Z\sim T_{z_\xi+w(\varepsilon,\xi)}\tilde{Z}\qquad \text{for } \varepsilon \text{ small},$$

where $\tilde{Z}=\{z+w\}$. Suppose $z_{\xi_0}+w(\varepsilon,\xi_0)$ is a critical point of $f_\varepsilon|_{\tilde{Z}}$. Then $\nabla f_\varepsilon(z_{\xi_0}+w(\varepsilon,\xi_0))$ is perpendicular to $T_{z_{\xi_0}+w(\varepsilon,\xi_0)}\tilde{Z}$, and hence almost perpendicular to $T_{z_{\xi_0}}Z$. Since, by construction of $\tilde{Z}$, it is $\nabla f_\varepsilon(z_{\xi_0}+w(\varepsilon,\xi_0))\in T_{z_{\xi_0}}Z$, it must be $\nabla f_\varepsilon(z_{\xi_0}+w(\varepsilon,\xi_0))=0$. This concludes the proof.

Remark 3.4 *From (30) it immediately follows that*

$$\|w\|\le C\left(\varepsilon|\nabla V(\varepsilon\xi)|+\varepsilon^2\right), \tag{32}$$

where $C>0$.

3.2 Study of the reduced functional and applications

The main purpose of this subsection is to use the estimates on w established above to find an expansion of $\Phi_\varepsilon(\xi)$ and $\nabla\Phi_\varepsilon(\xi)$, where $\Phi_\varepsilon(\xi)=f_\varepsilon(z_\xi+w(\varepsilon,\xi))$. In the sequel, to be short, we will often write z instead of z_ξ and w instead of $w(\varepsilon,\xi)$. It is always understood that ε is taken so small that all the results discussed in the preceding subsection hold.

We have

$$\Phi_\varepsilon(\xi)=\frac{1}{2}\|z+w\|^2+\frac{1}{2}\int_{\mathbb{R}^n}V(\varepsilon x)(z+w)^2-\frac{1}{p+1}\int_{\mathbb{R}^n}(z+w)^{p+1}.$$

Since $-\Delta z+V(\varepsilon\xi)z=z^p$ we infer that

$$\begin{aligned}\|z\|^2 &= -V(\varepsilon\xi)\int_{\mathbb{R}^n}z^2+\int_{\mathbb{R}^n}z^{p+1};\\ (z|w) &= -V(\varepsilon\xi)\int_{\mathbb{R}^n}zw+\int_{\mathbb{R}^n}z^pw.\end{aligned}$$

Then we find

$$\begin{aligned}\Phi_\varepsilon(\xi) &= \left(\frac{1}{2}-\frac{1}{p+1}\right)\int_{\mathbb{R}^n}z^{p+1}+\frac{1}{2}\int_{\mathbb{R}^n}\left[V(\varepsilon x)-V(\varepsilon\xi)\right]z^2\\ &+ \int_{\mathbb{R}^n}\left[V(\varepsilon x)-V(\varepsilon\xi)\right]zw+\frac{1}{2}\int_{\mathbb{R}^n}V(\varepsilon x)w^2\\ &+ \frac{1}{2}\|w\|^2-\frac{1}{p+1}\int_{\mathbb{R}^n}\left[(z+w)^{p+1}-z^{p+1}-(p+1)z^pw\right].\end{aligned}$$

Since $z(x) = \alpha(\varepsilon\xi)\overline{u}(\beta(\varepsilon\xi)x)$, where $\alpha = V^{1/(p-1)}$ and $\beta = V^{1/2}$, see (7), it follows that

$$\int_{\mathbb{R}^n} z^{p+1} dx = C_0 V(\varepsilon\xi)^\theta, \quad C_0 = \int_{\mathbb{R}^n} U^{p+1}; \qquad \theta = \frac{p+1}{p-1} - \frac{n}{2}.$$

Letting $C_1 = C_0[1/2 - 1/(p+1)]$ one has

$$\begin{aligned} \Phi_\varepsilon(\xi) &= C_1 V(\varepsilon\xi)^\theta + \frac{1}{2}\int_{\mathbb{R}^n} [V(\varepsilon x) - V(\varepsilon\xi)]\, z^2 + \int_{\mathbb{R}^n} [V(\varepsilon x) - V(\varepsilon\xi)]\, zw \\ &+ \frac{1}{2}\int_{\mathbb{R}^n} V(\varepsilon x) w^2 + \frac{1}{2}\|w\|^2 - \frac{1}{p+1}\int_{\mathbb{R}^n} \left[(z+w)^{p+1} - z^{p+1} - (p+1) z^p w\right]. \end{aligned} \tag{33}$$

The function Φ_ε can be estimated as follows.

Lemma 3.5 *Let $a(\varepsilon\xi) = \theta C_1 V(\varepsilon\xi)^{\theta-1}$. Then one has*

$$\Phi_\varepsilon(\xi) = C_1 V(\varepsilon\xi)^\theta + \rho_\varepsilon(\xi), \quad C_1 > 0, \quad \theta = \frac{p+1}{p-1} - \frac{n}{2}, \tag{34}$$

where $|\rho_\varepsilon(\xi)| \le C\left(\varepsilon|\nabla V(\varepsilon\xi)| + \varepsilon^2\right)$, and

$$\nabla\Phi_\varepsilon(\xi) = a(\varepsilon\xi)\varepsilon\nabla V(\varepsilon\xi) + \varepsilon^{1+\gamma} R_\varepsilon(\xi), \tag{35}$$

where $|R_\varepsilon(\xi)| \le C$ and $\gamma = \min\{1, p-1\}$.

PROOF. The first four error terms in (33) can be estimated as in Lemma 3.2, using the Hölder inequality and (32). Let us focus on the last term. Using the uniform boundedness of z and some elementary inequalities one finds

$$\left|(z+w)^{p+1} - z^{p+1} - (p+1)z^p w\right| \le C\left(|w|^2 + |w|^{p+1}\right).$$

Hence, from the Sobolev inequality we deduce

$$\left|\int_{\mathbb{R}^n} \left[(z+w)^{p+1} - z^{p+1} - (p+1)z^p w\right]\right| \le C\left(\|w\|^2 + \|w\|^{p+1}\right).$$

Then, using (32), we obtain (34). We do not treat (35) here, details are given in [3]. ∎

As an application of the previous results, we give the following theorem, regarding concentration at non-degenerate critical points of V.

Theorem 3.6 *Assume $(V1)$ and $(V2)$ hold, and suppose ξ_0 is a non-degenerate critical point of V, namely for which $D^2V(\xi_0)$ is non-singular. Then there exists a solution u_ε of (NLS_ε) which concentrates at ξ_0.*

PROOF. Using a Taylor expansion for V, one can find a small positive number δ_0 such that

$$\text{(36)} \qquad \nabla V \neq 0 \text{ on } \partial B_{\delta_0}(\xi_0) \qquad \text{and} \qquad \deg(\nabla V, 0, \partial B_{\delta_0}(\xi_0)) = (-1)^{\operatorname{sgn} \det D^2 V(\xi_0)}.$$

For $t \in [0,1]$, consider the function $\Phi_\varepsilon^t(\xi) = t\Phi_\varepsilon(\xi) + (1-t)a(\xi_0)V(\varepsilon\xi)$. From (35) and the first part of (36) one deduces that $\nabla\Phi_\varepsilon^t \neq 0$ on $\partial B_{\delta_0}(\xi_0)$ for all $t \in [0,1]$. By the homotopy property of the degree, it follows that

$$\deg(\nabla\Phi_\varepsilon, 0, \partial B_{\delta_0}(\xi_0)) = \deg(a(\xi_0)\nabla V, 0, \partial B_{\delta_0}(\xi_0)) = \deg(\nabla V, 0, \partial B_{\delta_0}(\xi_0)) \neq 0.$$

As a consequence Φ_ε possesses a critical point in $B_{\delta_0}(\xi_0)$ and hence, by Proposition 3.1, f_ε has a critical point of the form $z_{\xi_0} + o(1)$. Scaling back in the variable x, we obtain the conclusion. ■

4 Concentration at spheres for (NLS_ε)

In this section we study concentration at spheres for (NLS_ε). We assume that V is radial, namely $V(x) = V(|x|)$, and we will work in the space H^1_r of radial functions in $H^1(\mathbb{R}^n)$. Using this space, the Euler functional f_ε becomes, up to a constant factor

$$(37) \qquad f_\varepsilon(u) = \frac{1}{2}\int_0^\infty \left((u')^2 + v(\varepsilon r)u^2\right) r^{n-1} dr - \frac{1}{p+1}\int_0^\infty |u|^{p+1} r^{n-1} dr; \qquad u \in H^1_r.$$

To give an idea why (NLS_ε) might possess solutions concentrating on a sphere, let us make the following heuristic considerations. A concentrated solution of (NLS_ε) carries a *potential* energy due to V and a *volume energy.* The former would lead the region of concentration to approach the minima of V. On the other hand, unlike for the case of spike-layer solutions where the volume energy does not depend on the location, the volume energy of solutions concentrating on spheres tends to shrink the sphere. In the region where V is decreasing, there could possibly be a balance, that gives rise to solutions concentrating on a sphere. This phenomenon is quantitatively reflected by an *auxiliary weighted* potential M defined as follows. Let

$$\theta = \frac{p+1}{p-1} - \frac{1}{2},$$

and define M by setting

$$M(r) = r^{n-1} V^\theta(r).$$

We have the following result.

Theorem 4.1 *Let $(V1) - (V2)$ hold, let $p > 1$ and suppose that M has a point of local strict maximum or minimum at $r = \overline{r}$. Then, for $\varepsilon > 0$ small enough, (NLS_ε) has a radial solution which concentrates near the sphere $|x| = \overline{r}$.*

This existence results is complemented by showing that concentration at spheres necessarily occurs on stationary points of M.

Theorem 4.2 *Suppose that, for all $\varepsilon > 0$ small, (NLS_ε) has a radial solution u_ε concentrating on the sphere $|x| = \widehat{r}$, in the sense that $\forall\,\delta > 0$, $\exists\,\varepsilon_0 > 0$ and $R > 0$ such that*

$$(38) \qquad u_\varepsilon(r) \leq \delta, \quad \textit{for}\ \varepsilon \leq \varepsilon_0,\ \textit{and}\ |r - \widehat{r}| \geq \varepsilon R.$$

Then u_ε has a unique maximum at $r = r_\varepsilon$, $r_\varepsilon \to \widehat{r}$ and $M'(\widehat{r}) = 0$.

Theorem 4.2 is the counterpart of known results dealing with necessary conditions for concentration of spikes.

In the case $n = 1$ one obviously has $M'(r) = 0$ iff $V'(r) = 0$. Otherwise, when $n > 1$ one has

$$M'(r) = r^{n-2} V^{\theta-1}(r)\ \left[(n-1)V(r) + \theta r V'(r)\right].$$

Therefore, critical points of M belong to the region $V' < 0$, as remarked before.

Remark 4.3 *Note that in Theorem 4.1 we do not require any upper bound on the exponent p, namely we can deal with the supercritical case as well. Roughly, the reason is the following. The asymptotic profile of a radial concentrating function is the the solution of (P_λ) for $n=1$ (and a suitable λ). In this case there is no restriction on p for the existence of a solution.*

4.1 The finite-dimensional reduction

Setting $Stat(M)=\{r>0: M'(r)=0\}$, let us fix $\rho_0>0$ with $8\rho_0<\min Stat(M)$ and let $\phi_\varepsilon(r)$ denote a smooth non-decreasing function such that

$$\phi_\varepsilon(r)=\begin{cases}0, & \text{if} \quad r\le \frac{\rho_0}{2\varepsilon},\\ \\ 1, & \text{if} \quad r\ge \frac{\rho_0}{\varepsilon}.\end{cases} \qquad |\phi_\varepsilon'(r)|\le \frac{4\varepsilon}{\rho_0}, \quad |\phi_\varepsilon''(r)|\le \frac{16\varepsilon^2}{\rho_0^2}.$$

For $\rho\ge 4\rho_0/\varepsilon$, set

$$z_{\rho,\varepsilon}(r)=\phi_\varepsilon(r)\,\overline{u}_\lambda(r-\rho); \qquad \lambda^2=V(\varepsilon\rho). \tag{39}$$

Fixed $\ell>\overline{r}$, see Theorem 4.1, consider the compact interval $\mathcal{T}_\varepsilon=[4\varepsilon^{-1}\rho_0,\varepsilon^{-1}\ell]$ and let

$$Z=Z_\varepsilon=\{z=z_{\rho,\varepsilon}:\rho\in\mathcal{T}_\varepsilon\}.$$

Recall that Proposition 2.1 yields

$$z(r)\le Ce^{-\lambda_0|r-\rho|}, \qquad (\lambda_0^2=\inf\{V(r):r\in\mathbb{R}^+\}).$$

Choose $\eta>0$ such that

$$\lambda_1:=\lambda_0-\eta>\frac{\lambda_0}{\min\{p,2\}}.$$

Given a positive constants $\gamma>0$ (to be fixed later), we define

$$\mathcal{C}_\varepsilon=\left\{w\in H_r^1:\ \|w\|_{H_r^1}\le\gamma\varepsilon\|z\|_{H_r^1},\ |w(r)|\le\gamma\, e^{-\lambda_1(\rho-r)} \text{ for } r\in[0,\rho]\right\}, \tag{40}$$

We will look for critical points of f_ε of the form

$$u=z+w, \quad z=z_{\rho,\varepsilon}\in Z, \quad w\perp T_zZ, w\in\mathcal{C}_\varepsilon.$$

We have indeed the following Proposition.

Proposition 4.4 *For ε sufficiently small there exists a positive constant γ such that for $\rho\in\mathcal{T}_\varepsilon$, there exists and a function $w=w(z_{\rho,\varepsilon})\in W=(T_zZ)^\perp$ satisfying $Pf_\varepsilon'(z+w)=0$ and*

i) $w\in\mathcal{C}_\varepsilon=\left\{w\in H_r^1:\ \|w\|\le\gamma\varepsilon\|z\|,\ |w(r)|\le\gamma\, e^{-\lambda_1|r-\rho|} \textit{ for } r\in[0,\rho]\right\}$,

ii) $\|w\|\le C\,\|f_\varepsilon'(z_\rho)\|$.

Setting

$$\Phi_\varepsilon(\rho) = f_\varepsilon(z_{\rho,\varepsilon} + w_{\rho,\varepsilon}),$$

if, for some $\varepsilon \ll 1$, ρ_ε *is stationary point of* Φ_ε, *then* $\widetilde{u}_\varepsilon = z_{\rho_\varepsilon,\varepsilon} + w_{\rho_\varepsilon,\varepsilon}$ *is a critical point of* f_ε.

The reason for the introduction of the set $\mathcal{C}_\varepsilon$ is the following. The norm of the function $z_{\rho,\varepsilon}$ and of the gradient $I'_\varepsilon(z_{\rho,\varepsilon})$ may diverge as ε goes to zero, see (42) and Lemma 4.6. For this reason it is not possible to perform the contraction argument using only norm estimates, as in the proof of Proposition 3.1. By means of the set $\mathcal{C}_\varepsilon$ we keep the function w small in L^∞, see (44), and the function $z + w$ concentrated near $|x| = \rho$.

Let us recall that, by the Strauss Lemma, if $u \in H^1_r$ one has

$$|u(r)| \le C\, r^{(1-n)/2}\, \|u\|_{H^1(\mathbb{R}^n)}, \quad r \ge 1. \tag{41}$$

Lemma 4.5 *For* $\varepsilon > 0$ *small,* $w \in \mathcal{C}_\varepsilon$, $\rho \in \mathcal{T}_\varepsilon$ *and* $r > 0$ *one has*

$$\|z_{\rho,\varepsilon}\|_{H^1_r} \sim \varepsilon^{(1-n)/2}; \tag{42}$$

$$\|w\|_{H^1_r} \lesssim \varepsilon\, \|z_{\rho,\varepsilon}\|_{H^1_r} \sim \varepsilon^{(3-n)/2}; \tag{43}$$

$$|w(r)| \le C\,\varepsilon, \quad \forall\, r \ge 0, \tag{44}$$

where C *depends only on* n, p, V *and the constant* γ *in the definition of* $\mathcal{C}_\varepsilon$.

PROOF. By the exponential decay of $z_{\rho,\varepsilon}$ we have

$$\|z_{\rho,\varepsilon}\|^2_{H^1_r} = \int_0^{+\infty} r^{n-1}(|z'|^2 + V(\varepsilon r)z^2)dr \sim \rho^{n-1}.$$

Since $\rho \in \mathcal{T}_\varepsilon$, then $\rho \sim \varepsilon^{-1}$ and (42) follows. Equation (43) is an immediate consequence of (42) and of the fact that $w \in \mathcal{C}_\varepsilon$. To estimate $|w(r)|$ we first recall that, taking ε such that $\rho_0/\varepsilon > 1$ and using (41) we get

$$|w(r)| \le C\, r^{(1-n)/2}\|w\|_{H^1_r} \le C\, \varepsilon^{(n-1)/2}\|w\|_{H^1_r}, \qquad \text{for all } r \ge \rho_0/\varepsilon.$$

Then (43) implies

$$|w(r)| \le C\,\varepsilon \qquad \text{for all } r \ge \rho_0/\varepsilon.$$

Furthermore, according to the definition of $\mathcal{C}_\varepsilon$, and recalling that $\rho \in \mathcal{T}_\varepsilon$ implies $\rho \ge 4\frac{\rho_0}{\varepsilon}$, we have

$$|w(r)| \le \gamma e^{-\lambda_1(\rho - r)} \le \gamma e^{-3\frac{\lambda_1}{\varepsilon}} \le C\varepsilon, \qquad \text{for all } r \le \rho_0/\varepsilon.$$

Hence (44) follows. ■

4.1.1 Gradient estimates and invertibility

Lemma 4.6 *For $\rho \in \mathcal{T}_\varepsilon$ there holds*

(*E*1) $\quad \|f_\varepsilon'(z_{\rho,\varepsilon})\| \lesssim \varepsilon \, \|z_{\rho,\varepsilon}\|_{H_r^1} \sim \varepsilon^{(3-n)/2}$;

(*E*2) $\quad \|f_\varepsilon''(z_{\rho,\varepsilon} + sw)\| \leq C, \quad (0 \leq s \leq 1)$;

Proof of $(E1)$. For all $v \in H_r^1$ one has

$$
\begin{aligned}
f_\varepsilon'(z)[v] &= \int_0^{+\infty} r^{n-1}\left(z'v' + V(\varepsilon r) z v - z^p v\right) dr \\
&= -\int_0^{+\infty} v(r^{n-1}z')' dr + \int_0^{+\infty} r^{n-1}\left(V(\varepsilon r) z v - z^p v\right) dr \\
&= \underbrace{-(n-1)\int_0^{+\infty} r^{n-2} z' v dr}_{A_0(v)} \underbrace{- \int_0^{+\infty} r^{n-1} z'' v dr + \int_0^{+\infty} r^{n-1}\left(V(\varepsilon r) z v - z^p v\right) dr}_{A_1(v)} .
\end{aligned}
$$

Using the Hölder inequality we get

$$
|A_0(v)| \leq C \, \|v\|_{H_r^1} \left(\int_0^{+\infty} (r^{(n-3)/2} z')^2 dr \right)^{1/2} .
$$

Since z decays exponentially away from $r = \rho$ and since $\rho \in \mathcal{T}_\varepsilon$, it follows that

$$
\int_0^{+\infty} (r^{(n-3)/2} z')^2 dr = \int_0^{+\infty} r^{-2} \cdot r^{n-1} |z'|^2 dr \sim \rho^{-2} \|z\|_{H_r^1}^2 \sim \varepsilon^2 \|z\|_{H_r^1}^2 .
$$

Then, using (42) we find

$$
\sup\{|A_0(v)| : \|v\|_{H_r^1} \leq 1\} \sim \varepsilon \, \|z\| \sim \varepsilon^{(3-n)/2}. \tag{45}
$$

To estimate $A_1(v)$ we recall that, by definition, $z = \phi \cdot \overline{u}_\lambda(r - \rho)$ and hence

$$
\begin{aligned}
A_1(v) &= \underbrace{= \int_0^{+\infty} r^{n-1} (\phi'' U + 2\phi' U') v dr}_{A_2(v)} \\
&+ \underbrace{\int_0^{+\infty} r^{n-1}\left(\phi U v + V(\varepsilon r) \phi U v - (\phi U)^p v\right) dr - \int_0^{+\infty} r^{n-1} \phi U'' v dr}_{A_3(v)},
\end{aligned}
$$

where U stands for $\overline{u}_\lambda(r - \rho)$. Since the support of ϕ' is the interval $[\rho_0/2\varepsilon, \rho_0/\varepsilon]$ and U decays exponentially to zero as $r \to \infty$ we get

$$
\sup\{|A_2(v)| : \|v\|_{H_r^1} \leq 1\} \sim e^{-\frac{1}{C\varepsilon}}. \tag{46}
$$

Finally, using equation (56) we infer

$$A_3(v) = \int_0^{+\infty} r^{n-1}\left(V(\varepsilon r) - V(\varepsilon\rho)\right)\phi U v dr = \int_0^{+\infty} r^{n-1}\left(V(\varepsilon r) - V(\varepsilon\rho)\right) z v dr.$$

Since V is bounded,

$$|A_3(v)| \leq C\ \|v\|_{H_r^1}\left(\int_0^{+\infty} r^{n-3} z^2 dr\right)^{1/2},$$

and hence, in view of (42),

$$\sup\{|A_3(v)| : \|v\|_{H_r^1} \leq 1\} \sim \varepsilon\ \|z\| \sim \varepsilon^{(3-n)/2}. \tag{47}$$

Putting together (45), (46) and (47), we find $(E1)$. ∎

Proof of $(E2)$. For $v \in H_r^1$ we get

$$\begin{aligned} |f_\varepsilon''(z+sw)[v,v]| &= \left|\int_0^{+\infty} r^{n-1}\left(|v'|^2 + v^2 + V(\varepsilon r)v^2 - p|z+sw|^{p-1}v^2\right) dr\right| \\ &\leq C\ \|v\|_{H_r^1}^2 + p\left|\int_0^{+\infty} r^{n-1}|z+sw|^{p-1}v^2 dr\right|. \end{aligned}$$

According to (44) one has that $|z(r) + sw(r)| \leq C$ and thus

$$\left|\int_0^{+\infty} r^{n-1}|z+sw|^{p-1}v^2 dr\right| \leq c\ \|v\|_{H_r^1}^2.$$

Then $|f_\varepsilon''(z+sw)[v,v]| \leq c\ \|v\|_{H_r^1}^2$ and $(E2)$ follows. ∎

As in the previous section, with only minor modifications, one can prove the following result.

Lemma 4.7 *There exists a positive constant C such that, for every $\rho \in \mathcal{T}_\varepsilon$ and for ε sufficiently small there holds*

$$f_\varepsilon''(z)[v,v] \geq C^{-1}\|v\|^2, \qquad \text{for all } v \perp \{tz\} \oplus T_z Z.$$

4.1.2 Decay estimates

In this section we just give a brief motivation of the choice of the set $\mathcal{C}_\varepsilon$, and in particular of the number λ_1. The function w is obtained as a fixed point of a map $N_{\varepsilon,\rho}$. If $\tilde{w} = N_{\varepsilon,\rho} w$, with $w \in \mathcal{C}_\varepsilon$, then $\tilde{w}$ satisfies the equation

$$\begin{aligned} -\Delta\tilde{w} + V(\varepsilon r)\tilde{w} - pz^{p-1}\tilde{w} &= -\left((z+w)^p - z^p - pz^{p-1}w\right) + \beta\left(-\Delta\dot{z} + V(\varepsilon r)\dot{z}\right) \\ &+ \left(-\Delta z + V(\varepsilon r) z - z^p\right), \qquad \text{in } \mathbb{R}^n, \end{aligned} \tag{48}$$

where β is a real number with $|\beta| \lesssim \varepsilon$, and where $\dot{z} = \frac{\partial z}{\partial \rho}$, see [1]. From the decay of w and some elementary inequalities we have

$$\left|(z+w)^p - z^p - pz^{p-1}w\right|(x) \leq C\max\{|w|^p, |w|^2\} \leq Ce^{-(2\wedge p)\lambda_1|\rho-|x||} \leq Ce^{-\lambda_0|\rho-|x||}.$$

The function on the right-hand side has a decay comparable to that of z.

4.2 Proof of Theorem 4.1

First of all we expand the functional Φ_ε.

Lemma 4.8 *For $\varepsilon > 0$ small, there is a constant $C_0 > 0$ such that:*

$$\varepsilon^{n-1} f_\varepsilon(z_{\rho,\varepsilon} + w_{\rho,\varepsilon}) = C_0 M(\varepsilon\rho) + O(\varepsilon^2), \quad \rho \in \mathcal{T}_\varepsilon.$$

PROOF. For brevity, we write z instead of $z_{\rho,\varepsilon}$ and w instead of $w_{\rho,\varepsilon}$. One has

$$f_\varepsilon(z+w) = f_\varepsilon(z) + f_\varepsilon'(z)[w] + \int_0^1 f_\varepsilon''(z+sw)[w]^2 ds.$$

Using (43), $(E1)$ and $(E2)$ we infer

$$f_\varepsilon(z+w) = f_\varepsilon(z) + O(\varepsilon^{3-n}).$$

On the other hand, recall that by definition $z_{\rho,\varepsilon}(r) = \phi_\varepsilon(r)\overline{u}_\lambda(r-\rho)$. Then z concentrates near ρ and one finds

$$\begin{aligned} f_\varepsilon(z) &= \int_0^\infty r^{n-1}\left(\frac{|z'|^2 + V(\varepsilon r)z^2}{2} - \frac{z^{p+1}}{p+1}\right) dr \\ &= \rho^{n-1}\int_{\mathbb{R}}\left(\frac{|\overline{u}_\lambda'|^2 + V(\varepsilon\rho)\overline{u}_\lambda^2}{2} - \frac{\overline{u}_\lambda^{p+1}}{p+1}\right) dr\,(1+o(1)). \end{aligned}$$

We recall that

$$\overline{u}_\lambda(r) = \lambda^{2/(p-1)}\overline{u}(\lambda r), \qquad \lambda^2 = V(\varepsilon\rho).$$

It follows by a straightforward calculation that

$$\int_{\mathbb{R}}\left(\frac{|U'|^2 + V(\varepsilon r)U^2}{2} - \frac{U^{p+1}}{p+1}\right) dr = C_0 V^\theta(\varepsilon\rho),$$

where θ and C_0 have been defined in the previous section. Substituting into the preceding equations we find

$$f_\varepsilon(z+w) = C_0\rho^{n-1}V^\theta(\varepsilon\rho) + O(\varepsilon^{3-n}).$$

Recalling the definition of M we get

$$f_\varepsilon(z+w) = \frac{C_0}{\varepsilon^{n-1}}(\varepsilon\rho)^{n-1}V^\theta(\varepsilon\rho) + O(\varepsilon^{3-n}) = \frac{C_0}{\varepsilon^{n-1}} M(\varepsilon\rho) + O(\varepsilon^{3-n}),$$

and the lemma follows. ∎

Proof of Theorem 4.1 completed. Let us first consider the case $p \in \left(1, \frac{n+2}{n-2}\right]$. By Lemma 4.8, if $\overline{r}$ is a maximum (resp. minimum) of M then $\Phi_\varepsilon(\rho) = f_\varepsilon(z_{\rho_\varepsilon,\varepsilon} + w_{\rho_\varepsilon,\varepsilon})$ will possess a maximum (resp. minimum) at some $\rho_\varepsilon \sim \overline{r}/\varepsilon$, with $\rho_\varepsilon \in \mathcal{T}_\varepsilon$. Using Proposition 4.4, such a stationary point of Φ_ε gives rise to a critical point $\widetilde{u}_\varepsilon = z_{\rho_\varepsilon,\varepsilon} + w_{\rho_\varepsilon,\varepsilon}$, which is a (radial) solution of (NLS_ε). Since $\widetilde{u}_\varepsilon(r) \sim \overline{u}_\lambda(r-\rho_\varepsilon) \sim \overline{u}_\lambda(r-\overline{r}/\varepsilon)$, then $u_\varepsilon(r) \sim \overline{u}_\lambda\left((r-\overline{r})/\varepsilon\right)$ and hence u_ε, rescaled in r, concentrates near the sphere $|x| = \overline{r}$.

Let us now consider the case $p > \frac{n+2}{n-2}$. The proof is done using some truncation for the nonlinear term, and then proving a-priori L^∞ estimates on the solutions. We list the modifications to the above proof which are necessary to handle this case.

For $K > 0$, we define a smooth positive function $F_K : \mathbb{R} \to \mathbb{R}$ such that

$$F_K(t) = |t|^{p+1} \quad \text{for } |t| \le K; \qquad F_K(t) = (K+1)^{p+1} \quad \text{for } |t| \ge K+1.$$

Let $f_{\varepsilon,K} : H^1_r \to \mathbb{R}$ be the functional obtained substituting $|u|^{p+1}$ with $F_K(u)$ in f_ε, and let $K_0 = (\sup V)^{\frac{1}{p-1}}$. Since the non-linear term in $f_{\varepsilon,K}$ is sub-critical, this is a well-defined functional on H^1_r.

We note that by the definition of $\overline{u}_\lambda$ and $z_{\rho,\varepsilon}$, it is $\|z_{\rho,\varepsilon}\| \le K_0$ for all $\rho \in \mathcal{T}_\varepsilon$ and ε sufficiently small.

In the above notation, if $K \ge K_0$, the operator $Pf''_{\varepsilon,K}(z)$ remains invertible and its inverse A_ε has uniformly bounded norm, independent of K. In fact, Lemmas 4.5, 4.6, 4.7 and Proposition 4.4 are based on local arguments and remain unchanged. Moreover, if $K \ge K_0 + \gamma$ (see the definition (40) of $\mathcal{C}_\varepsilon$) and using the pointwise bounds on $|w(r)|$ stated in equation (44), one readily checks that the estimates $(E1)-(E2)$ involving $f'_\varepsilon(z)$ and $f''_\varepsilon(z+w)$ are also independent of K. Hence the above method produces a solution u_ε of $f'_{\varepsilon,K} = 0$ for which $\|u_\varepsilon\|_\infty \le K$. Hence u_ε also solves (NLS_ε). This completes the proof of Theorem 4.1. ∎

We remark that, since Φ_ε depends only on one variable, it is not necessary to study its derivative with respect to ρ in order to get critical points.

4.3 Bifurcation of non-radial solutions

In this section we study bifurcation of non-radial solutions from a family of radial solutions concentrating on spheres. We will assume first that the exponent p lies in the range $\left(1, \frac{n+2}{n-2}\right]$, so all the functionals involved are well-defined. The general case $p > \frac{n+2}{n-2}$ will be handled by a truncation procedure as before.

Proposition 4.9 *Let u_ε be the family of solutions radial solutions u_ε of (NLS_ε) having the form*

$$u_\varepsilon = z_{\rho_\varepsilon,\varepsilon} + w_{\rho_\varepsilon,\varepsilon}, \qquad \text{for some } \rho_\varepsilon \sim \frac{r_0}{\varepsilon},$$

where $w_{\rho_\varepsilon,\varepsilon} \in \mathcal{C}_\varepsilon$ (see equation (40)). Then the Morse index of u_ε in $H^1(\mathbb{R}^n)$ tends to infinity as ε goes to zero.

PROOF. Let $\{\varphi_j\}_j, j \in \mathbb{N}$ denote the eigenfunctions of $-\Delta$ on S^{n-1} with eigenvalues $\{\mu_j\}_j$, where the μ_j's are chosen to be non-decreasing in j. In particular we have

$$\int_{S^{n-1}} |\nabla \varphi_j|^2 = \mu_j \int_{S^{n-1}} \varphi_j^2, \qquad j \ge 1. \tag{49}$$

Recalling the definition of the cutoff function ϕ_ε in Section 2, let us define $v_j \in H^1(\mathbb{R}^n)$ in the following way

$$v_j(r,\theta) = u_\varepsilon(r)\phi_\varepsilon(r)\varphi_j(\theta), \qquad r \geq 0, \theta \in S^{n-1}, j \geq 1.$$

We note that, in polar coordinates

$$\nabla v_j = \left(\varphi_j(\theta) \left(\phi_\varepsilon u_\varepsilon\right)'(r), \frac{1}{r} u_\varepsilon(r)\phi_\varepsilon(r)\nabla_{S^{n-1}}\varphi_j(\theta) \right), \tag{50}$$

and hence we deduce

$$(v_i, v_j) = 0 \qquad \text{for } i \neq j. \tag{51}$$

Moreover it turns out that

$$I_\varepsilon''(u_\varepsilon)[v_j, v_j] = C_{1,\varepsilon} \int_{S^{n-1}} \varphi_j^2 + C_{2,\varepsilon} \int_{S^{n-1}} |\nabla \varphi_j|^2, \tag{52}$$

where

$$\begin{aligned} C_{1,\varepsilon} &= \left[\int r^{n-1} \left((\phi_\varepsilon u_\varepsilon)' \right)^2 + \int r^{n-1} V(\varepsilon r)(\phi_\varepsilon u_\varepsilon)^2 - p \int r^{n-1} u_\varepsilon^{p+1} \phi_\varepsilon^2 \right]; \\ C_{2,\varepsilon} &= \int r^{n-3} (\phi_\varepsilon u_\varepsilon)^2. \end{aligned}$$

From the estimates of the previous sections one easily finds

$$C_{1,\varepsilon} \sim \left(\frac{r_0}{\varepsilon}\right)^{n-1} \overline{I}_{V(r_0)}''[\overline{u}_\lambda, \overline{u}_\lambda] < 0; \qquad C_{2,\varepsilon} \sim \left(\frac{r_0}{\varepsilon}\right)^{n-3} \int_{\mathbb{R}} \overline{u}_\lambda^2.$$

From (49) and the last equations it follows that, for any fixed $j \geq 1$, for ε small one has

$$I_\varepsilon''(u_\varepsilon)[v_i, v_i] \leq \frac{1}{2}\omega_{n-1} \left(\frac{r_0}{\varepsilon}\right)^{n-1} \overline{I}_{V(r_0)}''[\overline{u}_\lambda, \overline{u}_\lambda] < 0, \qquad \text{for all } i \leq j,$$

where $\lambda^2 = V(r_0)$. Hence setting

$$\mathcal{M}_j = \text{ span}\{v_i \; : \; i = 1, \ldots, j\},$$

from the orthogonality relation (51) it follows that $I_\varepsilon''(u_\varepsilon)$ is negative-definite on $\mathcal{M}_j$ for ε sufficiently small. This concludes the proof. ■

With some delicate analysis of the small eigenvalues of $f_(''u_\varepsilon)$, one can prove the following result.

Proposition 4.10 *Suppose $M''(r_0) \neq 0$, and suppose u_ε is a solution concentrating at $|x| = r_0$ constructed with the above method. Then, for ε small, u_ε is non-degenerate in H_r^1.*

By means of these two propositions, we can prove the following result.

Theorem 4.11 *Suppose that, in addition to the assumption of Theorem 4.1, the potential V is smooth and that at a point $\overline{r} > 0$ of local strict maximum or minimum of M there holds*

$$M''(\overline{r}) \neq 0. \tag{53}$$

Then there exists $\varepsilon_0 \in (0, \overline{\varepsilon})$ such that $\Lambda = \Lambda_{\overline{r},\varepsilon_0}$ is a smooth curve. Moreover, there exist a sequence $\varepsilon_j \downarrow 0$ such that from each $u_{\varepsilon_j} \in \Lambda$ bifurcates a family of non-radial solutions of (NLS_ε).

PROOF. By Proposition 4.10 the solution $u_\varepsilon = \overline{z}_\omega + \overline{w}$ of (NLS_ε) is non-degenerate and locally unique in the class of radial functions. This implies that the set Λ in Theorem 4.11 is a smooth curve. By Proposition 4.9 the Morse index of $f''_\varepsilon(u_\varepsilon)$, in the space $H^1(\mathbb{R}^n)$, diverges as $\varepsilon \to 0$. To obtain the conclusion it is sufficient to apply a bifurcation result of Kielhofer. When $p > \frac{n+2}{n-2}$ it is sufficient to consider a cutoff function F_K as in the proof of Theorem 4.1. The above argument yields bifurcation of non-radial solutions of $f'_{\varepsilon,K} = 0$. The L^∞ bounds on the radial solutions and standard regularity results imply that non-radial solutions which are sufficiently close to the radial ones (in $H^1(\mathbb{R}^n)$) are also uniformly bounded. Hence these critical points are also solutions of (NLS_ε). ∎

5 Concentration at spheres for (P_ε)

In this section we study concentration at spheres for (P_ε) in the case of the unit ball $\Omega = B_1 = \{x \in \mathbb{R}^n : |x| < 1\}$, highlighting that new phenomena take place, due to the imposed boundary conditions. Our main result on problem (N) is the following theorem.

Theorem 5.1 *Let $p > 1$, and consider the problem*

$$(\tilde{N}) \qquad \begin{cases} -\varepsilon^2 \Delta u + u = u^p, & \text{in } B_1, \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial B_1, & u > 0. \end{cases}$$

Then there exists a family of radial solutions u_ε of $(\tilde{N})$ concentrating on $|x| = r_\varepsilon$, where r_ε is a local maximum point of u_ε for which $1 - r_\varepsilon \sim \varepsilon |\log \varepsilon|$.

A similar result holds in the case of the annulus, $\Omega = \{a < |x| < 1\}$, with $a \in (0, 1)$. Also, related phenomena occur in annuli when one imposes Dirichlet boundary conditions, see [2].

As before, it is convenient to scale the problem to the set $\Omega_\varepsilon = \frac{1}{\varepsilon} B_1$, and to use the functional I_ε introduced above

$$I_\varepsilon(u) = \frac{1}{2} \int_{\Omega_\varepsilon} \left(|\nabla u|^2 + V(\varepsilon |x|) u^2 \right) - \frac{1}{p+1} \int_{\Omega_\varepsilon} |u|^{p+1}, \qquad u \in H^1_r.$$

5.1 Abstract setting and preliminary estimates

For any $r_0 < \frac{1}{2}$, let $\phi_\varepsilon(r)$ be a smooth cutoff function such that

$$(54) \qquad \phi_\varepsilon(r) = \begin{cases} 0 & \text{for } r \in \left[0, \frac{r_0}{8\varepsilon}\right]; \\ 1 & \text{for } r \in \left[\frac{r_0}{4\varepsilon}, \frac{1}{\varepsilon}\right]; \\ |\phi_\varepsilon'(r)| \leq C\varepsilon & \text{for } r \in \left[\frac{r_0}{8\varepsilon}, \frac{r_0}{4\varepsilon}\right]; \\ |\phi_\varepsilon''(r)| \leq C\varepsilon^2 & \text{for } r \in \left[\frac{r_0}{8\varepsilon}, \frac{r_0}{4\varepsilon}\right]. \end{cases}$$

We define Z^N to be the following manifold

$$(55) \qquad Z^N = \left\{ \phi_\varepsilon \left(z_\rho + \alpha_{1,p} e^{-\left(\frac{1}{\varepsilon} - \rho\right)} e^{-\left(\frac{1}{\varepsilon} - \cdot\right)} \right) \right\}_\rho := \left\{ z^N_\rho = \phi_\varepsilon \left(z_\rho + v_\rho \right) \right\}_\rho; \qquad \rho \geq \frac{3}{4\varepsilon}.$$

Here $\alpha_{1,p}$ is the constant in Proposition 2.1 for $n = 1$, and $z_\rho(r) = \overline{u}(r - \rho)$. The range of ρ will be chosen appropriately later. The function z^N has been chosen in such a way that it has a small normal derivative at $\partial \Omega_\varepsilon$. In fact, we have the following estimate

$$(56) \qquad (z^N)' \left(\frac{1}{\varepsilon} \right) = z_\rho' \left(\frac{1}{\varepsilon} \right) - \alpha_{1,p} e^{-\left(\frac{1}{\varepsilon} - \rho\right)}$$

$$(57) \qquad = z_\rho \left(\frac{1}{\varepsilon} \right) \left(\frac{z_\rho'}{z_\rho} \left(\frac{1}{\varepsilon} \right) - \frac{\alpha_{1,p} e^{-\left(\frac{1}{\varepsilon} - \rho\right)}}{z_\rho \left(\frac{1}{\varepsilon} \right)} \right) = o \left(e^{-\left(\frac{1}{\varepsilon} - \rho\right)} \right).$$

The difference between z^N and z can be heuristically viewed as an addition of a *virtual spike* outside Ω.

We need first the following lemma, which can be proved by reasoning as in [1], Section 3.

Lemma 5.2 *Let Z^N be as above, and let $w \in \mathcal{C}_\varepsilon$. Then the following properties hold true*

$(E1)$ $\qquad \|I''_\varepsilon(z^N_\rho + sw)\| \leq C, \quad (0 \leq s \leq 1)$;

$(E2)$ $\qquad \|I''_\varepsilon(z^N_\rho + sw) - I_\varepsilon(z^N_\rho)\| \leq C \max\left\{\|w\|_\infty, \|w\|_\infty^{(p-1)}\right\}, \quad (0 \leq s \leq 1)$;

The same arguments of [1] Section 4, with some minor modifications, yield the following result.

Proposition 5.3 *There exists a positive constant μ with the following property. For ε sufficiently small, and for all $\rho \in \left[\frac{r_0}{\varepsilon}, \frac{1}{\varepsilon} - \mu\right]$ there exist a function $w(z^N_\rho) = w^N_\rho$ satisfying*

i) $I'_\varepsilon(z^N_\rho + w(z^N_\rho)) = \alpha^N_\rho \frac{\partial}{\partial\rho} z^N_\rho$;

ii) $w(z^N_\rho) \perp T_{z^N_\rho} Z^N$;

iii) $\|w(z^N_\rho)\| \leq C\|I'_\varepsilon(z^N_\rho)\|$;

iv) $\|w(z^N_\rho)\|_\infty \leq C\left(\varepsilon + \varepsilon^{\frac{n-1}{2}}\|I'_\varepsilon(z^N_\rho)\|\right)$.

where $\alpha^N_\rho \in \mathbb{R}$ and C is a positive constant depending only on n, p and μ. Moreover, if for some $\varepsilon \ll 1$, ρ_ε is stationary point of $\Psi_\varepsilon(\rho) = I_\varepsilon(z^N_\rho + w^N_\rho)$, then $\widetilde{u}_\varepsilon = z^N_{\rho_\varepsilon} + w^N_{\rho_\varepsilon}$ is a critical point of I_ε.

Proposition 5.3 is proved using the Contraction Mapping Theorem in the set $\mathcal{C}_\varepsilon$. The condition $\rho \in \left[\frac{r_0}{\varepsilon}, \frac{1}{\varepsilon} - \mu\right]$, for μ sufficiently large and for ε sufficiently small, makes the functional $I''_\varepsilon(z^N_\rho)$ qualitatively similar to $\overline{I}''(\overline{u})$ and allows to conclude

$$I''_\varepsilon(z^N_\rho)[v, v] \geq \delta\|v\|^2; \qquad \text{for all } v \perp z^N_\rho \oplus T_{z^N_\rho} Z^N,$$

for some $\delta > 0$, as in [1], Lemma 4.1.

We note that in the case $\Omega = A$ the functions in H^1_r belong to L^∞. The argument carries over directly in H^1_r and there is no need to introduce the set $\mathcal{C}_\varepsilon$.

For proving Theorem 5.1 we need to be careful in the expansion of Ψ_ε, since we want to consider values of ρ which are *close* to the exterior boundary of Ω_ε. In the next Lemma we estimate the quantity $\|I'_\varepsilon(z^N_\rho)\|$, which by **iii)** is an upper bound for $\|w(z^N_\rho)\|$.

Lemma 5.4 *Let Z^N be defined in* (55). *Then there holds*

$$\|I_\varepsilon'(z_\rho^N)\| \leq C\varepsilon^{\frac{1-n}{2}}\left(\varepsilon + o\left(e^{-\left(\frac{1}{\varepsilon}-\rho\right)}\right)\right), \qquad \textit{for every } z_\rho^N \in Z^N.$$

for some constant C depending only and n and p.

PROOF. In the proof we will often omit the index ρ in z_ρ and v_ρ. Since $z_\rho = \overline{u}(\cdot - \rho)$ and v_ρ satisfy respectively the equations $-z_\rho'' + z_\rho = z_\rho^p$ and $-v_\rho'' + v_\rho = 0$, we have

$$\begin{aligned} I_\varepsilon'(z^N)[u] &= \int\left(-\Delta z^N + V(\varepsilon r)z^N - (z^N)^p\right)u + \left(\frac{1}{\varepsilon}\right)^{n-1}(z^N)'(1/\varepsilon)u(1/\varepsilon) \\ &= \varepsilon^{1-n}(z^N)'(1/\varepsilon)u(1/\varepsilon) - (n-1)\int\frac{1}{r}(z^N)'u \\ &- \int\left(2\phi_\varepsilon'(z_\rho+v_\rho)' + \phi_\varepsilon''(z_\rho+v_\rho)\right)u - \int\left((z^N)^p - \phi_\varepsilon z_\rho^p\right). \end{aligned}$$

From formulas (41) and (56) we find

$$\varepsilon^{1-n}|(z^N)'(1/\varepsilon)u(1/\varepsilon)| = \varepsilon^{\frac{1-n}{2}}o\left(e^{-\left(\frac{1}{\varepsilon}-\rho\right)}\right)\|u\|. \tag{58}$$

On the other hand, since the function Z^N is supported in $\left\{r \geq \frac{r_0}{8\varepsilon}\right\}$, one has also

$$\left|\int\frac{1}{r}(z^N)'u\right| \leq C\varepsilon^{\frac{3-n}{2}}\|u\|. \tag{59}$$

From the exponential decay of z_ρ and v_ρ, from the fact that $\phi_\varepsilon', \phi_\varepsilon''$ have support in $\left[\frac{r_0}{8\varepsilon}, \frac{r_0}{4\varepsilon}\right]$ and from $\rho \geq \frac{3}{4\varepsilon}$, one deduces the estimates

$$\left|\int\phi_\varepsilon'(z_\rho+v_\rho)'u\right| \leq C\varepsilon\varepsilon^{\frac{1-n}{2}}e^{-\frac{r_0}{4\varepsilon}}\|u\|; \qquad \left|\int\phi_\varepsilon''(z_\rho+v_\rho)u\right| \leq C\varepsilon^2\varepsilon^{\frac{1-n}{2}}e^{-\frac{r_0}{4\varepsilon}}\|u\|. \tag{60}$$

Let us consider now the term $\int\left((z^N)^p - \phi_\varepsilon z_\rho^p\right)u$. We can write

$$(z^N)^p - \phi_\varepsilon z_\rho^p = \phi_\varepsilon^p\left((z+v)^p - z^p\right) + \phi_\varepsilon^p\left(\phi_\varepsilon^p z^p - \phi_\varepsilon z_\rho^p\right).$$

Since z is uniformly bounded, we have

$$\left|(z+v)^p - z^p - pz^{p-1}v\right| \leq C\max\{|v|^2, |v|^p\}.$$

It follows that

$$\left|\int\left[(z+v)^p - z^p\right]u\right| \leq p\left|\int z^{p-1}v|u|\right| + C\left|\int|u|\max\{|v|^2,|v|^p\}\right|.$$

Again from the Hölder inequality we obtain

$$\left|\int|v|^{2\wedge p}|u|\right| \leq e^{-(2\wedge p)\left(\frac{1}{\varepsilon}-\rho\right)}\int e^{-\left(\frac{1}{\varepsilon}-r\right)}|u| \leq Ce^{-(2\wedge p)\left(\frac{1}{\varepsilon}-\rho\right)}\varepsilon^{\frac{1-n}{2}}\|u\|.$$

We have also $\left|\int z^{p-1}v|u|\right| \leq \left(\int z^{2(p-1)}v^2\right)^{\frac{1}{2}} \|u\|$. We divide the last integral in the two regions $r \leq \frac{\rho+\varepsilon^{-1}}{2}$ and $r \geq \frac{\rho+\varepsilon^{-1}}{2}$. When $r \leq \frac{\rho+\varepsilon^{-1}}{2}$, v satifies $|v| \leq e^{-\frac{3}{2}\left(\frac{1}{\varepsilon}-\rho\right)}$ and hence

$$\left(\int_{r\leq\frac{\rho+\varepsilon}{2}} z^{2(p-1)}v^2\right)^{\frac{1}{2}} \leq Ce^{-\frac{3}{2}\left(\frac{1}{\varepsilon}-\rho\right)} \left(\int_{r\leq\frac{\rho+\varepsilon}{2}} z^{2(p-1)}\right)^{\frac{1}{2}} \leq Ce^{-\frac{3}{2}\left(\frac{1}{\varepsilon}-\rho\right)}\varepsilon^{\frac{1-n}{2}}.$$

On the other hand when $r \geq \frac{\rho+\varepsilon^{-1}}{2}$, z satisfies $|z(r)| \leq e^{-\frac{1}{2}\left(\frac{1}{\varepsilon}-\rho\right)}$ so we have

$$\left(\int_{r\geq\frac{\rho+\varepsilon}{2}} z^{2(p-1)}v^2\right)^{\frac{1}{2}} \leq Ce^{-\frac{p-1}{2}\left(\frac{1}{\varepsilon}-\rho\right)} \left(\int |v|^2\right)^{\frac{1}{2}} \leq Ce^{-\left(1+\frac{p-1}{2}\right)\left(\frac{1}{\varepsilon}-\rho\right)}\varepsilon^{\frac{1-n}{2}}.$$

We have also the estimate

$$\left|\int \left(\phi_\varepsilon^p z_\rho^p - \phi_\varepsilon z_\rho^p\right) u\right| \leq C\left(\int \left(\phi_\varepsilon^p - \phi_\varepsilon\right)^2 \overline{u}^{2p}\right)^{\frac{1}{2}} \|u\| \leq Ce^{-\frac{pr_0}{2\varepsilon}}\varepsilon^{\frac{1-n}{2}}\|u\|.$$

The above estimates yield

$$\left|\int \left((z^N)^p - \phi_\varepsilon \overline{u}^p\right) u\right| \leq C\varepsilon^{\frac{1-n}{2}}\left(e^{-\left(\frac{3\wedge(p+1)}{2}\right)\left(\frac{1}{\varepsilon}-\rho\right)} + e^{-\frac{pr_0}{4\varepsilon}}\right)\|u\|. \tag{61}$$

Hence (58)-(61) imply

$$\|I_\varepsilon'(z_\rho^N)\| \leq C\varepsilon^{\frac{1-n}{2}}\left(\varepsilon + o\left(e^{-\left(\frac{1}{\varepsilon}-\rho\right)}\right) + e^{-\frac{r_0}{4\varepsilon}}\right).$$

This concludes the proof of the lemma. ∎

5.2 Expansion of I_ε on Z^N

In this subsection we expand $I_\varepsilon(z_\rho^N)$ as a function of ρ and ε. Integrating by parts and using the equations satisfied by z and v (see the proof of Lemma 5.4), we find

$$\begin{aligned} I_\varepsilon(z^N) &= \frac{1}{2}\int\left(|\nabla z^N|^2 + (z^N)^2\right) - \frac{1}{p+1}\int |z^N|^{p+1} = \frac{1}{2}\int\left(-\Delta z^N + z^N\right)z^N \\ &+ \frac{1}{2}\left(\frac{1}{\varepsilon}\right)^{n-1} z^N(1/\varepsilon)(z^N)'(1/\varepsilon) - \frac{1}{p+1}\int |z^N|^{p+1} \\ &= \frac{1}{2}\left(\frac{1}{\varepsilon}\right)^{n-1} z^N(1/\varepsilon)(z^N)'(1/\varepsilon) + \frac{1}{2}\int \phi_\varepsilon z^p z^N - \frac{1}{p+1}\int |z^N|^{p+1} \\ &- \frac{n-1}{2}\int \frac{(z^N)'z^N}{r} - \int \phi_\varepsilon' z^N (z+v)' - \frac{1}{2}\int \phi_\varepsilon'' z^N(z+v). \end{aligned} \tag{62}$$

Let us estimate each of the seven terms in the last expression. From equations (2) and (56) we deduce

$$\varepsilon^{1-n}\left|z^N(1/\varepsilon)(z^N)'(1/\varepsilon)\right| = \varepsilon^{1-n}o\left(e^{-2\left(\frac{1}{\varepsilon}-\rho\right)}\right). \tag{63}$$

To estimate the second and the third term, we can write

$$
\begin{aligned}
\frac{1}{2}\int \phi_\varepsilon z^p z^N - \frac{1}{p+1}\int |z^N|^{p+1} &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\int \phi_\varepsilon^{p+1} z^{p+1} \\
&+ \frac{1}{2}\int \left(\phi_\varepsilon^2 - \phi_\varepsilon^{p+1}\right) z^p (z+v) - \frac{1}{2}\int \phi_\varepsilon^{p+1} z^p v \\
&- \frac{1}{p+1}\int \phi_\varepsilon^{p+1}\left(|z+v|^{p+1} - z^{p+1} - (p+1) z^p v\right).
\end{aligned} \tag{64}
$$

We have

$$
\begin{aligned}
\left|\int \phi_\varepsilon^{p+1} z^{p+1} - \rho^{n-1}\int_{\mathbb{R}} \overline{u}^{p+1}\right| &\leq \rho^{n-1}\int_{r\geq 1/\varepsilon} \overline{u}^{p+1}(r-\rho) + \int_{[0,1/\varepsilon]} r^{n-1}(1-\phi_\varepsilon^{p+1}) z^{p+1} \\
&+ \left|\int_{[0,1/\varepsilon]} (r^{n-1} - \rho^{n-1}) \overline{u}^{p+1}(r-\rho)\right|.
\end{aligned}
$$

Using a Taylor expansion for the function $r^{n-1} - \rho^{n-1}$ and the fact that $r \leq C(r_0)\rho$ (since $\rho \geq r_0/\varepsilon$), we obtain

$$
\left|\int_{[0,1/\varepsilon]} (r^{n-1} - \rho^{n-1}) \overline{u}^{p+1}(r-\rho)\right| \leq C(n, r_0)\rho^{n-2}\int_{[0,1/\varepsilon]} |r-\rho| \overline{u}^{p+1}(r-\rho) \leq C\rho^{n-2}.
$$

On the other hand, from (2) we get

$$
\rho^{n-1}\int_{r\geq 1/\varepsilon} \overline{u}^{p+1}(r-\rho) \leq C\varepsilon^{1-n}\left(e^{-(p+1)\left(\frac{1}{\varepsilon}-\rho\right)} + e^{-\frac{(p+1)r_0}{4\varepsilon}}\right)
$$

$$
\int_{[0,1/\varepsilon]} r^{n-1}(1-\phi_\varepsilon^{p+1}) \overline{u}^{p+1} \leq C\varepsilon^{1-n} e^{-\frac{(p+1)r_0}{4\varepsilon}}.
$$

Hence from the last three equations we deduce

$$
\left|\int \phi_\varepsilon^{p+1} z^{p+1} - \rho^{n-1}\int_{\mathbb{R}} \overline{u}^{p+1}\right| \leq C\varepsilon^{1-n}\left(e^{-(p+1)\left(\frac{1}{\varepsilon}-\rho\right)} + \varepsilon\right). \tag{65}
$$

The term $\int \phi_\varepsilon^{p+1}\left(|z+v|^{p+1} - z^{p+1} - (p+1)z^p v\right)$ in (64) can be estimated as follows. From the inequality

$$
\left||z+v|^{p+1} - z^{p+1} - (p+1)z^p v - p(p+1) z^{p-1} v^2\right| \leq C\max\{|v|^3, |v|^{p+1}\},
$$

one finds

$$
\int_{[0,1/\varepsilon]} \left||z+v|^{p+1} - z^{p+1} - (p+1)z^p v\right| \leq C\int_{[0,1/\varepsilon]} z^{p-1} v^2 + C\int_{[0,1/\varepsilon]} \max\{|v|^3, |v|^{p+1}\}.
$$

The first integral in the last expression can be estimated dividing the domain in the two regions $r \leq \frac{\rho+\varepsilon^{-1}}{2}$ and $r \geq \frac{\rho+\varepsilon^{-1}}{2}$, as before, while for the second it is sufficient to use the explicit expression of v. In this way we find

$$
\begin{aligned}
&\left|\int_{[0,1/\varepsilon]} \phi_\varepsilon^{p+1}\left(|z+v|^{p+1} - z^{p+1} - (p+1)z^p v\right)\right| \\
&\leq C\varepsilon^{1-n}\left(e^{-3\left(\frac{1}{\varepsilon}-\rho\right)} + e^{-\frac{(p+3)}{2}\left(\frac{1}{\varepsilon}-\rho\right)} + e^{-(3\wedge(p+1))\left(\frac{1}{\varepsilon}-\rho\right)}\right).
\end{aligned} \tag{66}
$$

The term $\int_{A_\varepsilon} \phi_\varepsilon^{p+1} z^p v$ in (64) turns out to be of order $\varepsilon^{1-n} e^{-2\left(\frac{1}{\varepsilon}-\rho\right)}$. We need to have a rather precise expansion of this term, so we treat it in some detail. There holds

$$\begin{aligned}
\int_{[0,1/\varepsilon]} \phi_\varepsilon^{p+1} z^p v &= \alpha_{1,p} \rho^{n-1} e^{-2\left(\frac{1}{\varepsilon}-\rho\right)} \int_{\mathbb{R}} \overline{u}^p e^r \\
&- \alpha_{1,p} \rho^{n-1} e^{-2\left(\frac{1}{\varepsilon}-\rho\right)} \int_{r \geq 1/\varepsilon} \overline{u}^p (r-\rho) e^{(r-\rho)} \\
&+ \int_{[0,1/\varepsilon]} (r^{n-1} - \rho^{n-1}) z^p v + \int_{[0,1/\varepsilon]} (\phi_\varepsilon^{p+1} - 1) z^p v.
\end{aligned}$$

Reasoning as above, we obtain

$$\rho^{n-1} e^{-2\left(\frac{1}{\varepsilon}-\rho\right)} \int_{r \geq 1/\varepsilon} \overline{u}^p (r-\rho) e^{(r-\rho)} \leq C \varepsilon^{1-n} e^{-(p+1)\left(\frac{1}{\varepsilon}-\rho\right)};$$

$$\left| \int_{[0,1/\varepsilon]} (r^{n-1} - \rho^{n-1}) z^p v \right| \leq C \varepsilon^{2-n} e^{-2\left(\frac{1}{\varepsilon}-\rho\right)} \qquad \int_{[0,1/\varepsilon]} (1 - \phi_\varepsilon^{p+1}) z^p v \leq C \varepsilon^{1-n} e^{-\frac{(p+1) r_0}{4\varepsilon}}.$$

Hence the last three equations and the expression of $\overline{u}$ imply

$$\begin{aligned}
\int \phi_\varepsilon^{p+1} z^p v &= \alpha_{1,p} \rho^{n-1} e^{-2\left(\frac{1}{\varepsilon}-\rho\right)} \int_{\mathbb{R}} \overline{u}^p e^r + \varepsilon^{1-n} e^{-2\left(\frac{1}{\varepsilon}-\rho\right)} O\left(\varepsilon + e^{-(p-1)\left(\frac{1}{\varepsilon}-\rho\right)}\right) \\
&= \alpha_{1,p} \varepsilon^{1-n} (\varepsilon\rho)^{n-1} e^{-2\left(\frac{1}{\varepsilon}-\rho\right)} \int_{\mathbb{R}} \overline{u}^p e^r + \varepsilon^{1-n} e^{-2\left(\frac{1}{\varepsilon}-\rho\right)} O\left(\varepsilon + e^{-(p-1)\left(\frac{1}{\varepsilon}-\rho\right)}\right), \qquad (67)
\end{aligned}$$

for ε small. The fourth term in (62) can be estimated as for (59), and gives

$$\left| \int \frac{(z^N)' z^N}{r} \right| \leq C \varepsilon^{2-n}. \qquad (68)$$

The fifth and the sixth terms in (62) can be estimated in the following way

$$\left| \int \phi_\varepsilon' z^N (z+v)' \right| \leq C \varepsilon^{2-n} e^{-\frac{r_0}{2\varepsilon}} \qquad \left| \int \phi_\varepsilon'' z^N (z+v) \right| \leq C \varepsilon^{3-n} e^{-\frac{r_0}{2\varepsilon}}. \qquad (69)$$

From (63)-(69) we deduce the following result.

Lemma 5.5 *Let z^N be defined in (55). Then one has*

$$\begin{aligned}
I_\varepsilon(z_\rho^N) &= \varepsilon^{1-n} (\varepsilon\rho)^{n-1} \left[\left(\frac{1}{2} - \frac{1}{p+1} \right) \int_{\mathbb{R}} \overline{u}^{p+1} - \frac{1}{2} \alpha_{1,p} e^{-2\left(\frac{1}{\varepsilon}-\rho\right)} \int_{\mathbb{R}} \overline{u}^p e^r \right] \\
&+ O(\varepsilon^{2-n}) + \varepsilon^{1-n} o\left(e^{-2\left(\frac{1}{\varepsilon}-\rho\right)}\right)
\end{aligned}$$

for all $\rho \in \left[\frac{3}{4\varepsilon}, \frac{1}{\varepsilon}\right]$.

5.3 Proof of Theorem 5.1

For $s \in [0,1]$, using $(E1)$ and $(E2)$ in Lemma 5.2, we have

$$\begin{aligned}\left\|I'_\varepsilon(z^N + sw^N) - I'_\varepsilon(z^N)\right\| &\leq \|I''_\varepsilon(z^N)[sw^N]\| + \left\|\int_0^1 \left(I''_\varepsilon(z^N + \zeta s w^N) - I''_\varepsilon(z^N)\right)[w]d\zeta\right\| \\ &= O(\|w^N\|) + O\left(\max\left\{\|w^N\|^2, \|w^N\|^p\right\}\right).\end{aligned}$$

Hence using property **iii)** in Proposition 5.3 and the smallness of $\|I'_\varepsilon(z^N)\|$, we deduce

$$\begin{aligned}I_\varepsilon(z^N + w(z^N)) &= I_\varepsilon(z^N) + I'_\varepsilon(z^N)[w(z^N)] + \int_0^1 \left(I'_\varepsilon(z^N + sw^N) - I'_\varepsilon(z^N)\right)[w]ds \\ &= I_\varepsilon(z^N) + O\left(\|I'_\varepsilon(z^N)\|^2\right).\end{aligned}$$

Hence from Lemmas 5.4 and 5.5 it turns out that

$$(70) \qquad I_\varepsilon(z_\rho^N + w_\rho^N) = \rho^{n-1}\left[\alpha - \beta e^{-2\left(\frac{1}{\varepsilon}-\rho\right)}\right] + O(\varepsilon^{2-n}) + \varepsilon^{1-n} o\left(e^{-2\left(\frac{1}{\varepsilon}-\rho\right)}\right)$$

where

$$(71) \qquad \alpha = \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}} \overline{u}^{p+1}; \qquad \beta = \frac{1}{2}\alpha_{1,p}\int_{\mathbb{R}} \overline{u}^p e^r.$$

We are going to show that the function $\rho \mapsto I_\varepsilon(z_\rho^N + w_\rho^N)$ possesses a critical point ρ_ε with $\left|\frac{1}{\varepsilon} - \rho_\varepsilon\right| \sim |\log \varepsilon|$. We give first an heuristic argument, which justifies the choice of the numbers $\rho_{0,\varepsilon}, \rho_{1,\varepsilon}$ and $\rho_{2,\varepsilon}$ below. The main term in (70) is $\rho^{n-1}\left[\alpha - \beta e^{-2\left(\frac{1}{\varepsilon}-\rho\right)}\right]$. Differentiating with respect to ρ we obtain

$$(n-1)\rho^{n-2}\left[\alpha - \beta e^{-2\lambda\left(\frac{1}{\varepsilon}-\rho\right)}\right] - 2\beta\rho^{n-1}e^{-2\left(\frac{1}{\varepsilon}-\rho\right)}.$$

Since $\left|\frac{1}{\varepsilon} - \rho_\varepsilon\right| \sim |\log \varepsilon|$, the term $e^{-2\left(\frac{1}{\varepsilon}-\rho\right)}$ converges to 0 as ε goes to 0, hence to get a critical point we must require, roughly

$$(n-1)\rho^{n-2} = 2\beta\rho^{n-1}e^{-2\left(\frac{1}{\varepsilon}-\rho\right)}.$$

Taking the logarithm, and using the fact that all the terms except ε and $e^{-2\left(\frac{1}{\varepsilon}-\rho\right)}$ are uniformly bounded from above and from below by positive constants, we obtain the condition

$$(72) \qquad |\log \varepsilon| \sim 2\left(\frac{1}{\varepsilon} - \rho\right) \quad \Leftrightarrow \quad \left(\frac{1}{\varepsilon} - \rho\right) \sim \frac{|\log \varepsilon|}{2}.$$

We now begin our justification of the above arguments. Given $C_0 > 0$ (to be fixed later sufficiently large), consider the three numbers

$$(73) \qquad \rho_{0,\varepsilon} = \frac{1}{\varepsilon} - \frac{1}{2}|\log \varepsilon|; \qquad \rho_{1,\varepsilon} = \frac{1}{\varepsilon} - \frac{1}{C_0}|\log \varepsilon|; \qquad \rho_{2,\varepsilon} = \frac{1}{\varepsilon} - C_0|\log \varepsilon|.$$

By condition (72) we expect $\rho_{0,\varepsilon}$ to be almost critical for the function $\rho \mapsto \Psi_\varepsilon(\rho) = I_\varepsilon(z^N_\rho + w^N_\rho)$. Using Lemma 5.5 and some elementary computations, one finds

$$\begin{aligned}\Psi_\varepsilon(\rho_{0,\varepsilon}) &= \varepsilon^{1-n}\left(1+o(\varepsilon|log\varepsilon|)\right)\left[\alpha - \beta\varepsilon^{\left(1-\frac{\varepsilon|\log\varepsilon|}{2}\right)}\right] \\ &+ O(\varepsilon^{2-n}) + \varepsilon^{1-n} o\left(\varepsilon^{\left(1-\frac{\varepsilon|\log\varepsilon|}{2}\right)}\right)\end{aligned}$$

We have $\varepsilon^{\left(1-\frac{\varepsilon|\log\varepsilon|}{2}\right)} = \varepsilon^{1+O(\varepsilon|log\varepsilon|)} = O(\varepsilon) \ll \varepsilon|\log\varepsilon|$, and hence

$$\Psi_\varepsilon(\rho_{0,\varepsilon}) = \varepsilon^{1-n}\alpha\left(1+o(\varepsilon|log\varepsilon|)\right).$$

On the other hand, there holds

$$\begin{aligned}\Psi_\varepsilon(\rho_{1,\varepsilon}) &= \varepsilon^{1-n}\left(1+o(\varepsilon|log\varepsilon|)\right)\left[\alpha - \beta\varepsilon^{2\left(1-\frac{\varepsilon|\log\varepsilon|}{C_0}\right)/C_0}\right] \\ &+ O(\varepsilon^{2-n}) + \varepsilon^{1-n} o\left(\varepsilon^{2\left(1-\frac{\varepsilon|\log\varepsilon|}{C_0}\right)/C_0}\right).\end{aligned}$$

If $C_0 > 2$, we use the estimate $\varepsilon^{2\left(1-\frac{\varepsilon|\log\varepsilon|}{C_0}\right)/C_0} = \varepsilon^{2/C_0+O(\varepsilon|log\varepsilon|)} = \varepsilon^{2/C_0}(1+o(1)) \gg \varepsilon|\log\varepsilon|$, to obtain

$$\Psi_\varepsilon(\rho_{1,\varepsilon}) = \varepsilon^{1-n}\left[\alpha - \beta\varepsilon^{\frac{2}{C_0}} + o\left(\varepsilon^{\frac{2}{C_0}}\right)\right].$$

For the third term, we can write

$$\begin{aligned}\Psi_\varepsilon(\rho_{2,\varepsilon}) &= \varepsilon^{1-n}\left(1+o(\varepsilon|log\varepsilon|)\right)\left[\alpha - \beta\varepsilon^{2C_0(1-C_0\varepsilon|\log\varepsilon|)}\right] \\ &+ O(\varepsilon^{2-n}) + \varepsilon^{1-n} o\left(\varepsilon^{2C_0(1-C_0\varepsilon|\log\varepsilon|)}\right).\end{aligned}$$

If $C_0 > \frac{1}{2}$, we obtain $\varepsilon^{2C_0(1-C_0\varepsilon|\log\varepsilon|)} = \varepsilon^{2C_0+O(\varepsilon|log\varepsilon|)} = O(\varepsilon^{2C_0+O(\varepsilon|log\varepsilon|)}) \ll \varepsilon|\log\varepsilon|$, and hence

$$\Psi_\varepsilon(\rho_{2,\varepsilon}) = \varepsilon^{1-n}\alpha\left(1+o(\varepsilon|log\varepsilon|)\right),$$

for ε sufficiently small. If C_0 is chosen sufficiently large, the last three equations imply

$$\sup_{[\rho_{2,\varepsilon},\rho_{1,\varepsilon}]} \Psi_\varepsilon > \sup_{\partial[\rho_{2,\varepsilon},\rho_{1,\varepsilon}]} \Psi_\varepsilon.$$

Hence it follows that the reduced functional Ψ_ε possesses a critical point (maximum) ρ in the interval $\left(\frac{1}{\varepsilon} - C_0|\log\varepsilon|, \frac{1}{\varepsilon} - \frac{1}{C_0}|\log\varepsilon|\right)$. By Proposition 5.3, we obtain a critical point of I_ε with the desired asymptotic profile. By construction, this solution is close in L^∞ to a positive function. Then from the maximum principle it is easy to conclude that u_ε is strictly positive. This concludes the proof of the theorem.

Considering a more general equation, with a radial potential V

$$(N) \qquad \begin{cases} -\varepsilon^2\Delta u + V(|x|)u = u^p & \text{in } \Omega; \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, & u > 0 \text{ in } \Omega, \end{cases}$$

the above result admits the following extension.

Theorem 5.6 *Let* $(V1) - (V2)$ *hold,* $p > 1$ *and let* $\Omega \subseteq \mathbb{R}^n$ *be the ball* B_1 *(resp. the annulus* A*). Suppose that the function* M *satisfies the condition*

$$M'(1) > 0 \qquad (\textit{resp. } M'(a) < 0). \tag{74}$$

Then there exists a family of radial solutions u_ε *of* (N) *concentrating on* $|x| = r_\varepsilon$*, where* r_ε *is a local maximum for* u_ε *such that* $1 - r_\varepsilon \sim \varepsilon|\log\varepsilon|$ *(resp.* $r_\varepsilon - a \sim \varepsilon|\log\varepsilon|$*).*

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