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SMR1486/11

Workshop and Conference on Recent Trends in Nonlinear Variational Problems

(22 April - 9 May 2003)

Topology and Sobolev spaces
Part II: Higher dimensions

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These are preliminary lecture notes, intended only for distribution to participants

Topology and Sobolev spaces
Part II : Higher dimensions

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Abstract. This course further continues the study of S^1-valued maps. Two questions are detailed : existence of a lifting and density of smooth maps.

I. The main problems

We consider maps from a domain$\Omega \subset \mathbb{R}^N$, $N \geq 2$, into the unit circle S^1. To start with, we consider the simplest possible domains, e.g., balls or cubes. More complicated domains will be examined later. However, Ω will always be assumed **connected**. We consider that these maps have some Sobolev regularity, i.e., that they belong to some (integer or fractional) Sobolev space $W^{s,p}(\Omega; S^1)$, $0 < s < \infty$, $1 < p < \infty$ (for the definition of these spaces when s is not an integer, see [1]). We address two questions :
(i) (**Lifting**) Given an S^1-valued map $u \in W^{s,p}(\Omega; S^1)$, can one find a **real**-valued map $f \in W^{s,p}(\Omega; \mathbb{R})$ such that $u = e^{\imath f}$? If so, is f **unique** modulo constants in $2\pi\mathbb{Z}$?
(ii) (**Density**) Given an S^1-valued map $u \in W^{s,p}(\Omega; S^1)$, can one find a sequence of smooth maps $(u_n) \subset C^\infty(\overline{\Omega}; S^1)$ such that $u_n \to u$ in $W^{s,p}$?

Comments

a) These questions have been completely settled in the papers [2] and [3]. See the references therein for previous results concerning the same questions. We will sketch below part of the proofs. Sections II-VI deal with the relatively simple cases. The delicate cases are discussed, without proofs, in Sections VII-VIII. Some details about how the proof goes in these cases will be given during the lecture.
b) Question (i) for **continuous** maps is a well known exercice. When Ω is, e.g., a ball (or, more generally, a simply connected domain), the answer is yes. However, for a general Ω, the answer may be no : consider, e.g., the case where Ω is a 2D-annulus. Thus, one may expect (and this turns out to be true), the answer to depend on the topology of Ω.
c) The key point in question (ii) is that we ask the maps u_n to be S^1-valued. Indeed, any map $u \in W^{s,p}(\Omega; S^1)$ can be approximated by smooth maps : e.g., we mollify u. However, the sequence of smooth maps converging to u obtained in this way **needs not** be S^1-valued. Actually, we will see that, in general, the answer is **no**.

II. Uniqueness

Assume we may write $u = e^{\imath f} = e^{\imath g}$, with $f, g \in W^{s,p}(\Omega; \mathbb{R})$. Thus the map $k = (f - g)/(2\pi)$ belongs to $W^{s,p}(\Omega; \mathbb{R})$ and it is $\mathbb{Z}$-valued a.e. Therefore, uniqueness is equivalent to the following

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Question. Is every map $k \in W^{s,p}(\Omega;\mathbb{Z})$ constant a.e. ?
When $sp < 1$, the answer is **no**. Indeed, take Q any cube properly contained in Ω. It is easy to see that the map $k = \chi_Q$ is $\mathbb{Z}$-valued, not constant a.e., and belongs to $W^{s,p}(\Omega;\mathbb{Z})$. However, this is the only case of nonuniqueness.

Lemma 1. Assume $sp \geq 1$ and Ω connected. Then every map $k \in W^{s,p}(\Omega;\mathbb{Z})$ constant a.e.

Proof. Start with $N = 1$. Then, by the Sobolev embeddings, $W^{s,p} \subset W^{1/p,p} \subset$VMO. Approximate k by smooth (not necesarily $\mathbb{Z}$-valued) maps, by mollifying k. The argument used in the Part I of this course for VMO$(S^1;S^1)$ maps shows that, for large n, k_n is almost $\mathbb{Z}$-valued, i.e., $\mathrm{dist}(k_n(x),\mathbb{Z}) \to 0$ uniformly in x. Take n_0 such that, for $n \geq n_0$, this distance is uniformly less than $1/3$. Thus, for $n \geq n_0$, k_n takes values into $\cup_{m\in\mathbb{Z}}(m-1/3, m+1/3)$. Since k_n is continuous, there must be some integer m_n such that k_n takes its values into $(m_n - 1/3, m_n + 1/3)$. The sequence (m_n) is bounded. Indeed, on the one hand we have $m_n - 1/3 \leq \fint k_n \leq m_n + 1/3$. On the other hand, we have $\fint k_n \to \fint k$, since convergence in $W^{s,p}$ implies convergence in L^1. Up to a subsequence, we may thus assume $m_n \equiv m$. Since $k_n \to k$ a.e., we find that $k(x) \in (m-1/3, m+1/3)$ a.e., so that $k \equiv m$ a.e.
We now consider the case $N = 2$; the case $N \geq 2$ is identical. Since Ω is connected, it suffices to prove that k is locally constant a.e. We may thus assume Ω to be a square, e.g. the unit square $(0,1)^2$. For a.e. $x, y \in (0,1)$, the maps $u(x,.)$ and $u(.,y)$ belong to $W^{s,p}$ (see [1]). Thus, for any such x or y, these maps are constant a.e., by the case $N = 1$. Now write

$$|k(a,b) - k(c,d)| \leq |k(a,b) - k(a,d)| + |k((a,d) - k(c,d)|$$

and integrate this inequality over a, b, c, d. For a.e. a, we have

$$\int\int |k(a,b) - k(a,d)|db\, dd = 0,$$

so that

$$\int\int\int\int |k(a,b) - k(a,d)|da\, db\, dc\, dd = 0.$$

Using a similar argument, we find that

$$\int\int\int\int |k(a,b) - k(c,d)|da\, db\, dc\, dd = 0,$$

so that k is constant a.e.

Final conclusion. Uniqueness holds if and only if $sp \geq 1$.

III. Case of continuous maps : $sp > N$

Recall that, when $sp > N$, then $W^{s,p} \subset C^0$, by the Sobolev embeddings. In particular, this implies the following simple

Lemma 2. Assume $sp > N$. Then smooth S^1-valued maps are dense in $W^{s,p}(\Omega; S^1)$.

Proof. Approximate u by mollifying it. The sequence (u_n) obtained in this way converges to u in $W^{s,p}$, and in particularly in C^0. Thus, in particular, $|u_n| \to |u| = 1$ uniformly. Consider the map $\Phi : \mathbb{R}^2 \setminus D_{1/2} \to \mathbb{R}^2$, $\Phi(z) = z/|z|$. Then Φ is smooth and, for large n, $\Phi(u_n)$ is well-defined. Using the following general result, due to Peetre [4],

Let $\Phi \in C^\infty$, $sp > N$. If $u \in W^{s,p}$, then $\Phi(u) \in W^{s,p}$. Moreover, if $u_n \to u$ in $W^{s,p}$, then $\Phi(u_n) \to \Phi(u)$ in $W^{s,p}$

we find that the sequence of smooth S^1-valued maps $(\Phi(u_n))$ converges to u.

Concerning the existence of a lifting, it is easy to see that in general the answer is **no**.

Example. Take $\Omega = D_1 \setminus D_{1/2} \subset \mathbb{R}^2$ and $u(z) = z/|z|$. Let any s, p be such that $sp > 2$. The there is no $f \in W^{s,p}(\Omega; \mathbb{R})$ such that $u = e^{\imath f}$.

Proof. First of all, $u \in W^{s,p}$, since u is smooth. For the nonexistence of f, argue by contradiction. Then f is continuous. Since $u = e^{\imath f}$ a.e, we actually have $u = e^{\imath f}$ everywhere. In particular u has a continuous lifting. But this is well known to be false.

However, we have

Lemma 3. Assume Ω **simply connected**. Let $sp > N$. Then every $u \in W^{s,p}(\Omega; S^1)$ has a lifting $f \in W^{s,p}(\Omega; \mathbb{R})$.

Proof. For simplicity, we prove the fact that $f \in W^{s,p}(K)$ for each K compact in Ω. By adapting the argument below, one may obtain the full Lemma. Recall that u, being continuous and S^1-valued on a simply connected domain, has a **continuous** lifting f. We claim that this f is actually in $W^{s,p}$. Indeed, pick some $x_0 \in \Omega$. Assume, e.g., $u(x_0) = 1$. There is a ball $B \in x$ such that $|u(x) - 1| \leq 1$ for $x \in B$. Thus $-\imath \log u$ is well-defined, continuous, and clearly is a lifting of u in B. Therefore, up to a multiple of 2π, we have $f = -\imath \log u$ in B. By Peetre's result, we find that $f \in W^{s,p}(B)$. Thus f is locally in $W^{s,p}$.

IV. The bad case : $1 \leq sp < 2$
This is the **no** situation : in **any** domain, there is, for a general u, no lifting, and smooth S^1-valued maps are not dense. The following example concerns a specific domain. However, it can be adapted to the general case.

Example. Let Ω be the unit disc in $\mathbb{R}^2$ and $1 \leq sp < 2$. Let $u(z) = z/|z|$. Then $u \in W^{s,p}(\Omega; S^1)$. However,
a) there is no $f \in W^{s,p}(\Omega; \mathbb{R})$ such that $u = e^{\imath f}$;
b) u cannot be approximated by smooth S^1-valued maps.

Proof. The fact that $u \in W^{s,p}(\Omega; S^1)$ can be checked directly from the definition of $W^{s,p}$ (it is a rather long computation). To prove a), argue by contradiction. Then, for a.e. $1/2 < r < 1$, we have that both $u_{|C_r}$ and $f_{|C_r}$ belong to $W^{s,p}(C_r)$ and that $u = e^{\imath f}$ a.e. on C_r. Pick any such r. Then, on C_r, the VMO map $u(z) = z/|z|$ has a VMO lifting f. Cf Part I, this contradicts the fact that, on C_r, we have deg $u = 1$.
The proof of b) follows also by contradiction. Assume that there is a sequence (u_n) of smooth S^1-valued maps approximating u. Then, up to a subsequence, we may assume that, for a.e. r, we have $u_{n|C_r} \to u_{|C_r}$ in $W^{s,p}(C_r)$. Pick any such r. Then, in particular, $u_{n|C_r} \to u_{|C_r}$ in VMO $(C_r; S^1)$, so that deg $(u_{n|C_r}) \to$ deg $(u_{|C_r})$. On the one hand, recall that deg $(u_{|C_r}) = 1$. On the other hand, we claim that deg $(u_{n|C_r}) = 0$. This will lead to a contrdiction. To justify the claim, note that Ω is simply connected, so that we may write $u_n = e^{\imath f_n}$ for some smooth f_n. Taking restrictions to C_r, we find that $u_{n|C_r}$ has a continuous lifting, so that it has degree 0.

V. At least one derivative : $s \geq 1$, $sp \geq 2$

We start with the case of a **simply connected domain** Ω. We will turn later to the general case.

Theorem 1. Assume Ω simply connected, $s \geq 1$, $sp \geq 2$. Then :
a) Every $u \in W^{s,p}(\Omega; S^1)$ can be written as $u = e^{\imath f}$ for some $f \in W^{s,p}(\Omega; \mathbb{R})$;
b) Smooth S^1-valued maps are dense in $W^{s,p}(\Omega; S^1)$.

The proof is rather technical. However, the main idea, which originated in [5], is simple. We present it when $s = 1$. The general case is more involved.

Proof of a) when $s = 1$. The idea is to assume that f is known and to derive some consequences. Writing $u = u_1 + \imath u_2$, with $u_1 = \cos f$ and $u_2 = \sin f$, we have

$$Du_1 = -(\sin f)Df = -u_2 Df$$

and

$$Du_2 = (\cos f)Df = u_1 Df.$$

Hence

$$Df = u_1 Du_2 - u_2 Du_1. \tag{1}$$

The strategy is now to find f by solving (1) with the help of a generalized form of Poincaré's lemma,

Lemma 4. Let $1 \leq p < \infty$ and let $F \in L^p(\Omega; \mathbb{R}^N)$. The following properties are equivalent:

$a)$ there is some $f \in W^{1,p}(\Omega; \mathbb{R})$ such that

$$F = Df,$$

$b)$ one has

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i} \quad \forall\, i,j, \;\; 1 \leq i,j \leq n \tag{2}$$

in the sense of distributions, i.e.,

$$\int F_i \frac{\partial \psi}{\partial x_j} = \int F_j \frac{\partial \psi}{\partial x_i} \qquad \forall\, \psi \in C_0^\infty(\Omega).$$

We emphasize that the assumption that Ω is simply connected is needed in this lemma.

Proof of Lemma 4. The implication $a) \Rightarrow b)$ is obvious. To prove the converse, let $\bar{F}$ be the extension of F by 0 outside Ω and let $\bar{F}_\varepsilon = \rho_\varepsilon \star \bar{F}$ where (ρ_ε) is a sequence of mollifiers. The $\bar{F}_\varepsilon$'s satisfy (2) on every compact subset of Ω (for ε sufficiently small). In particular, on every smooth simply connected domain $\omega \subset \Omega$ with compact closure in Ω, there is a function $\tilde{f}_\varepsilon$ such that

$$D\tilde{f}_\varepsilon = \bar{F}_\varepsilon \quad \text{in } \omega$$

(Here we have used the standard Poincaré lemma). Passing to the limit we obtain some $\tilde{f} \in W^{1,p}(\omega)$ such that $D\tilde{f} = F$ in ω. Finally, we write Ω as an increasing union of ω_n as above and obtain a corresponding sequence f_n. In the limit we find some $f \in L^1_{\text{loc}}(\Omega)$ with $Df = F$ in Ω. Using the regularity of Ω and a standard property of Sobolev spaces (see e.g. Maz'ja [6], Corollary in Section 1.1.11) we conclude that $f \in W^{1,p}(\Omega)$.

Proof of a) for $s = 1$ completed. We will first verify condition $b)$ of the lemma for

$$F = u_1 Du_2 - u_2 Du_1 \tag{3}$$

i.e.,

$$F_i = u_1 \frac{\partial u_2}{\partial x_i} - u_2 \frac{\partial u_1}{\partial x_i}.$$

Formally, property (2) is clear. Indeed, if u_1 and u_2 are smooth, then

$$\frac{\partial F_i}{\partial x_j} - \frac{\partial F_j}{\partial x_i} = 2\left(\frac{\partial u_1}{\partial x_j}\frac{\partial u_2}{\partial x_i} - \frac{\partial u_1}{\partial x_i}\frac{\partial u_2}{\partial x_j}\right).$$

On the other hand, if we differentiate the relation

$$|u|^2 = u_1^2 + u_2^2 = 1$$

we find

$$u_1\frac{\partial u_1}{\partial x_i} + u_2\frac{\partial u_2}{\partial x_i} = 0 \quad \forall\, i = 1, 2, \dots, n. \tag{4}$$

Thus, in $\mathbb{R}^2$, the vector $(\frac{\partial u_1}{\partial x_i}, \frac{\partial u_2}{\partial x_i})$ is orthogonal to (u_1, u_2). It follows that the vectors $(\frac{\partial u_1}{\partial x_i}, \frac{\partial u_2}{\partial x_i})$ and $(\frac{\partial u_1}{\partial x_j}, \frac{\partial u_2}{\partial x_j})$ are colinear and therefore

$$\det\begin{pmatrix} \frac{\partial u_1}{\partial x_i} & \frac{\partial u_2}{\partial x_i} \\ \frac{\partial u_1}{\partial x_j} & \frac{\partial u_2}{\partial x_j} \end{pmatrix} = \frac{\partial u_1}{\partial x_i}\frac{\partial u_2}{\partial x_j} - \frac{\partial u_1}{\partial x_j}\frac{\partial u_2}{\partial x_i} = 0. \tag{5}$$

Hence (2) holds. To make this argument rigorous we rely on the density of smooth functions in the Sobolev space $W^{1,p}(\Omega;\mathbb{R})$: there are sequences (u_{1n}) and (u_{2n}) in $C^\infty(\overline{\Omega};\mathbb{R})$ such that $u_{1n} \to u_1$ and $u_{2n} \to u_2$ in $W^{1,p}(\Omega;\mathbb{R})$ and $\|u_{1n}\|_{L^\infty} \le 1, \|u_{2n}\|_{L^\infty} \le 1$.

[**Warning:** We do not claim that $u_n = (u_{1n}, u_{2n})$ takes its values in S^1.]

Set

$$F_n = u_{1n}Du_{2n} - u_{2n}Du_{1n},$$

so that

$$F_n \to F \quad \text{in } L^p$$

and

$$\frac{\partial F_{in}}{\partial x_j} - \frac{\partial F_{jn}}{\partial x_i} = 2\left(\frac{\partial u_{1n}}{\partial x_j}\frac{\partial u_{2n}}{\partial x_i} - \frac{\partial u_{1n}}{\partial x_i}\frac{\partial u_{2n}}{\partial x_j}\right) \tag{6}$$

converges in $L^{p/2}$ to $2\left(\frac{\partial u_1}{\partial x_j}\frac{\partial u_2}{\partial x_i} - \frac{\partial u_1}{\partial x_i}\frac{\partial u_2}{\partial x_j}\right)$. Multiplying (6) by $\psi \in C_0^\infty(\Omega)$, integrating by parts and passing to the limit (using the fact that $p \ge 2$) we obtain

$$-\int_\Omega \left(f_i\frac{\partial \psi}{\partial x_j} - f_j\frac{\partial \psi}{\partial x_i}\right) = 2\int_\Omega \left(\frac{\partial u_1}{\partial x_j}\frac{\partial u_2}{\partial x_i} - \frac{\partial u_1}{\partial x_i}\frac{\partial u_2}{\partial x_j}\right)\psi.$$

On the other hand (4) and (5) hold a.e. (even for any $u \in W^{1,p}(\Omega; S^1)$, $1 \le p < \infty$). It follows that F satisfies $b)$ of Lemma 3, and therefore there is some $\varphi \in W^{1,p}(\Omega; \mathbb{R})$ such that

$$F = Df.$$

We will now prove that this f is essentially the one we are looking for.

Recall that if $g, h \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $1 \le p < \infty$, then $gh \in W^{1,p}$ and

$$\frac{\partial}{\partial x_i}(gh) = g\frac{\partial h}{\partial x_i} + h\frac{\partial g}{\partial x_i}.$$

Set

$$v = ue^{-\imath f},$$

so that $v \in W^{1,p}$ and

$$\begin{aligned} Dv &= e^{-\imath f}(Du - \imath Df) = ue^{-\imath f}(\bar{u}Du - \imath Df) \\ &= ue^{-\imath f}(\bar{u}Du - \imath F) = ue^{-\imath f}(u_1 Du_1 + u_2 Du_2) = 0 \quad \text{by (4).} \end{aligned}$$

We deduce that v is a constant and since $|v| = 1$ we may write $v = e^{\imath C}$ for some constant $C \in \mathbb{R}$. Hence $u = e^{i(f+C)}$ and the function $f + C$ has the desired properties.

Idea of the proof of a) for a general $s \ge 1$. The strategy is the same, i.e., we consider the same vector field F. Using the Gagliardo-Nirenberg inequalities, one may see that F verifies condition b) of Lemma 4. Moreover (this is the key and more delicate point), F belongs to $W^{s-1,p} \cap L^{sp}$. A variant of Lemma 4 implies that we may write $F = Df$ for some $f \in W^{s,p} \cap W^{1,sp}$. As above, this f is essentially the one needed. This proof yields thus the following refined version of a)

Part a) sharpened. Any u has a lifting in $W^{s,p} \cap W^{1,sp}$.

Proof of b) when $s = 1$. Let $u \in W^{1,p}(\Omega; S^1)$ and let $f \in W^{1,p}(\Omega; \mathbb{R})$ be a lifting of u. Let (f_n) be a sequence of smooth real functions such that $f_n \to f$ in $W^{1,p}$. Using the following standard simple property

Let Φ be a C^1 functions such that Φ' is bounded. If $u \in W^{1,p}$, then $\Phi(f) \in W^{1,p}$. Moreover, if $f_n \to f$ in $W^{1,p}$, then $\Phi(f_n) \to \Phi(f)$ in $W^{1,p}$

it is obvious that the sequence $(e^{\imath f_n})$ of smooth S^1-valued maps approximates u in $W^{1,p}$.

Idea of the proof of b) for a general $s \ge 1$. Big problem ! When f belongs to $W^{s,p}$, $\Phi(f)$ need not belong to $W^{s,p}$, even for very nice maps Φ. In particular, one can not use

Part a) anymore in order to prove Part b). Instead, one has to rely on the following much more delicate result ([7])

Let Φ be a C^∞ function with bounded derivatives and let $s \geq 1$. If $f \in W^{s,p} \cap W^{1,sp}$, then $\Phi(f) \in W^{s,p}$. Moreover, if $f_n \to f$ in $W^{s,p} \cap W^{1,sp}$, then $\Phi(f_n) \to \Phi(f)$ in $W^{s,p}$

Thus Part b) follows from Part a) sharpened.

General domains. In general, one can not expect existence of a lifting. Consider, e.g., the 2D-annulus $\Omega' = D_1 \setminus D_{1/2}$ and the smooth map $u(z) = z/|z|$. Assume by contradiction that $u = e^{\imath f}$ for some $f \in W^{s,p}(\Omega'; \mathbb{R})$. Then for a.e. r with $1/2 < r < 1$ we have $u_{|C_r} = e^{\imath f_{|C_r}}$ and, on C_r, u and f belong to $W^{s,p}$. For any such r, $u_{|C_r}$ has thus a continuous lifting. This contradicts the fact that $u_{|C_r}$ has degree 1. In higher dimension, a similar counterexample holds : consider, on $\Omega' \times (0,1)^{N-2}$, the map $u(z,x) = z/|z|$. Then u has no lifting in $W^{s,p}$.

This time, the existence of a lifting is related to topological properties of Ω :

Theorem 2. Assume $s \geq 1$, $1 < p < \infty$, $sp \geq 2$. Then :
a) Every map $u \in W^{s,p}(\Omega; S^1)$ has a lifting $f \in W^{s,p}(\Omega; \mathbb{R})$ if and only if every continuous map $u \in C^0(\overline{\Omega}; S^1)$ has a continuous lifting $f \in C^0(\overline{\Omega}; \mathbb{R})$;
b) Smooth S^1-valued maps are dense in $W^{s,p}(\Omega; S^1)$.

Proof of Theorem 2 when $s = 1$. The main tool is the following

Lemma 5. Let $p \geq 2$. Then every $u \in W^{1,p}(\Omega; S^1)$ can be written as $u = ve^{\imath f}$ for some $v \in C^\infty(\Omega; S^1)$ and $f \in W^{1,p}(\Omega; \mathbb{R})$.

Proof of Lemmma 5. Consider again the vector field $F \in L^p(\Omega; \mathbb{R}^N)$. Let f be the solution of

$$\Delta f = \text{ div } F \quad \text{ in } \Omega \ , \quad f = 0 \quad \text{ on } \partial\Omega.$$

Then $f \in W^{1,p}(\Omega; \mathbb{R})$ (see [8]). We claim that $v = ue^{-\imath f} \in C^\infty$. Indeed, recall that, by the proof of Theorem 1, we may write, on each ball $B \subset \Omega$, $u = e^{\imath g}$, for some $g \in W^{1,p}(B; \mathbb{R})$ such that $Dg = F$ on B. Then, in B, we have $v = e^{\imath(g-f)}$ and, clearly, $\Delta(g - f) = 0$ in B. Thus $g - f \in C^\infty$, by Weyl's Lemma. It follows that $v \in C^\infty$.

Proof of Theorem 2 when $s = 1$ **completed.** "$\Rightarrow$" Take $u \in C^0(\overline{\Omega}; S^1)$. By mollifying u, we may find some $v \in C^\infty(\overline{\Omega}; S^1)$ such that $|u\overline{v} - 1| < 1$. Thus we may write $u\overline{v} = e^{\imath k}$, wher k is the continuous map Arg $u\overline{v}$. On the other hand, $v = e^{\imath g}$ for some $g \in W^{1,p}(\Omega; \mathbb{R})$. Take B any ball in Ω. Then, on B, we may write the smooth map v as $v = e^{\imath h}$ for some smooth h. Thus, in B, the difference $g - h$ is $2\pi\mathbb{Z}$-valued and belongs to $W^{1,p}$. By Lemma

1, this difference must be constant a.e. Therefore, g is smooth. Finally, $u = e^{\imath(g+k)}$, with $g + k$ continuous.
"$\Leftarrow$" We will make use of the following intuitively clear geometric property:

If $\varepsilon > 0$ is sufficiently small, the domains $\overline{\Omega}$ and $\overline{\Omega}_\varepsilon = \{x \in \Omega; \text{dist } (x, \partial\Omega) \le \varepsilon\}$ are diffeomorphic through some smooth diffeomorphism Φ_ε. Moreover, assume, e.g. $0 \in \Omega$. Then we may construct Φ_ε such that $\Phi_\varepsilon(0) = 0$ for sufficiently small ε. Moreover, we may construct Φ_ε in order to have the additional properties $\Phi_{\varepsilon|\Omega_{2\varepsilon}} =$id and $\|D\Phi_\varepsilon - \text{id}\| \le C\varepsilon$

Let $u \in W^{1,p}(\Omega; S^1)$ and write $u = ve^{\imath f}$ as in Lemma 5. Since $v_\varepsilon = v_{|\overline{\Omega}_\varepsilon} \circ \Phi_\varepsilon$ is S^1-valued and continuous, we may write $v_\varepsilon = e^{\imath g_\varepsilon}$ for some continuous g_ε. Assume, e.g., $v(0) = 1$. Then, for small ε, $v_\varepsilon(0) = 1$ and we may assume $g_\varepsilon(0) = 0$. Let now $0 < \varepsilon < \delta$ be sufficiently small. Then clearly on the connected domain Ω_δ we have $g_\varepsilon - g_\delta \equiv$ const, and this constant must be 0, by our normalization condition $g_\varepsilon(0) = 0$. Thus the map $g(x) = g_\varepsilon(x)$ if $x \in \Omega_\varepsilon$ is well-defined and continuous, and $v = e^{\imath g}$. Actually, we even have $g \in C^\infty$, by an argument already used above. In particular, $|Dg| = |Dv|$. On the other hand, recall that $v = ue^{-\imath f}$, so that $|Dv| \le |Du| + |D(e^{-\imath f})| = |Du| + |Df| \in L^p$. Therefore, $g \in W^{1,p}$. Finally, $u = e^{\imath(f+g)}$, with $f + g \in W^{1,p}$.
Proof of b) Recall that we already proved that $e^{\imath f}$ can be approximated by smooth S^1-valued maps. The idea is to make use of the following property of $W^{1,p}$

If $f_n \to f$, $g_n \to g$ in $W^{1,p}$ and $\|f_n\|_{L^\infty} \le C$, $\|g_n\|_{L^\infty} \le C$, then $f_n g_n \to fg$ in $W^{1,p}$

In view of this property, it suffices to write $u = ve^{\imath f}$ as in Lemma 5 and approximate v with smooth S^1-valued maps. [**Warning :** v need not be smooth up to the boundary.] By the above arguments, we have $v \in C^\infty(\Omega; S^1) \cap W^{1,p}(\Omega; S^1)$. Let v_ε be as above, so that clearly v_ε is S^1-valued and smooth up to the boundary. We claim that $v_\varepsilon \to v$ in $W^{1,p}$. Clearly, $v_\varepsilon \to$ uniformly on compacts and thus in L^1_{loc} (actually, convergence holds also in L^1, since the maps are uniformly bounded). Therefore, it suffices to prove that $|Dv_\varepsilon - Dv| \to 0$ in L^p. Now clearly

$$\int_\Omega |Dv_\varepsilon - Dv|^p dx = \int_{\Omega\setminus\Omega_{2\varepsilon}} |Dv_\varepsilon - Dv|^p dx \le C \int_{\Omega\setminus\Omega_{2\varepsilon}} |Dv|^p dx \to 0 \quad \text{as } \varepsilon \to 0.$$

Idea of the proof for a general $s \ge 1$**.** The proof goes along the same lines. One has to use instead of Lemma 5 its following straightforward variant

Lemma 5'. Let $s \ge 1$, $sp \ge 2$. Then every $u \in W^{s,p}(\Omega; S^1)$ can be written as $u = ve^{\imath f}$ for some $v \in C^\infty(\Omega; S^1)$ and $f \in W^{s,p}(\Omega; \mathbb{R}) \cap W^{1,sp}(\Omega; \mathbb{R})$.

As for the property of products, one has to rely instead on the following variant, usually named "$W^{s,p} \cap L^\infty$ is an algebra" :

If $f, g \in W^{s,p} \cap L^\infty$, then $fg \in W^{s,p}$. Moreover, if $f_n \to f$, $g_n \to g$ in $W^{s,p}$ and $\|f_n\|_{L^\infty} \leq C$, $\|g_n\|_{L^\infty} \leq C$, then $f_n g_n \to fg$ in $W^{s,p}$

VI. Lifting when we have less than one derivative : $0 < s < 1$, $2 \leq sp < N$

Lemma 6. Assume $0 < s < 1$, $2 \leq sp < N$. Then there is somme $u \in W^{s,p}(\Omega; S^1)$ which can not be lifted, i.e., such that there is no $f \in W^{s,p}(\Omega; \mathbb{R})$ with $u = e^{\imath f}$ a.e.

Proof of Lemma 6. Assume, e.g., that the unit ball B is contained in Ω. Let $u(x) = e^{2\imath\pi/|x|^a}$ in B, extended with the value 1 outside B. Here, $a > 0$ is to be determined later. It is easy to see that $u \in W^{1,q}$ provided $(a+1)q < N$. Using the following

Gagliardo-Nirenberg type inequality. If $u \in W^{r,q} \cap L^\infty$ and $0 < t < 1$, then $u \in W^{tr,q/t}$

(a proof of the full scale of the Gagliardo-Nirenberg inequalities may be foung, e.g., in [7]), we find that $u \in W^{s,q/s}$. Thus $u \in W^{s,p}$ as soon as $(a+1)sp < N$. On the other hand, a straightforward but long computation shows that the map $g(x) = 2\pi/|x|^a$ in B, extended with the value 2π outside B, belongs to $W^{s,p}$ if and only of $(a+s)p < N$. On the other hand, we have $g \in W^{s,p}_{\text{loc}}(\Omega \setminus \{0\})$. Pick now some a such that $(a+1)sp < N$, but $(a+s)p \geq N$ (there is enough room !) and consider the corresponding u. We claim that this u can not be lifted. Argue by contradiction, i.e, assume that $u = e^{\imath f}$ for some $f \in W^{s,p}(\Omega; \mathbb{R})$. Take Q be any cube such that $\overline{Q} \subset \Omega \setminus \{0\})$. Then, on Q, we have $f - g \in W^{s,p}$ is a $2\pi\mathbb{Z}$-valued map. By Lemma 1, this function must be constant a.e. Since $\Omega \setminus \{0\}$ is connected, we find that $f \equiv g+$ const a.e. However, we have $f \in W^{s,p}$ and $g \notin W^{s,p}$. Contradiction !

VI. Lifting when we have little regularity : $sp < 1$
This is the really difficult case.

Theorem 3. Assume $sp < 1$. Then :
a) Every $u \in W^{s,p}(\Omega; S^1)$ can be written as $u = e^{\imath f}$ for some $f \in W^{s,p}(\Omega; \mathbb{R})$;
b) Smooth S^1-valued maps are dense in $W^{s,p}(\Omega; S^1)$.

The delicate part is a). We refer to [2] for details. Part b) is a trivial consequence of a) and of the following elementary property

Let Φ be a Lipschitz map and $0 < s < 1$. If $f \in W^{s,p}$, then $\Phi(f) \in W^{s,p}$. Moreover, if $f_n \to f$ in $W^{s,p}$, then $\Phi(f_n) \to \Phi(f)$ in $W^{s,p}$

VII. Density in the remaining case : $0 < s < 1$, $2 \leq sp < N$

Recall that in this case there is no lfting, even in simply connected domains. Thus we may not use approximation of the phase f by smooth functions and some composition property in order to obtain density. However, we have the following

Theorem 4. Assume $0 < s < 1$, $sp \geq 2$. Then smooth S^1-valued maps are dense in $W^{s,p}(\Omega; S^1)$.

The proof is delicate ; see [3].

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