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SPRING SCHOOL ON SUPERSTRING THEORY AND RELATED TOPICS

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TOPOLOGICAL STRING AND ITS APPLICATION

Lectures 1 and 2

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Please note: These are preliminary notes intended for internal distribution only.

Topological string and its application

- Definition

Start with $N=2$ SCFT in 2d worldsheet Σ.

(e.g. sigma-model $\Sigma \to M$: M : Calabi-Yau n-fold)

$N=2$ SCA $\quad T,\ G^+,\ G^-,\ J$: left mover

$\bar{T},\ \bar{G}^+,\ \bar{G}^-,\ \bar{J}$: right mover

$$J(z)\, J(w) \sim \frac{\hat{c}}{(z-w)^2}$$

$$c = 3\hat{c}$$

$\hat{c}$ is related to the Virasoro central charge $\cancel{c = \frac{3}{2}\hat{c}}$.

For the sigma-model on CY_n, $\hat{c} = n$.

Topological twisting

$$T \to \tilde{T} = T + \frac{1}{2}\partial J$$

(we can also choose "−")

This changes conformal dimensions

G^+ : $(\frac{3}{2}, 0) \to (1, 0)$

G^- : $(\frac{3}{2}, 0) \to (2, 0)$

$$Q_{BRST} = \oint G^+ \quad : \text{ topological BRST operator}$$

$$(Q_{BRST})^2 = 0$$

$$\tilde{T} = \{ Q_{BRST}, G^- \}$$

geometrically, the twisting is generated by coupling J to an $U(1)$ gauge field $\bar{A}$ as $\int_\Sigma \bar{A} J$,

and by setting $\bar{A} = \frac{-i}{2} \bar{\omega}$. $\bar{\omega}$: spin connection.
$(A = \frac{i}{2} \omega)$

If the energy-momentum tensor for $\bar{A} = 0$ is T, the one for $\bar{A} = \frac{-i}{2} \bar{\omega}$ is given by $\tilde{T} = T + \frac{1}{2} \partial J$

since $\delta \bar{\omega} \sim \partial \delta g$. $\bar{\omega} = \partial_{\bar{z}} \log \det g$

———(o)———

An important example of $N=2$ SCFT : sigma-model on CY_n.

CY_n is Kähler $\Leftrightarrow g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$

Ricci-flat $\Leftrightarrow R_{i\bar{j}} = \partial_i \partial_{\bar{j}} \log \det g = 0$

$\Leftrightarrow \exists$ holomorphic n-form $\Omega_{i_1 \cdots i_n}$
$\det g = \Omega \bar{\Omega}$.

sigma-model on M, M : CY_n

$X^i : \Sigma \to M$, ψ^i : fermions.

$$J = g_{i\bar{j}}\, \psi^i \psi^{\bar{j}}$$

$$G^+ = g_{i\bar{j}}\, \psi^i \partial X^{\bar{j}}, \quad G^- = g_{i\bar{j}}\, \psi^{\bar{j}} \partial X^i$$

$$\bar{J} = g_{i\bar{j}}\, \bar{\psi}^{\bar{j}} \psi^i$$

$$G^+ = g_{i\bar{j}}\, \bar{\psi}^{\bar{j}} \bar{\partial} X^i, \quad \bar{G}^- = g_{i\bar{j}}\, \bar{\psi}^i \bar{\partial} X^{\bar{j}}$$

If we twist $\tilde{T} = T + \frac{1}{2} J$, $\tilde{\bar{T}} = \bar{T} + \frac{1}{2} \bar{\partial} \bar{J}$

$$G^+ : (1,0), \quad \bar{G}^+ : (0,1)$$

$$Q_{BRST} = \oint G^+ + \oint \bar{G}^+$$

$$\psi^i : (\tfrac{1}{2}, 0) \to (0,0), \quad \bar{\psi}^{\bar{i}} : (0, \tfrac{1}{2}) \to (0,0)$$

$$\delta X^i = [\varepsilon Q_{BRST}, X^i] = \varepsilon \psi^i$$

$$\delta X^{\bar{i}} = \varepsilon \bar{\psi}^{\bar{i}}$$

$$\delta \psi^{\bar{i}} = \varepsilon \partial X^{\bar{i}}, \quad \delta \bar{\psi}^i = \varepsilon \bar{\partial} X^i$$

BRST invariant configuration of $X : \Sigma \to M$
= holomorphic map.

This is called <u>A-model</u>.

* There are two more, but these are just complex conjugations.

It turns out that there is another way to twist the theory.*

$$\tilde{T} = T - \frac{1}{2}\partial J \ , \quad \tilde{\bar{T}} = \bar{T} + \frac{1}{2}\bar{\partial}\bar{J}$$

$$G^- : (1,0) \ , \quad \bar{G}^+ : (0,1)$$

$$Q_{BRST} = \oint G^- + \oint \bar{G}^+ .$$

$$\psi^{\bar{i}} : (0,0) \ , \quad \bar{\psi}^{\bar{i}} : (0,0)$$

$$\delta X^i = 0 \ , \quad \delta X^{\bar{i}} = \varepsilon(\psi^{\bar{i}} + \bar{\psi}^{\bar{i}})$$

$$\delta \psi^i = \varepsilon\, \partial X^i \ , \quad \delta \bar{\psi}^i = \varepsilon\, \bar{\partial} X^i$$

BRST invariant configuration of $X : \Sigma \to M$

= constant map.

This is called <u>B-model</u>.

(M, W) : mirror pair of CY_m

$\Leftrightarrow$ A-model on M = B-model on W.

BRST cohomology

Since $\widetilde{T} = \{ Q_{BRST}, * \}$,

we must look for operators of dimensions $(0,0)$.

A-model

$$\omega_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q}(X)\, \psi^{i_1} \cdots \psi^{i_p}\, \bar{\psi}^{\bar{j}_1} \cdots \bar{\psi}^{\bar{j}_q}$$

$Q_{BRST} = \oint G^+ + \bar{G}^+$ acts as $\partial + \bar{\partial} = d$.

$$\Rightarrow \bigoplus_{p,q=0}^{n} H^{(p,q)}(M). \qquad (p,q) = J, \bar{J} \text{ charges.}$$

"chiral primaries"

B-model

Since $\delta X^{\bar{i}} = \varepsilon (\psi^{\bar{i}} + \bar{\psi}^{\bar{i}})$

it is convenient to write

$$\eta^{\bar{i}} = \psi^{\bar{i}} + \bar{\psi}^{\bar{i}}$$
$$\theta_i = g_{i\bar{j}} (\psi^{\bar{j}} - \bar{\psi}^{\bar{j}})$$

$$V_{\bar{i}_1 \cdots \bar{i}_p}{}^{j_1 \cdots j_q}\, \eta^{\bar{i}_1} \cdots \eta^{\bar{i}_p}\, \theta_{j_1} \cdots \theta_{j_q}$$

$Q_{BRST} = \oint G^+ + \bar{G}^+$ act as $\bar{\partial}$ w/ coefficient θ's.

$$\Rightarrow \bigoplus_{p,q} H_{\bar{\partial}}^{(0,p)}(\Lambda^q T^{(1,0)} M)$$

In particular

A-model

elements of $\underset{\Downarrow}{H^{(1,1)}}(M)$ generate marginal deformations

$$k_a \qquad a = 1, \cdots, h_{1,1} = \dim H^{(1,1)}$$

$$\phi_a = k_{a\, i\bar{j}}\, \psi^i \bar{\psi}^{\bar{j}}$$

$$G_{-1}^{-}\bar{G}_{-1}^{-}\phi_a = k_{a\, i\bar{j}}\, \partial X^i \bar{\partial} X^{\bar{j}} + \cdots$$

$\left\{ \begin{array}{l} \text{This is } (1,1) \text{ form on } \Sigma \\ [Q_{BRST}, \ ] \text{ of this is total derivative on } \Sigma \end{array} \right.$

$$\sum_a t^a \int G_{-1}^{-}\bar{G}_{-1}^{-}\phi_a$$

t^a : (complexified) Kähler moduli.

To preserve the reality of the actions

add $\displaystyle \sum_a \bar{t}^a \int \underbrace{G_0^+ \bar{G}_0^+ \bar{\Phi}_a}_{= \{Q_{BRST}, [\bar{Q}_{BRST}, \bar{\Phi}_a]\}}$: BRST trivial

$$\Downarrow$$

$$(t^a + \bar{t}^a)\, k_{a\, i\bar{j}}\, (\partial X^i \bar{\partial} X^{\bar{j}} + \bar{\partial} X^i \partial X^{\bar{j}}) \leftarrow \text{"real" Kähler moduli}$$

$$+ (t^a - \bar{t}^a)\, k_{a\, i\bar{j}}\, (\partial X^i \bar{\partial} X^{\bar{j}} - \bar{\partial} X^i \partial X^{\bar{j}}) \leftarrow \text{instanton action.}$$

B-model

$H_{\bar{\partial}}^{(0,1)}(T^{1,0}M) \longrightarrow$ complex moduli

(analogue of the Beltrami differential on Σ).

A-model amplitudes depend on t

but not on $\bar{t}$ or the complex moduli

i.e. they are holomorphic sections

of a bundle over the (complexified)

Kähler moduli space.

This sometime is not quite true

--- holomorphic anomaly.

B-model amplitudes depend holomorphically on

complex moduli, but not on $(t, \bar{t})$.

A-model : $\bar{t} \to \infty \Rightarrow$ instanton approx exact.

$$\bar{t}^a k_{a i \bar{j}}\, \bar{\partial} X^i\, \partial X^{\bar{j}}$$

B-model : $t, \bar{t} \to \infty \Rightarrow$ constant map.

(again, almost true ---)

What we have seen so far apply to any value of $\hat{c}$.

Something special happens at $\hat{c}=3$.

e.g. sigma-model on CY_3.

Coupling $\int \bar{A} J + A \bar{J}$ terms on the chiral anomaly.

$$\bar{\partial} J \sim \hat{c}\, F_{z\bar{z}}$$

If we set $A = \frac{1}{2}\omega$, we have $\bar{\partial} J \sim \hat{c} R \sqrt{g}$.

We can also see this by writing $J = i\sqrt{\hat{c}}\, \partial\phi$.

$$J(z)\, J(w) \sim \frac{\hat{c}}{(z-w)^2} \quad \Rightarrow \quad \phi(z)\,\phi(w) \sim \log \frac{1}{z-w}$$

Kinetic term $\int \frac{1}{2} \partial\phi \bar{\partial}\phi$.

add $\sqrt{\hat{c}} \int \frac{1}{2}\bar{\omega}\, \partial\phi + \frac{1}{2}\omega\, \bar{\partial}\phi \;\rightarrow\; \frac{\sqrt{\hat{c}}}{2} \int R\, \phi \sqrt{g}$

The e.o.m. for ϕ $\Rightarrow$ $\bar{\partial} J \sim \hat{c} R \sqrt{g}$.

Since $\frac{1}{2}\int R\sqrt{g} = 1-g$. g : genus of Σ,

the total amount of J charge violation is $\hat{c}(1-g)$.

(In particular, for sigma-model on CY_n, $n=\hat{c}$

ψ^i : $(0,0)$-form, $\psi^{\bar{i}}$: $(1,0)$-form

In the weak coupling ; # ψ^i zeromodes : n

$\psi^{\bar{i}}$ 〃 〃 : ng.

$\Rightarrow n(1-g)$.

The index does not change by the perturbation.)

In the following, we use the A-model notation,

So $\quad Q_{BRST} = \oint G^+ + \oint \bar{G}^+$

$$\tilde{T} = T + \frac{1}{2}\partial J \quad , \quad \tilde{\bar{T}} = \bar{T} + \frac{1}{2}\bar{\partial}\bar{J} .$$

If $\hat{c} = 3$, we can define

$$\langle G^-(z_1) \cdots G^-(z_{3g-3}) \bar{G}^-(\bar{w}_1) \cdots \bar{G}^-(\bar{w}_{3g-3}) \rangle$$

each G^- can be folded with a Beltrami diff.

$$\int G^-_{zz}\, \eta^z{}_{\bar{z}}\, d^2z$$

$$\int \bar{G}_{\bar{z}\bar{z}}\, \bar{\eta}^{\bar{z}}{}_z\, d^2z$$

η can be regarded as an element of $T\mathcal{M}_g$.

So $\langle (G^-)^{3g-3} (\bar{G}^-)^{3g-3} \rangle$

is $(3g-3, 3g-3)$ form on $\mathcal{M}_g$.

So we can integrate it over $\mathcal{M}_g$.

This defines

$$F_g = \int_{\mathcal{M}_g} \langle (G^-)^{3g-3} (\bar{G}^-)^{3g-3} \rangle$$

For $g=1$, we need to fix conformal invariance.

Define $$\int_{\mathcal{M}_{g=1}} \langle \phi_a(z)\, G^- \bar{G}^- \rangle = \frac{\partial F_{g=1}}{\partial t^a}$$

This defines F_1 up to an additional const.

It turns out that we can define

$$F_{g=1} = \int_{\mathcal{M}_{g=1}} \underbrace{\langle J, \bar{J} \rangle}_{\mathrm{tr}\left[(-1)^{J_0+\bar{J}_0} J_0 \bar{J}_0\, q^{L_0-\frac{c}{24}} \bar{q}^{\bar{L}_0-\frac{c}{24}}\right]}.$$

For $g=0$, we need to fix 3-points.

$$C_{abc} = \langle \phi_a(z_1)\, \phi_b(z_2)\, \phi_c(z_3) \rangle$$

Prepotential F_0 : $\quad C_{abc} = \dfrac{\partial^3 F_0}{\partial t^a \partial t^b \partial t^c}$

Special geometry of the moduli space of CY_3.
$\mathcal{M}$.

$\mathcal{M}$ is locally a product of $\mathcal{M}_K \times \mathcal{M}_C$
Kähler complex.

We will discuss $\mathcal{M}_K$, but $\mathcal{M}_C$ is the same.

- $\mathcal{M}_K$ is a Kähler manifold

$$G_{a\bar{b}} = \partial_a \partial_{\bar{b}} K(t, \bar{t}).$$

- $\exists$ line bundle $\mathcal{L}$ s.t. $G_{a\bar{b}}\, dt^a \wedge dt^{\bar{b}}$ gives 2π times its Chern class

<u>or</u> e^{-K} gives the metric on $\mathcal{L}$.

- C_{abc} is a holomorphic section of $\mathrm{Sym}^3 T^{*(1,0)}\mathcal{M} \otimes \mathcal{L}^2$.

<u>and</u>

"Flatness of moduli space"
$[\nabla - c, \nabla - c] = 0$

$$R_{a\bar{b}c\bar{d}} = G_{a\bar{d}} G_{\bar{b}c} + G_{a\bar{b}} G_{c\bar{d}} - e^{2K} C_{ace} \bar{C}_{\bar{b}\bar{d}\bar{f}} G^{e\bar{f}}.$$

These can be derived by either of.

- geometry
- Topological field theory computation
- Spacetime supersymmetry on $CY_3 \times \mathbb{R}^4$

In general F_g is a section of $\mathcal{L}^{2-2g}$.

Is it a holomorphic section?

Not quite

$$F_g = \int_{\mathcal{M}_g} \langle (G^-)^{3g-3} (\bar{G}^-)^{3g-3} \rangle$$

$$\frac{\partial}{\partial \bar{t}^a} F_g = \int_{\mathcal{M}_g} \langle (G^-)^{3g-3} (\bar{G}^-)^{3g-3} \int G^+ \bar{G}^+ \bar{\phi}_a \rangle$$

$$\{ \oint G^+ , G^- \} = \tilde{T}$$

⇒ total derivative on $\mathcal{M}_g$

⇒ boundary contribution.

$$\frac{\partial}{\partial \bar{t}^a} F_g = \frac{1}{2} \bar{C}_{\bar{a}\bar{b}\bar{c}} \, e^{2K} G^{\bar{b}b} G^{\bar{c}c} \times \left\{ D_b D_c F_{g-1} + \sum_{r=1}^{g-1} D_b F_r \, D_c F_{g-r} \right\}$$

This, together with appropriate boundary conditions, can be used to compute F_g exactly.

$$\partial_a \bar{\partial}_{\bar{b}} F_1 = \frac{1}{2} C_{aef} \bar{C}_{\bar{b}\bar{e}\bar{f}} \, e^{2K} G^{e\bar{e}} G^{f\bar{f}} - \left(\frac{\chi}{24} - 1 \right) G_{a\bar{b}}$$

Geometric meaning of topological string amplitudes.

B-model

On CY_3, there is a unique non-zero hol 3-fm

$$\Omega_{ijk}.$$

(In CFT, $\Omega_{ijk}\psi^i\psi^j\psi^k = e^{i\sqrt{\hat{c}}\,\phi}$ where $J = i\sqrt{\hat{c}}\,\partial\phi$)

The quantities that define the special geometry

are expressed as

$$e^{-K} = \int_M \Omega \wedge \bar{\Omega}$$

$$C_{abc} = -\int_M \Omega \wedge \partial_a \partial_b \partial_c \Omega$$

These are classical geometric quantities — not stringy

This is because B-model amplitudes

are computed by (almost) constant map $\Sigma \to M$.

$F_{g=1}$ for B-model is given by determinants of Laplacians

$$F_{g=1} = \frac{1}{2} \sum_{p,q} (-1)^{p+q}\, p \cdot q \,\log \det \Delta^{(p,q)}$$

where $\Delta^{(p,q)}$ is associated to $\bar\partial : \Lambda^p T^{*(0,1)} M \otimes \Lambda^q T^{(1,0)} M \to \Lambda^{p+1} T^{*(0,1)} M \otimes \Lambda^q T^{(1,0)} M$.

$F_{g=1}$ is a linear combination of log (generalized Ray-Singer torsion).

In this case, Quillen's anomaly formula reproduces the holomorphic anomaly of $F_{g=1}$.

For $F_{g>1}$, there are Feynman diagram computations $\Leftarrow$ Kodaira-Spencer theory of gravity.

A-model

$\bar{t} \to \infty$: a sum over worldsheet instantons.

$$\sum_d (\text{top invariant}) \times e^{-dt}$$

more explicitly, if we normalize k_a as

$$\int_{\gamma_a} k_b = \delta_a{}^b \ , \quad \gamma_a \in H_2(M)$$

- $$C_{abc} = \int_M k_a \wedge k_b \wedge k_c + \sum_{d=0}^{\infty} N_d^{(0)} d_a d_b d_c \frac{\exp(-\sum_a t^a d_a)}{1 - \exp(-\sum t^a d_a)}$$

$N_d^{(0)} =$ "#" of $\mathbb{P}^1 \underset{hol}{\subset} M$.

with degrees $d = (d_1, \cdots, d_{h^{1,1}})$

- $$\frac{\partial}{\partial t^a} F_1 \underset{\bar{t}\to\infty}{=} \frac{-1}{24} \int_M k_a \wedge C_2 + \sum_d d_a N_d^{(1)} \sum_{n=1}^{\infty} \frac{e^{-ndt}}{1 - e^{-nd\cdot t}}$$
$$+ \frac{1}{12} \sum_d d_a N_d^{(0)} \frac{e^{-dt}}{1 - e^{-dt}} .$$

etc.

As $t \to 0$, the instanton expansion starts diverging

$\Rightarrow$ singularity.

e.g. Conifold $\quad x^2 + y^2 + z^2 + w^2 = \mu \quad \mu \to 0$

blow up $\mathbb{P}^1 \leftarrow$ Kähler moduli t.

As $t \to 0$, $\quad F_g \sim \dfrac{B_{2g}}{2g(2g-2)} \dfrac{1}{t^{2g-2}}$

B_{2g} : Bernoulli number.

Euler characteristic
of moduli space of Σ_g
(up to $(-1)^{g-1}$)

Relation to physical string

Consider Type IIA / IIB string on $CY_3 \times \mathbb{R}^4$

$\Rightarrow$ $\mathcal{N}=2$ SUSY on $\mathbb{R}^4$

There are low energy effective theory terms that can be computed using the topological A/B model.

moduli of CY_3 (Kähler for IIA / Complex for IIB)

$\hookrightarrow$ vector multiplet $T^a(\theta^1, \theta^2) = t^a + \cdots$

$\mathcal{N}=2$ supergravity

$\hookrightarrow$ chiral superfield $W_{\alpha\beta} = F_{\alpha\beta} + R_{\alpha\beta\gamma\delta}\,\theta_1^\gamma \theta_2^\delta + \cdots$

$F_{\alpha\beta}$: graviphoton

Then $\int d^2\theta_1 d^2\theta_2 \, (W_{\alpha\beta} W^{\alpha\beta})^{2g} \, F_g(T)$

is in the low energy effective action,

where F_g is the g-loop top string amplitude

This contains, among other terms,

$R^2 (F^2)^{2g-2} F_g(t)$. for $g \geq 1$.

why?

As we mentioned, if we write $J = i\sqrt{\hat{c}}\,\partial\phi$,

we have

$$\frac{i}{2}\sqrt{\hat{c}} \int R\sqrt{g}\,\phi\, d^2z \qquad \text{coupling}$$

Since $\int R\sqrt{g} = 2-2g$, we can choose

the WS metric so that $R\sqrt{g} = \sum_{i=1}^{2g-2} -\delta(z-z_i)$

So, effectively the topological twist inserts

$$\prod_{i=1}^{2g-2} \underbrace{e^{-\frac{i}{2}\sqrt{\hat{c}}\,\phi(z_i)}}_{\text{R vertex}}$$

$\because$) $J = i\sqrt{\hat{c}}\,\partial\phi$

$= g_{i\bar{j}}\,\psi^i\psi^{\bar{j}}$

So, if we write

$\psi^i = e^{i\varphi_i}$

$e^{-\frac{i}{2}\sqrt{\hat{c}}\,\phi} = e^{-\frac{i}{2}\sum_i \varphi_i}$

BCOV
Antoniadis, Gava, Narain + Taylor / 9307158
Berkovits + Vafa

D branes

Look for boundary conditions that preserve $\frac{1}{2}$ of the topological BRST sym.

A-model

$$J = \bar{J} \quad , \quad G^+ = \pm \bar{G}^+ \qquad \Rightarrow \qquad \text{A branes}$$

B-model

$$J = -\bar{J} \quad , \quad G^+ = \pm \bar{G}^- \qquad \Rightarrow \qquad \text{B branes}$$

A $\Rightarrow$ D wraps on a Lagrangian submfd $\gamma \subset M$

$$g_{i\bar{j}} \, dz^i \wedge dz^{\bar{j}} \,\big|_\gamma = 0$$

B $\Rightarrow$ γ : holomorphic $\subset M$

If we require the space time supersymmetry,

we need $\Omega = e^{i\sqrt{c}\,\phi}$ to be glued together nicely.

$\left(\because \text{ SUSY generator} \sim e^{i\frac{\sqrt{c}}{2}\phi} \right)$

A + SUSY $\Rightarrow$ Special Lagrangian.

$$\Omega \,\big|_\gamma = \text{volume form} \,\big|_\gamma$$

generalization, precise definition.

- gauge field on γ
- K theory
- Kapustin + Orlov.
- Kontsevich, Fukaya, Douglas.
- Calibrated submanifold.

---(o)---

Topological open string, when $\hat{c} = 3$.

$$F_{g,n} = \int_{\mathcal{M}_{g,n}} \langle (G^-)^{6g-6+3n} \rangle$$

n = # boundaries

g = genus.

String F T

$$S = \frac{1}{2} \langle \Phi \,|\, Q_{BRST} \,|\, \Phi \rangle + \frac{1}{3} g^{\lambda} \langle \Phi \,|\, \Phi * \Phi \rangle$$

In the B-model, this reduces to a field theory

since Vol $\to \infty$.

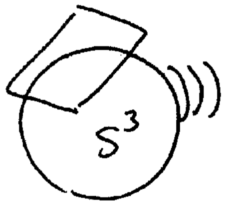
A model amplitudes may receive instanton corrections.

But it can also reduce to a field theory if there is no compact 2 cycle.

e.g. $M = T^* S^3$ No compact Kähler moduli

Wrap N D branes on S^3.

$\Rightarrow$ Chern-Simons gauge theory w/ $U(N)$ gauge group.



$$\log Z = \sum_{g,n} F_{g,n} \cdot (\lambda^2)^{2g-2} \cdot (\lambda^2 N)^n.$$

$$\lambda^2 = \frac{1}{k+N}$$

This actually is not quite the partition function on S^3.

Missing terms

$$\sum_{g=2}^{\infty} \frac{B_{2g}}{2g(2g-2)N^{2g-2}} - \frac{1}{12}\log N + \frac{1}{2}N^2 \log N$$

$$= -\log \mathrm{vol}\, U(N).$$

$\because$) $$\mathrm{vol}\, U(N) = \frac{(2\pi)^{\frac{1}{2}N(N+1)}}{(N-1)!(N-2)!\cdots 3!2!1!}$$

These terms do not correspond to Feynman diagrams.
They come from the measure.

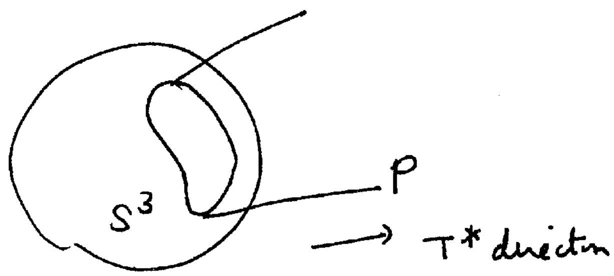
How about knot invariants?

Suppose $\gamma = \{ \xi(s) \mid \xi(s) \in S^3 ,\ 0 \leq s \leq 2\pi \}$

$\xi(0) = \xi(2\pi)$

$\Downarrow$ loop on S^3

$$C_\gamma = \{ (\xi(s), p) \in T^* S^3 \mid p_i \frac{d\xi^i}{ds} = 0 \}$$



C_γ has the topology of $\mathbb{R}^2 \times S^1$.

Turn on a holonomy U on S^1 in C_γ

$$\Rightarrow \quad Z_\gamma(U) \equiv \langle \exp \sum_{n=1}^{\infty} \frac{1}{n} \mathrm{tr}\, U^{-n} \, \mathrm{tr} (P e^{\oint_\gamma A})^n \rangle$$

generating fun of

$\langle \mathrm{tr}_R \, e^{\oint_\gamma A} \rangle$ for any rep R.

large N duality

$$Z(S^3) = \frac{e^{i\frac{\pi}{8}(N-1)N}}{(k+N)^{N/2}} \sqrt{\frac{k+N}{N}} \prod_{s=1}^{N-1} \left[2\sin\left(\frac{s\pi}{k+N}\right)\right]^{N-s}$$

$$= \exp\left[- \sum_{g=0}^{\infty} \lambda^{2g-2} F_g(t) \right]$$

$$t = \frac{iN}{k+N} \qquad \lambda^2 = \frac{1}{k+N}$$

$$F_0 = -\zeta(3) + \frac{i\pi^2}{6} t - i\left(m+\frac{1}{4}\right)\pi t^2 + \frac{i}{12} t^3 + \sum_{n=1}^{\infty} n^{-3} e^{-nt}$$

m some $\mathbb{Z}$

$$F_1 = \frac{1}{24} t + \frac{1}{12} \log\left(1 - e^{-t}\right)$$

$$F_g = \frac{(-1)^{g-1}}{2g(2g-2)} B_{2g} \left\{ \frac{(-1)^{g-1}}{(2\pi)^{2g-2}} 2\zeta(2g-2) - \frac{1}{(2g-3)!} \sum_{n=1}^{\infty} n^{2g-3} e^{-nt} \right\}$$

Note:
$$\chi_g = \frac{(-1)^{g-1}}{2g(2g-2)} B_{2g}$$

$$\int_{M_g} c_{g-1}^3 = \frac{(-1)^{g-1}}{(2\pi)^{2g-2}} 2\zeta(2g-2) \times \chi_g$$

Thus we can write

$$F_g = \int_{\mathcal{M}_g} c_{g-1}^3 - \frac{\chi_g}{(2g-3)!} \sum_{n=1}^{\infty} n^{2g-3} e^{-nt}$$

This is a g-loop closed top string amp

on resolved conifold.