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SPRING SCHOOL ON SUPERSTRING THEORY AND RELATED TOPICS

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STRINGY THERMODYNAMICS IN LARGE N GAUGE THEORIES

Lectures 1 and 2

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Please note: These are preliminary notes intended for internal distribution only.

INTRODUCTION

In these lectures I will describe the thermodynamics of a very simple system: weakly coupled $SU(N)$ Yang Mills on a sphere. Quite surprisingly, we will see that this system has very rich behaviour. In the $N \to \infty$ limit it actually has a sharp 'phase transition as a function of temperature. This phase transition is related ~~analogous~~, in ~~some~~ certain respects, to each of four classic phase transitions

→ The Hagedorn Transition in String Theory
→ The Deconfinement Transition in Yang Mills Theory
→ The Gross Witten Transition in large N Matrix ~~Gauge~~ Theories
→ The Hawking Page Transition in AdS Space

In these lectures I will first review these four classic phase transitions. I will then describe (and solve) the system of principle interest.

Plan of Lectures

① The Hagedorn Transition in String Theory (Review)

② Deconfinement in Gauge Theories; Hawking Page + Gross Witten (Review)

③ The Thermodynamics of weakly
④ coupled Yang Mills on a sphere

And lessons therefrom.

Lecture I

The Hagedorn Transition in String Theory.

(A) Thermodynamics of 1+1 dim CFTs

(B) The THERMODYNAMICS of the Bosonic String.

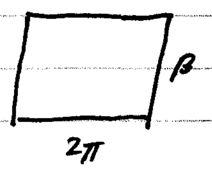
(C) Order Parameter and Generalization to the Superstring.

A) THERMODYNAMICS OF 1+1 Dim CFTs on S_1

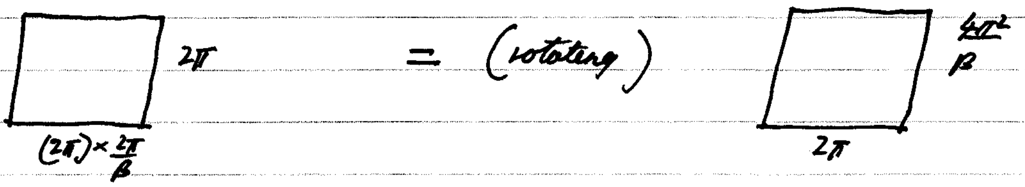
CONSIDER a CFT ON S^1, of unit radius. The partition f^n of this CFT is given by the path integral

$$Z(\beta) = \int D\phi \, \exp[-S]$$

where S is the Euclidean action on the torus [We will worry about world sheet fermions later]



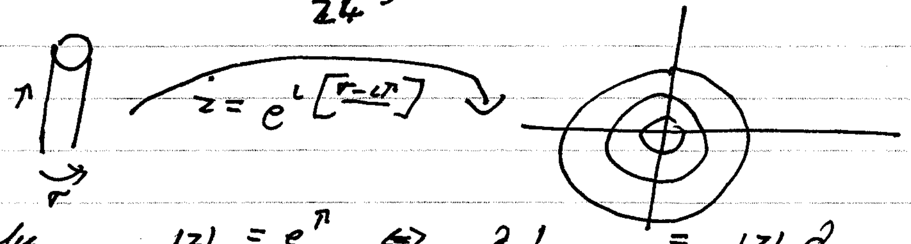
Since ~~the~~ a CFT is, ~~conformal~~ in particular, scale invariant this is the same as the same path integral on



$$\Rightarrow \quad Z[\beta] = Z[4\pi^2/\beta] \qquad \text{—— (1)}$$

We now assume that the CFT in question is unitary, and that ~~(on S^1)~~ it has a mass gap on S_1. ~~but the ground~~ Recall that the ground state energy of any unitary CFT is $-\left(\frac{c+\tilde{c}}{24}\right)$

[Why?



In particular $|z| = e^{\tau} \Leftrightarrow \frac{\partial}{\partial \tau}\Big|_{\sigma} = |z| \frac{\partial}{\partial |z|}$

$\Rightarrow$ Energy = Scaling dimension [classical]

However the correction from the Schwarzian derivative ($\because T_{zz}$ is not a tensor) gives $E = \text{Dim} - \left(\frac{c+\tilde{c}}{12}\right)$

Now ~~every unitary CFT~~ all scaling dimensions are ≥ 0 in a unitary CFT. Also $\exists$ a unique operator, I, with $\Delta = 0$. By the state operator map, thus, the vacuum state on S^1 has $E = -\left(\frac{c+\tilde{c}}{12}\right)$]

GAPPED CFTs

We now further assume that the CFT is gapped (Eg σ model on grp spaces)
Let the energy of the first excited state be $-\left(\frac{c+\tilde{c}}{12}\right)+\delta$

Then at low temperatures (β large) we know

$$Z[\beta] = e^{+\beta\left[\frac{c+\tilde{c}}{24}\right]}\left(1+O\left(e^{-\beta\delta}\right)\right) \qquad (\beta \text{ large})$$

From ① we find

$$Z[\beta] = e^{+\frac{4\pi^2}{\beta}\frac{c}{12}}\left(1+O\left(e^{-\frac{4\pi^2\delta}{\beta}}\right)\right) \qquad (\beta \text{ small})$$

(we have assumed $c=\tilde{c}$.)

$$Z[\beta] = e^{+\frac{c\pi^2}{3\beta}}\left(1+O\left(e^{-\frac{4\pi^2\delta}{\beta}}\right)\right) \qquad (\beta \text{ small}) \quad \text{—②}$$

It is easy to use ~~We will now use~~ ② ~~this~~ to compute the density of states at high energies. We find

$$\rho'(E) = \frac{c^{1/4}}{6}E^{-3/4}\exp\left[2\pi\sqrt{\frac{cE}{3}}\right] + O\left(e^{2\pi\sqrt{\frac{(c+12\delta)E}{3}}}\right)$$

Exercise ⓐ : $\rightarrow$ Show the exponent follows from Thermodynamics
(Hint : $\langle E\rangle = -\frac{\partial}{\partial\beta}\ln Z$; $S = \ln Z - \langle\beta\rangle$; $\rho = e^{S}$)

$\rightarrow$ In order to find the subleading piece assume $\rho'(E) = K E^{\alpha}\, e^{-\left[2\pi\sqrt{\frac{cE}{3}}\right]}$
and evaluate
$Z[\beta] = \int dE\, \rho(E)\, e^{-\beta E}$ at $\beta\to 0$ using saddle point.
(Hint: ~~In order~~ The exponent is minimized at $E=E_0 = \frac{\pi^2 c}{3\beta^2}$. The second derivative of the ~~potential~~ exponent is $F'' = -\frac{3}{2}\frac{\beta^3}{\pi c}$ (evaluated at minimum energy)

The Gaussian integral then gives

$$Z[\beta] = K\,(E_0)^{\alpha}\sqrt{2\pi\left(\frac{2\pi c}{3\beta^3}\right)}\; e^{-\frac{\pi^2 c}{3\beta}}.$$ Equating with ② gives
$\alpha = -3/4$; $K = \frac{c^{1/4}}{6}$)

CERTAIN UNGAPPED CFTS

It will turn out that (for string theory) we will be interested in CFTs whose spectrum includes a continuum that descends to zero : however the continuum will be very simple, and may be dealt with explicitly.

Infact, the theories we will study all have ~~to~~ Hamiltonians of the form

$$H = \frac{\alpha' P_i P^i}{2} + H' \qquad (i = 1 \cdots q \text{ and contracted over})$$

where P_i is $-i\frac{\partial}{\partial x_0^i}$ (x_0^i = the zero mode of a field X^i that parameterizes flat space)

$$Z[\beta] = \mathrm{Tr}\, e^{-\beta H} = \int \frac{d^q p}{(2\pi)^q}\, e^{-\frac{\alpha' P_i P^i}{2}}\, \mathrm{Tr}\left[e^{-\beta H'}\right]$$

$$= \left(\frac{2\pi}{\alpha'\beta}\right)^{q/2} Z'[\beta]$$

where $Z'[\beta] = \mathrm{Tr}\, e^{-\beta H'}$

Now $Z[\beta] = Z\left[\frac{(2\pi)^2}{\beta}\right] \Leftrightarrow Z'[\beta] = \left(\frac{\beta}{2\pi}\right)^q Z'\left[\frac{4\pi^2}{\beta}\right]$

The method behind then implies

$$\boxed{\rho'(E^*) = \frac{1}{6\times 12^{q/2}} \frac{c^{\frac{q}{2}+\frac{1}{4}}}{E^{\frac{3}{4}+\frac{q}{2}}}\, e^{2\pi\sqrt{\frac{cE}{3}}}}$$

Adding CFTs

Consider the direct sum of two CFTs with central charges c' and $\tilde{c}$, and parameters q' & $\tilde{q}$

The direct sum of these CFTs is a CFT with $c = c' + \tilde{c}$ and $q = q' + \tilde{q}$. Consequently we expect

$$\rho(E) = \cancel{(c+\tilde{c})^{\frac{q}{2}}} \frac{c^{\frac{q}{2}+\frac{1}{4}}}{E^{\frac{3}{4}+\frac{q}{2}}} e^{2\pi\sqrt{\frac{cE}{3}}}$$

On the other hand

$$\rho(E) = \int dx\, \rho'(x)\, \tilde{\rho}(E-x)$$

* We will now check the consistency of these formulas.

[We ignore constants. The Reader should check Constants as an Exercise]

$$\int dx \frac{c'^{\frac{q'}{2}+\frac{1}{4}}\, \tilde{c}^{\frac{\tilde{q}}{2}+\frac{1}{4}}}{x^{3/4+q'/2}\,(E-x)^{3/4+\frac{\tilde{q}}{2}}} e^{2\pi\left(\sqrt{\frac{c'}{3}x} + \sqrt{\frac{\tilde{c}}{3}(E-x)}\right)}$$

We solve this by saddle points.

$$\sqrt{\frac{c'}{3}}\frac{1}{\sqrt{x}} = \sqrt{\frac{\tilde{c}}{3}}\frac{1}{\sqrt{E-x}} \Leftrightarrow x = E\frac{c'}{(c'+\tilde{c})} \quad ; (E-x) = \frac{E\tilde{c}}{c'+\tilde{c}}$$

* The second derivative at the saddle point is

$$-\frac{\sqrt{3}}{2}\left[\frac{\sqrt{c'}}{x^{3/2}} + \frac{\sqrt{\tilde{c}}}{(E-x)^{3/2}}\right]$$

$$= -\frac{\sqrt{3}}{2E^3}(c+\tilde{c})^{3/2}\left[\frac{1}{c} + \frac{1}{\tilde{c}}\right] = \frac{-\sqrt{3}}{2E^{3/2}}\frac{(c+\tilde{c})^{5/2}}{c\tilde{c}}$$

Consequently the integral is

$$\frac{c'^{\frac{q'}{2}+\frac{1}{4}}\,\tilde{c}^{\frac{\tilde{q}}{2}+\frac{1}{4}}}{c'^{(\frac{q'}{2}+3/4)}\,\tilde{c}^{(\frac{\tilde{q}}{2}+3/4)}} (c+\tilde{c})^{\frac{6}{4}+\frac{q}{2}+\frac{\tilde{q}}{2}} \frac{\sqrt{c\tilde{c}}}{(c+\tilde{c})^{5/4}} \frac{\sqrt{2}}{3^{1/4}} \frac{E^{3/4}}{E^{3/4+q}E^{\frac{3}{4}+\tilde{q}}}$$

$$= \frac{(c+\tilde{c})^{\frac{1}{4}+\frac{q}{2}}}{E^{\frac{3}{4}+\frac{q}{2}}} e^{2\pi\sqrt{\frac{cE}{3}}}$$

Convolution

Lesson $\cancel{\frac{1}{E^{\alpha}} e^{2\pi\sqrt{\frac{cE}{3}}}}$ $\quad \frac{1}{E^{\alpha}} e^{\sqrt{aE}} * \frac{1}{E^{\beta}} e^{\sqrt{bE}} = \frac{1}{E^{\alpha+\beta-3/4}} e^{\sqrt{(a+b)E}}$

* For the 26 dimensional Bosonic string (without level matching) Hardy Ramanujan predicts the power as $\frac{1}{E^{48}} \times (E)^{3/4 \times 47} = \frac{1}{E^{\frac{192-141}{4}}} = \frac{1}{E^{\frac{51}{4}}}$

in agreement with $\frac{1}{E^{\frac{3}{4}+\frac{24}{2}}} = \frac{1}{E^{51/4}}$

~~Checks on the Formulae~~

Hardy Ramanujan Formula

For fun (and in order to check our work) we recall that the flat space sigma model in 1 dimension has an oscillator Hamiltonian consisting of left and right moving ~~oscill~~ oscillators.

According to the Hardy Ramanujan formula, each of these sectors has

$$\rho(E) \propto \frac{1}{E} e^{2\pi\sqrt{\frac{E}{6}}}$$

Using our compounding rule behind, we find that the total oscillator ~~Hamiltonian~~ ρ is

$$\propto \frac{E^{3/4}}{E^2} e^{2\pi\sqrt{\frac{E}{3}}}$$

$$\propto \frac{1}{E^{5/4}} e^{2\pi\sqrt{\frac{E}{3}}}$$

in agreement with $\rho(E')$ at $c=1$, $q=1$

* behind.

Level Matched Spectrum

In what follows ahead, we will be interested in $\rho(E)$ not for the full theory, but only for those states that are level matched.

Recall $\rho_{Full} \propto \rho_L \rho_R E^{3/2}$

(as far as power laws go)

$\rho_{Level\ matched} \propto \rho_L \rho_R$.

$$\Rightarrow \boxed{\rho_{Level\ Matched}(E) \propto \frac{\rho_{Full}(E)}{E^{3/4}} \propto \frac{e^{2\pi\sqrt{\frac{cE}{3}}}}{E^{3/2+q/2}}}$$

B THERMODYNAMICS OF THE CLOSED BOSONIC STRING

Consider the Bosonic String, with world sheet CFT
= (Sigma model on $R^{d-1,1}$) $\oplus$ CFT′
where CFT′ is any gapped CFT with central charge
26−d.

Moving to Lightcone gauge, we conclude (as usual) that the Hamiltonian H ~~for that results from~~ of the CFT $(c=24)$

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\, \partial x^i \bar{\partial} x_i + S'_{CFT} \qquad (i = 1 \cdots d-2)$$

takes the form $H = \frac{\alpha' p_i p^i}{2} + H'$

where the spectrum of H' ~~may be identified with~~ is the spectrum of $\frac{\alpha' M^2}{2}$ and M^2 is spacetime mass.

The density of states of this $c=24$ CFT (computed behind) is easily turned into a density function that measures how the number of particles ~~to~~ whose mass lies between M & $M+dM$

$$\rho(m) = \alpha' M\, \rho(E)$$

$$\propto \frac{(\alpha' m)\, e^{2\pi\sqrt{\frac{\alpha' m^2 24}{6}}}}{\left(\frac{\alpha' m^2}{2}\right)^{3/2+\frac{d-2}{2}}}$$

$$\boxed{\rho(m) \propto \frac{e^{4\pi\sqrt{\alpha'}\, m}}{m^d}}$$

The exponential growth in $\rho(m)$ might already make us suspect that Thermodynamics is poorly ~~de~~ defined for $T > \frac{1}{4\pi\sqrt{\alpha'}}$.

This is the basic "Hagedorn" observation, which we proceed to make precise

The Hagedorn Thermodynamics of the Closed String

Recall that the Grand Partition Function, $\mathrm{Tr}\, e^{-\beta H + \mu N}$, of a single free scalar field is

$$\ln Z = -\underbrace{\int \frac{d^{d-1}p\, V}{(2\pi)^{d-1}}}_{\text{Number of modes}} \underbrace{\ln\left[1 - e^{-\beta\sqrt{\vec{p}^2+m^2}+\mu}\right]}_{\text{G.P.F of each mode (Vacuum energy set to zero)}}$$

The string partition function, at $g=0$, is, consequently

$$\ln Z = -V\int \frac{d^{d-1}p\, dm}{(2\pi)^{d-1}}\, \rho(m) \ln\left[1 - e^{-\beta\sqrt{\vec{p}^2+m^2}+\mu}\right]$$

We will now estimate the effect of states whose mass $m^2\alpha' \gg 1$, on $\ln Z$ at $\beta = O(\sqrt{\alpha'})$. Three simplifications follow in this limit

① $$\ln\left[1 - e^{-\beta\sqrt{\vec{p}^2+m^2}+\mu}\right] \sim -e^{-\beta\sqrt{\vec{p}^2+m^2}+\mu}$$

② $$\int d^{d-1}p\; e^{-\beta\sqrt{\vec{p}^2+m^2}+\mu} \sim e^{\mu}\int d^{d-1}p\; e^{-\beta m + \frac{\beta p^2}{2m}}$$
$$= e^{\mu} e^{-\beta m}\left(\frac{2\pi m}{\beta}\right)^{\frac{d-1}{2}}$$

③ The estimate for $\rho(m) \propto \dfrac{e^{4\pi\sqrt{\alpha'}m}}{m^d}$, derived behind, applies. We find

$$\boxed{\ln Z \propto e^{\mu}\, T^{\frac{d-1}{2}} \int \frac{dm}{m^{\frac{d}{2}+1/2}}\; e^{-\left(\frac{1}{T}-\frac{1}{T_H}\right)m}}$$

This is Hagedorn Behaviour. We now pause for a bit to study the behaviour of such systems in more detail.

Hagedorn Thermodynamics

~~A system whose~~ We will refer to any system whose Grand Cannonical partition function is given by

$$\ln Z \propto \int dE\, E^{\alpha}\, e^{-(\beta-\beta_H)E + \mu}$$

as a Hagedorn system.
Clearly $\ln Z$ is not defined for such systems for $\beta < \beta_H$ $(T > T_H)$

The behaviour of this system as $T \to T_H$ however, depends in a crucial way on α.

Case a $\alpha \leq -1$

In this case

$$\ln Z \propto \frac{e^{\mu}}{(T_H - T)^{\alpha-1}}$$

[Note the closed string is in this case only for $d=1$. This will be significant later]

Thus, clearly

a) $\langle F \rangle$ Blows up like $(T_H - T)^{1+\alpha}$
b) $\langle E \rangle$ Blows up like $(T_H - T)^{\alpha}$
c) $\langle N \rangle$ Blows up like $(T_H - T)^{1+\alpha}$.

The system is a dense soup of energy and an increasing number of particles.

Case b $\alpha > -1$

[For $d > 1$, the closed string falls into this category]

In this case

$\ln Z$ is perfectly well defined at and upto the phase transition point. Nothing diverges while $\beta > \beta_H$.

Note: Why $\alpha = -1$? $\rho_{2ple}(E) = \int \rho(E-x)\,\rho(x)\,dx$

$$\Rightarrow \rho_{2ple} = \frac{e^{+\beta_H E}}{E^{2\alpha-1}}$$

Consequently $\rho_{2ple} \gg \rho_{1ple}$ $(E \gg 1)$ iff $\alpha < -1$.

~~Another cautionary note.~~

At the crudest level the thermodynamics of this system is $S = 4\pi\sqrt{\alpha'}E$

Note that: $\left(\partial S/\partial V\right)_E = P = 0$ (Hagedorn matter is pressureless)

However, in general we should be very careful in using thermodynamics. In particular, the equivalence between the canonical and micro canonical ensemble breaks down, because $Z[\beta]$ at fixed finite temperature gets important contributions from all energies as $\beta \to \beta_H$ (usual peaking in energy space fails). So we should ~~be very careful in using~~ very carefully state all questions we ask.

__Example__: Is this phase "stable" at some high temperature.

Interpret question in following manner: If we put a fixed amt of energy in an isolated ~~system~~ Hagedorn system, is it stable.

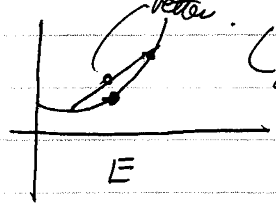
Use micro canonical ensemble

$S = \beta_H E + \alpha \ln E$; $\frac{1}{T(E)} = \left(\beta_H + \frac{\alpha}{E}\right)$

$\delta\left(\frac{1}{T}\right) = \alpha\, \delta\left(\frac{1}{E}\right)$

$\left(\frac{\partial E}{\partial T}\right)_V = C_V \sim \alpha\left(T_H^2 E^2\right)$ = $-$ve at large E

This indicates the system is unstable



$\left(\frac{\partial^2 S}{\partial E^2}\right.$ $\Rightarrow$ phase separation wants to happen)

However $\left(\frac{\partial E}{\partial T}\right)$ is always +ve in the canonical ensemble. So the system is very stable when put in contact with a heat bath

(C) Order Parameter, Tachyon Condensation, and Generalization to the Open String → Superstring

I start this part on an apparently unrelated note. Consider the thermodynamics of a single particle moving in one dimension, where target space is a circle of radius R.

$$\mathrm{Tr}\, e^{-\beta H} = (\text{we know}) \quad \sum_n e^{-\frac{n^2\beta}{2R^2 m}}$$

I wish to recover this result from path integrals. This is easy to do

$$\int \mathcal{D}x(t)\, \exp\left[-\int \frac{m\dot{x}^2}{2}\right] \qquad \left(x(\beta) = x(0) + 2\pi k R\right)$$

$$= \left(\int \mathcal{D}x(t)\, \exp\left[-\frac{m\dot{x}^2}{2}\right]\right) \sum_{k=0}^{\infty} e^{-\frac{M 4\pi^2 k^2 R^2}{2\beta}}$$

$x(\beta) = x(0)$ $\qquad\qquad$ $x = \dfrac{2\pi k R t}{\beta}$

$$= A \sum_{n=0}^{\infty} e^{-\frac{\beta n^2}{2R^2 m}} \qquad \text{(Poisson Resummation)}$$

(indep of R), absorbed into path integral measure).

> That was easy enough. In words we must sum over all "windings" of the particle around the time circle.

~~As an aside~~, Note that the Partition function for a free Fermion

$$\sum_n e^{-\frac{\beta (n+\frac{1}{2})^2}{2R^2 m}} = e^{-\frac{M 4\pi^2 k^2 R^2}{2\beta} + i\pi k}$$

(Because $i\pi k$ combines with $ik 2\pi n$ to give $n \to n + 1/2$)

> The slogan: For Fermions, weight the windings in time by $(-1)^w$

We will return to this soon.

Multy Particles from 1st Quantized

We wish to do something similar to the behind, only this time

(a) Deal with the relativistic action $\int \left(\frac{\dot{x}^2+m^2}{2}\right) d\tau$

(b) This time put the space time theory at finite temperature. x^0 is one of our world sheet fields (plays the role of x behind) is

$x^0 \equiv x^0+\beta$ (So $2\pi R$ from behind is β)

(c) Deal with multiparticles by integrating over module. (So $x(t) = x(0) + k\beta$; t is integrated over and plays the role of β behind.

(d) Because of the action we have used m (behind) $\to 1$. Also we have the additional constant term $e^{-\frac{m^2 t}{2}}$ in our path integral.

Making the replacements the sum from behind is

$$e^{-\frac{m^2 t}{2}} \sum_k e^{-\frac{\beta^2 k^2}{2t}}.$$

And the full answer is $\propto$

$$\sum_k \int \frac{dt}{t} \; \mathcal{D}x(t) \; \exp\left[-\int_0^t \frac{\dot{x}^2}{2} d\tau\right] \; e^{-\left(\frac{\beta^2 k^2}{2t} - \frac{m^2 t}{2}\right)}$$

Now the partition function $\int \mathcal{D}x(t) \exp\left[-\int_0^t \frac{\dot{x}^2}{2} d\tau\right]$

$$= \mathrm{Tr}\; e^{-tH}$$

$$= \int \frac{d^d p}{(2\pi)^d} e^{-\frac{tp^2}{2}}$$

$$= \frac{1}{(2\pi t)^{\frac{d}{2}}}$$

$$Z = -2\int \frac{d\tau\, d\bar\tau}{4\tau_2}\left(4\pi^2\alpha'\tau_2\right)^{-13} |\eta(\tau)|^{-48} \sum_{r=1}^{\infty} e^{-\frac{r^2}{4\pi T^2\alpha'\tau_2}} \Big) \quad 14$$

So we might expect that an equivalent form for the single pt'le partition f^{tn} is

$$\int \frac{dt}{t}\,\frac{1}{(2\pi t)^{d/2}} \sum_k e^{-\frac{m^2 t}{2} - \frac{\beta^2 k^2}{2t}}$$

This is indeed the case ~~as~~ one can demonstrate the equivalence of this formula with the one behind. (see inset).

Now summing over the string spectrum gives $Z(t)$ (a caveat about fermions) if we dont worry about level matching.

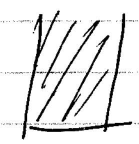
Actually worrying about level matching is very easy. Use

$$\int \frac{d\theta}{2\pi}\, e^{i(L_0-\bar L_0)\theta} = \delta_{L_0,\bar L_0}$$ (When L_0, $\bar L_0$ = integers as reqd by modular invariance)

We see that ~~inter~~ inserting this into our PF

→ Complexifies τ.

→ Causes the $\int$: $Z(\tau)\, Z(\bar\tau)$ to be integrated ~~on~~ over



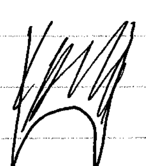
this is very much ~~like~~ ordinary the usual 1 loop string diagram, except that

a) We have included no winding modes

b) We have integrated over ∞ copies of the fund. domain

These two problems cancel each other out; our path int =

$$2\int \frac{d\tau\, d\bar\tau}{\tau_2^{\,2}}\, Z(\tau)\, Z(\bar\tau)$$ over Full



Ex: Recover the divergence structure (both with & without imposing level truncation) from this formula

Reinterpretation

~~In the~~ The Euclidean path int. may be reinterpreted as the 1 loop connected vacuum energy of a spatial thermal circle.

~~In this~~ **Question**: What is the Hagedorn divergence here?

Ans: Winding mode going tachyonic.

Recall $S = \frac{1}{4\pi\alpha'} \int d^2\sigma \; (\dot{x}^2 - x'^2) \; \cancel{\partial x \bar{\partial} x} \; +$

$$H = (N + \tilde{N} - 2) + \frac{k^2 R^2}{2\alpha'} + \cdots$$

Massless mode when $R = 2\sqrt{\alpha'}$

$$\frac{2\pi R}{2\pi} = \frac{1}{T\, 2\pi} = 2\sqrt{\alpha'}$$

or $\frac{1}{T} = 4\pi\sqrt{\alpha'}$

The divergence comes from doing a 1 loop expansion about a massless / tachyonic mode.

Order of Transition (1st)

(ϕ = dilaton)

$$F = \cancel{a|\phi|^2 + b|\phi|^2} \quad a|T|^2 + g\, b\, |T|^2 \phi$$

Can go to ϕ sufficiently large (and op. sign to b) to make $-$ve F, even when a is $\neq 0$.

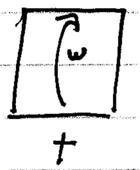
Thus the phase transition could happen considerably before the ~~transition temp~~ Hagedorn temp.

Super String

Most things go through unchanged, with $24 \to 12$ in various formulae.

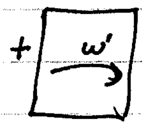
~~A the~~ Something we wish to explain in detail, however, is the tachyon in a winding mode.

Consider string theory on a Scherk Schwarz circle. Since we know that modes with no winding behave in the usual way (fermions have A.P. boundary conds) we conclude that ~~sta~~ R states should be be summed



$(-1)^{w}$

But by modular invariance that implies



$(-1)^{w'}$

Which $\Rightarrow$ we have reversed the GSO projection!

Note $\quad \frac{1}{T_H} = 2\sqrt{2}\,\pi\,\sqrt{\alpha'} \qquad (\because\ c \to c/2)$

Also the winding strings obey

$$\frac{\alpha' m^2}{2} = (N + \bar{N} - 1) + \frac{1}{2\alpha'}\underline{k^2 R^2}$$

$$\Rightarrow \quad \frac{1}{T_H} = 2\sqrt{2}\,\pi\,\sqrt{\alpha'}$$

It all goes through!

Lecture II

Deconfinement, Hawking Page & Gross Witten

In this lecture I will (very briefly) introduce you to other classic Phase Transitions

Deconfinement

Consider ordinary $d=4$ $SU(N)$ Yang Mills theory. At zero temperature this theory is believed to confine (~~All~~ all states are $SU(N)$ singlets).

However, at high energies all particles in this theory are almost free. So it seems reasonable that at high enough temperatures, colour non singlets are allowed states. Namely, thus, one expects a phase transition in Yang Mills theory as a function of temperature.

Polyakov introduced a sharp order parameter for this phase transition. ~~Consider~~ Yang Mills at temperature $\frac{1}{\beta}$ is described by the path integral of Euclidean Yang Mills on $M \times S^1$ (circumference $=\beta$)

The object $\langle \frac{1}{N} \mathrm{Tr}_{\text{Fund}}\, P\, e^{i\oint_{S^1} A_0} \rangle$

is referred to as the Polyakov loop. It represents the Free Energy of ~~a single, quark~~ ~~in the~~ of the system in the presence of a single classical quark.

In the low temperature phase ~~$\beta \to \infty$~~, we expect this to vanish. On the other hand, in the ~~limit $\beta \to 0$ we~~ high temp phase we expect to be finite.

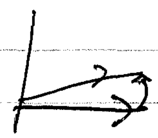
<u>Note</u>: ~~A no~~ ~~A theory~~ In a theory with ~~no~~ only adjoint fields one is allowed to make multivalued gauge transformation (around non contractible loops) provided the ~~gauge transformation~~ multivaluedness acts trivially on all fields; ie provided the G.T is well defined upto an element of the centre. Such gauge transformation <u>do not act</u> on physical fields, and so constitute a Z_N symmetry of the theory (that we do not gauge, to permit the possibility of adding fund quarks).

$$\frac{1}{N}\mathrm{Tr}\, P\exp\left[i\int_{S^1} A\right] \rightarrow \frac{1}{N}\mathrm{Tr}\left[Z\, P\exp\left(i\int_{S^1} A\right)\right]$$

So if $\mathrm{Tr}\,U$ = is nonzero, ~~then~~ that breaks Z_N invariance.

Consequently, the ~~confinement~~ confined phase is a symmetric phase; the deconfined phase the symmetry broken phase (opp of usual)

Finally note that as $N \to \infty$ $Z_N \to U(1)$



(can rotate by any angle)

So in the infinite N limit, deconfinement breaks a $U(1)$ symmetry.

Deconfinement in Strong Coupling Lattice Gauge Theory

Consider LGT

$$S = \frac{1}{2g^2}\sum \left[Tr\left(U_{x\vec\alpha}\, U_{x+\hat\alpha,\vec\beta}\, U_{x+\hat\alpha+\hat\beta,-\vec\alpha}\, U_{x+\hat\beta,-\vec\beta}\right) + cc - 2\right]$$

This is the simplest "symmetric" action. We could also study an ~~unsymmet~~ asymmetric version in a more

Naive Continuum Limit of symmetric action (Abelian)

Let $U_{x\vec\mu} = e^{i A_{x\mu} a}$ (a = lattice spacing)

~~Quadratizing S~~ For small g the U's will not fluctuate ~~too~~ far from I, and it ~~may be a~~ is a good approx to replace this by

$$S = \frac{1}{2g^2}\sum_{x\alpha\beta} -\left(F_{x\alpha\beta}\right)^2 a^4$$

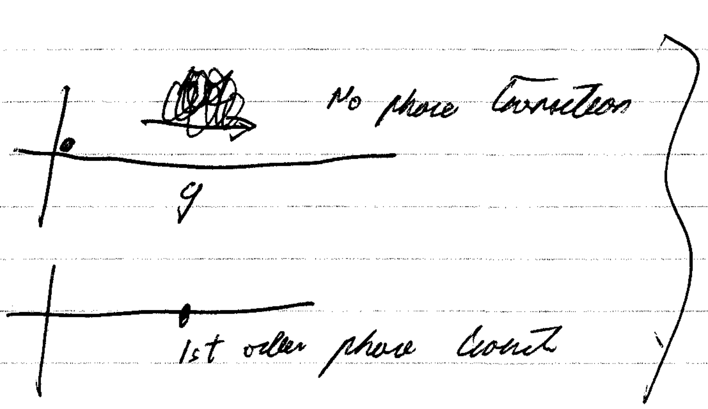
$$\sim \frac{1}{2g^2}\int F_{\mu\nu}F^{\mu\nu}$$

So for $g \to 0$ the model reproduces the Abelian Gauge Theory. Similar (slightly more ~~to~~ involved) statement is true of the NAGT.

Belief (hope?)

NAGT

Abelian



Why important? : lattice gauge theory easy to study at $g \to \infty$. Similar to behavior ~~at~~ of QCD at $g \to \infty$? ~~For U(1) Not true~~ Confinement. U(1))

Quickly show ~~of~~ you a deconfinement transition (T) in Abelian NAGT.

Many ways. Most transparent consider Asymmetric ~~Lattice~~ theory

$$S = \frac{\beta_0}{2} \sum_{\alpha,0} \left[\left(U_{x,\alpha}\, U_{x+\alpha,0}\, U_{x+\alpha+0,-\alpha}\, U_{x+0,-0} \right) + cc \right]$$

$$+ \frac{\beta_1}{2} \sum_{\alpha\beta} \Big[\qquad // \text{Same } 0 \Leftrightarrow \beta .$$

Consider $\beta_0 \gg (1, \beta_1)$

Let
$$U_{x,\alpha} = \exp[i A_{x\alpha}]$$
$$U_{x,0} = \exp[i \phi_x]$$

Since β_0 is large $(A_{x,\alpha} - A_{x+0,\alpha}) \ll 1$ and we are justified in taking the lattice in the 0 direction to a continuous parameter and replacing differences by derivatives

$$S = \int dt \Big[\frac{\beta_0}{2} \sum_{x,\alpha} \left(\dot{A}_{x\alpha} + \phi_x - \phi_{x+\alpha} \right)^2$$

$$+ \frac{\beta_1}{2} \sum_{x,\alpha,\beta} \left(\cos\left(A_{x,\alpha} + A_{x+\alpha,\beta} - A_{x+\beta,\alpha} - A_{x,\beta} \right) - 1 \right) \Big]$$

~~Now we set~~ $\mathcal{H}$

$$\mathcal{H} = \frac{1}{2\beta_0} \sum_{x,\alpha} (E_{x,\alpha})^2 + \beta_1 \left(\qquad \right)$$

$$C = D \sum_{\alpha} \left(E_{x,\alpha} - E_{x-\alpha,\alpha} \right) = 0$$

Strong coupling $\beta_0 \beta_1 \ll 1$ ignore magnetic term.

charge m at x.

$$\mathrm{Tr}\left[e^{-\beta' H}\right] = \int d\phi_x \sum_{n_x} e^{-\frac{\beta'}{\beta_0}\frac{n_{x\alpha}^2}{2} + i\sum_\alpha (n_{x\alpha} - n_{x-\alpha,\alpha})\phi_x + i m_x \phi_x}$$

Let $\beta'/\beta_0 = \beta$

$$Z[\beta] = \int \prod_x \frac{d\phi_x}{2\pi} \sum_{n_x} e^{-\beta\frac{n_{x\alpha}^2}{2} + i(\phi_{x+\alpha} - \phi_x) n_{x,\alpha} + i m_x \phi_x}$$

Do sum over $n_{x,\alpha}$

$$= \int \prod_x \frac{d\phi_x}{2\pi} \tilde{e}^{-\frac{1}{\beta}\frac{(\phi_{x+\alpha}-\phi_x)^2}{2} + i m_x \phi_x}.$$

Periodic Gaussian

$$\left(\tilde{e}^{-\alpha\frac{x^2}{2}} = \sum_m e^{-\frac{\alpha(x-2\pi m)^2}{2}} \right)$$

This Periodic Gaussian, for β small, is very like

$$\int \prod_x \frac{d\phi_x}{2\pi} e^{\frac{1}{\beta}\left[\cos(\phi_{x+\alpha} - \phi_x) - 1\right] + i m_x \phi_x}$$

So β small is like the low temperature phase of the planar Heisenberg ~~xxx~~ model

<u>Note</u>: $\phi_{x\alpha}$ aligned (Magnetization is spontaneously broken) $\langle \phi_x \rangle = 1$ (for unit.

$$\langle \phi(r)\, \phi(0) \rangle \sim \frac{1}{r}$$ (Spin waves, massless ple in 3 dim (Goldstone Boson)

So the ~~xxxxxxx~~ we see the Coulomb law for the force between poles.

On the other hand at β large ~~$\phi_{x+\alpha}$ ϕ_x~~ ~~vortex~~ vortex lines (monopoles) proliferate, and can ~~xxx~~ show

$$e^{i\lambda(0)} e^{i\lambda(r)} = c \exp\left[\quad \right]^{-\mu r}$$

string and string tension

Finally, the order parameter for the transition is clear; it is $\langle \phi \rangle$.

Actually same as Polyakov loop!

$\beta \to 0$ (High temp) $\langle \phi \rangle \neq 0$, deconfinement (breaking $U(1) \to$)

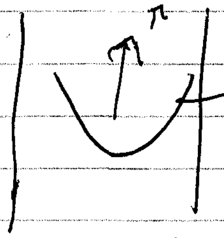
$\beta \to \infty$ Low temp, $\langle \phi \rangle = 0$, confinement ($U(1)$) unbroken

Similar statements for $O(N)$, will not review

Thermodynamics of IIB Theory on $AdS_5 \times S^5$

Numbers of order unity will be set to zero in this section

Consider $$ds^2 = R^2\left[-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\Omega_3^2 + d\Omega_5^2\right]$$



Potential; Effectively like a box of radius R

Spectrum discrete, ~~quan~~ E_0 quantized in units of $\frac{1}{R}$ (→ 1/2 integers)

~~R~~ $R = \lambda^{1/4}$, $E_0 =$ Energy of 1/m on sphere unit radius.

$g \to 0$, $N \to \infty$, λ fixed and large

~~Energy Regimes~~

$E \gg 1$, Parametrically small.

10 d Gravitons in a box of ~~size~~ unit size

$$\left.\begin{aligned} F &\propto -V \times T^{10} \\ E &= 9T^{10} \\ S &\sim T^9 \end{aligned}\right) \Rightarrow S \sim E^{9/10}$$

coefficients of order unity.

$E \gg \lambda^{1/4}$ $\left(\frac{E^2}{R^2} \gg \frac{1}{\alpha'}\right)$

Then Hagedorn like behaviour.

$S \sim \lambda^{1/4} E$

$F \sim |\ln(\beta - \lambda^{1/4})|$ (Assuming 1 dimensional)

$E \sim |\ln(\beta - \lambda^{1/4})|$

$$\left.\begin{aligned} &\frac{E}{R} \gg \frac{m_s}{g^{1/4}} \Rightarrow E \gg N^{1/4} \\ &S \sim \left(\frac{l_p E}{R}\right)^{8/7} = \left(\frac{E}{N^{1/4}}\right)^{9/7} \end{aligned}\right\} \text{10 dim Sch BHs}$$

$F = -\alpha$, −ve specific heat

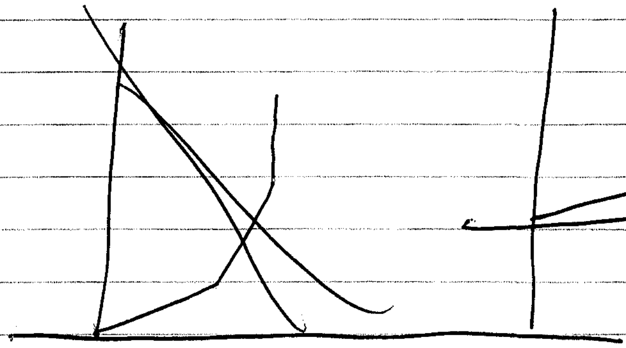
$E \sim N^2$ (AdS Schw BH)

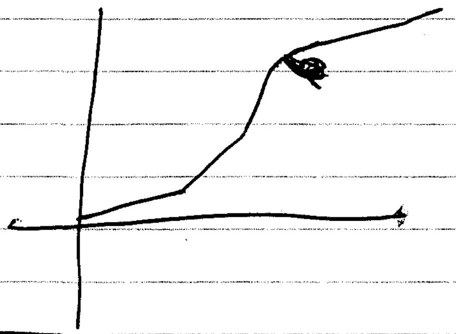
$S \sim N^2 T^4$

$S \propto N^{1/2} E^{3/4}$

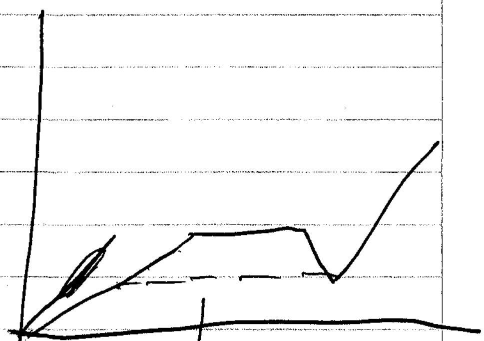
4 D CFT like Behaviour

$S(E)$





$T(E)$ $\qquad T = \dfrac{1}{\left(\frac{\partial S}{\partial E}\right)}$



1st order Phase

T const

In the 4 regimes $\sim E^{1/10}$

$\sim$ const

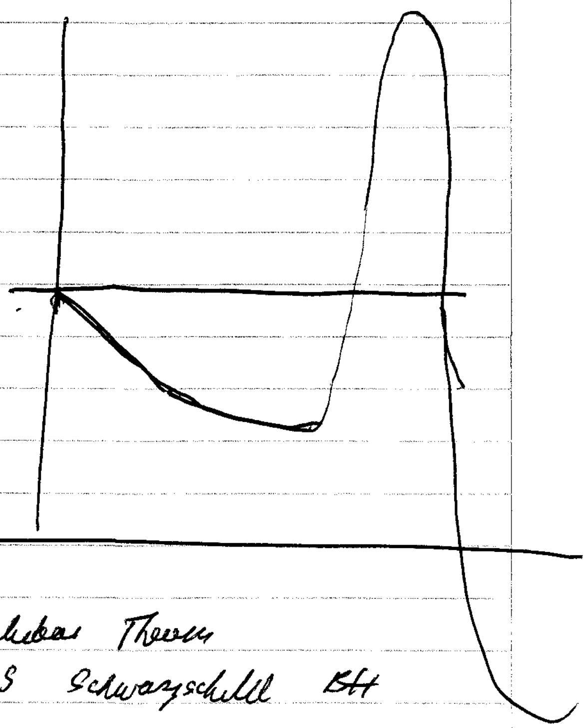
$\sim E^{-1/7}$

$\sim E^{1/4}$

$F(E)$ $\sim \tfrac{1}{4}E$

$\sim -|\ln E|)$

$\sim E/8$



Energy [illegible] points

$\lambda^{5/2}$

$N^2/\lambda^{7/4}$

N^2

Euclidean Theory

AdS Schwarzschild BH

The Gross Witten Wadia Model

$$\int \delta U \exp\left[\frac{N}{\lambda}\left(\mathrm{Tr}\, U + \mathrm{Tr}\, U^{+}\right)\right]$$

$$= \int \prod_i d\alpha_i \prod_{i<j} \sin^2\left(\frac{\alpha_i - \alpha_j}{2}\right) \exp\left[\frac{2N}{\lambda} \sum \cos\alpha_i\right]$$

$$= \int \prod_i d\alpha_i \exp\left[\sum_{i \neq j} \ln \sin\left(\frac{\alpha_i - \alpha_j}{2}\right) + \frac{2N}{\lambda} \cdot \sum_i \cos\alpha_i\right]$$

But $\ln \sin\left(\frac{\alpha_i - \alpha_j}{2}\right) = \ln 2 - \sum_{n=1}^{\infty} \frac{\cos\left(n(\alpha_i - \alpha_j)\right)}{n}$

$$= \ln 2 - \sum_{n=1}^{\infty} \cos\alpha_i \cos\alpha_j + \sin\alpha_i \sin\alpha_j$$

$$= \ln 2 - \sum_{n=1}^{\infty} e^{i\alpha_i} e^{-i\alpha_j}$$

Define $\rho_\alpha = \frac{1}{2\pi} \sum \rho_n e^{in\alpha}$; $\rho_n = \int e^{-in\alpha} \rho(\alpha)$

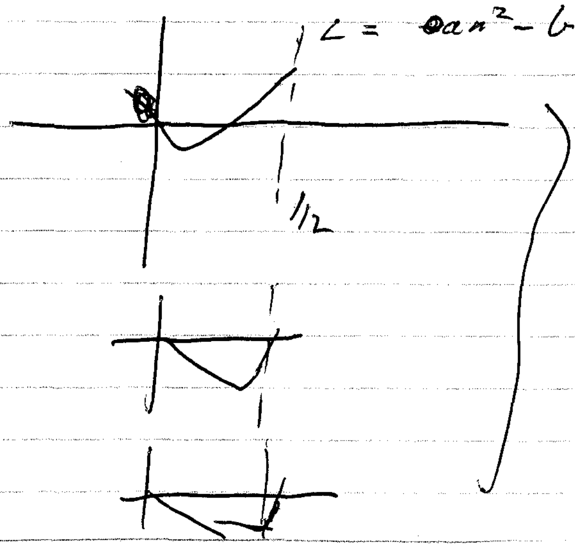
In terms of these

$$\int \mathcal{D}\rho \exp\left[+N^2\left(\frac{1}{\lambda}\left(\rho_1 + \rho_1^{*}\right) - \sum_n \frac{\rho_n \rho_n^{*}}{n}\right)\right]$$

Minimizing $\rho_1 = \rho_1^{*} = \frac{1}{\lambda}$.

$$\rho(\alpha) = \frac{1}{2\pi}\left[1 + \frac{2}{\lambda} \cos\alpha\right]$$

OK if $\lambda > 2$. Makes no sense for $\lambda < 2$



After this what?