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SMR.1498 - 8

SPRING SCHOOL ON SUPERSTRING THEORY AND RELATED TOPICS

31 March - 8 April 2003

TOPOLOGICAL STRING AND ITS APPLICATION

Lectures 3 and 4

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WORLDSHEET DERIVATION OF A LARGE N DUALITY

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BASED ON A WORK WITH CUMRUN VAFA

HEP-TH/0205297

STRING THEORY FROM A LARGE N LIMIT OF A GAUGE THEORY

'T HOOFT (1974)

$$S = \frac{1}{g_{YM}^2} \int \mathcal{L}(A) \,, \qquad A_\mu : U(N) \text{ GAUGE FIELD}$$

PROPAGATOR

$$\langle A_{\mu\, i}{}^{j} \, A_{\nu\, k}{}^{\ell} \rangle : \quad \begin{array}{l} i \longrightarrow k \\ j \longleftarrow \ell \end{array}$$

RIBBON GRAPH

EACH  GIVES A FACTOR OF N.

FILL  BY A DISK 

$\Rightarrow$ A CLOSED RIEMANN SURFACE

'T HOOFT COUNTING

$$(g_{YM}^2)^{-V+E} \cdot N^h = (g_{YM}^2)^{2g-2} \cdot (g_{YM}^2 N)^h$$

V = # VERTICES,

E = # PROPAGATORS,

h = # LOOPS = # HOLES.

g = GENUS OF THE SURFACE OBTAINED BY FILLING THE HOLES BY DISKS.

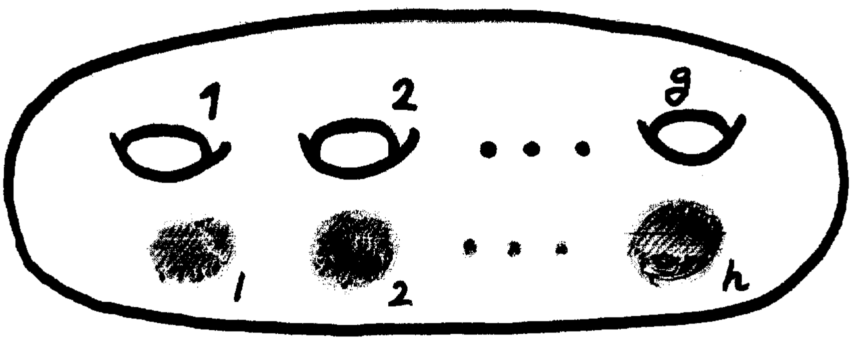
A PERTURBATIVE GAUGE THEORY AMPLITUDE IS EXPRESSED AS

$$F = \sum_{g=0}^{\infty} (g_{YM}^2)^{2g-2} F_g(t)$$

$$F_g(t) = \sum_{h=1}^{\infty} t^h F_{g,h} \quad , \quad t = g_{YM}^2 N$$

CONJECTURE

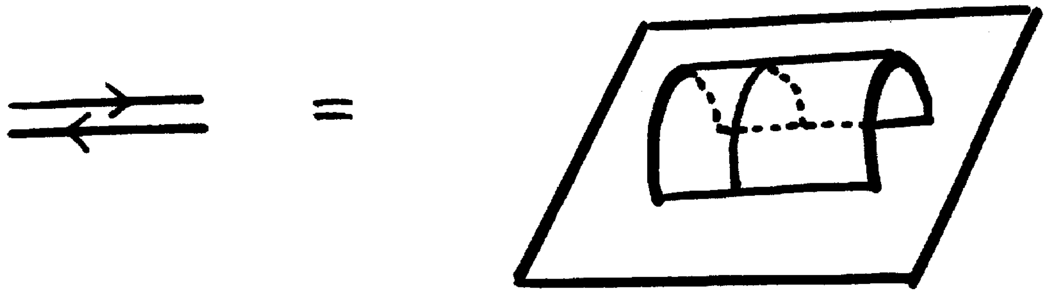
THERE IS A CLOSED STRING THEORY WHOSE g-LOOP AMPLITUDE IS GIVEN BY $F_g(t)$.



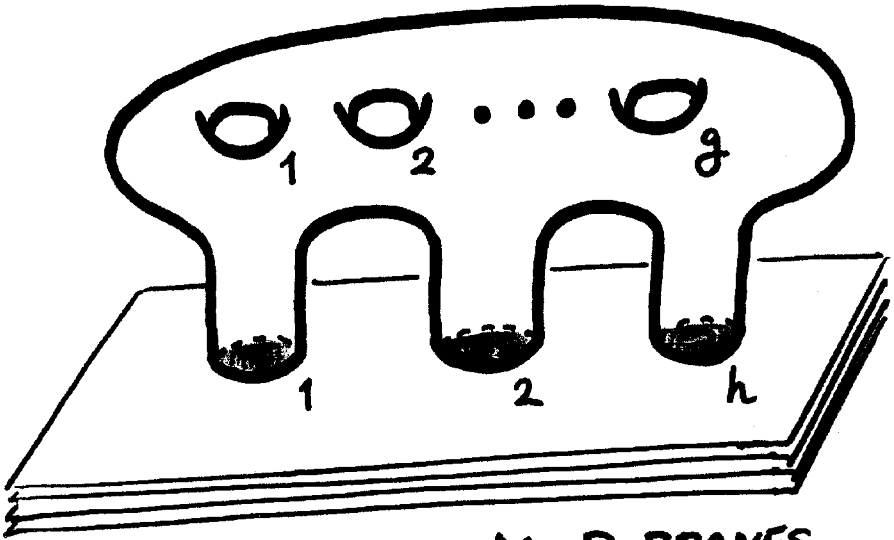
$\lambda_S = g_{YM}^2$: STRING COUPLING CONSTANT

$t = g_{YM}^2 N$: SOME PARAMETER, TYPICALLY GEOMETRIC MODULUS OF THE TARGET SPACE.

RIBBON GRAPHS COME TO LIFE ON D BRANES



D BRANES



N D BRANES

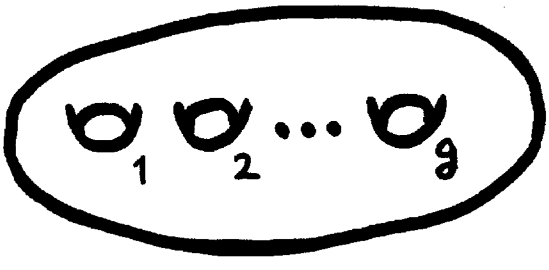
LOW ENERGY LIMIT

$\Downarrow$

GAUGE THEORY

$$\times\ \lambda_s^{2g-2}\cdot(N\lambda_s)^h$$

$\Updownarrow$ **CONJECTURE**



$$\times\ \lambda_s^{2g-2}$$

$t = N\lambda_s$: SOME GEOMETRIC MODULUS

THE 't HOOFT CONJECTURE IS THE LOW ENERGT LIMIT OF A MORE GENERAL CONJECTURE ABOUT THE EQUIVALENCE OF :

D BRANES $\Longleftrightarrow$ CLOSED STRING BACKGROUND

SEVERAL EXAMPLES OF LARGE N DUALITIES HAVE BEEN DISCOVERED ASSUMING:

- D BRANES $\longleftrightarrow$ CLOSED STRING BACKGROUND
- M THEORY DUALITIES

(1) AdS/CFT CORRESPONDENCE

MALDACENA

(2) TOPOLOGICAL STRING DUALITIES

- CHERN-SIMONS GAUGE THEORY $\leftarrow$ OPEN STRING ON A BRANES ON <u>DEFORMED</u> CY_3.

  $\Updownarrow$

  CLOSED STRING WITH A TWIST ON <u>RESOLVED</u> CY_3.

  GOPAKUMAR + VAFA

- HOLOMORPHIC MATRIX MODEL $\leftarrow$ OPEN STRING ON B BRANES ON <u>RESOLVED</u> CY_3.

  $\Updownarrow$

  CLOSED STRING WITH B TWIST ON <u>DEFORMED</u> CY_3.

  "KODAIRA-SPENCER THEORY OF GRAVITY"

  DIJKGRAAF + VAFA

IT IS DESIRABLE TO PROVE THESE CONJECTURES WITHOUT APPEALING TO M THEORY DUALITIES.

TOPOLOGICAL CLOSED STRING

START WITH AN $N=2$ SUPERCONFORMAL FIELD THEORY IN 2d WITH $\hat{c} = 6$.

e.g. THE SIGMA-MODEL ON CONIFOLD ($\dim_{\mathbb{C}} = 3$).

TOPOLOGICAL TWIST

$$\begin{array}{cccccc} & T & G^{+} & G^{-} & J & \\ & \downarrow & \downarrow & \downarrow & \downarrow & \\ (h_L, h_R) = & (2,0) & (1,0) & (2,0) & (1,0) & \end{array}$$

: CONFORMAL DIMENSIONS

G^{+}: TOPOLOGICAL BRST CURRENT

G^{-}: ANALOGUE OF THE ANTI-GHOST.

$$\mathcal{F}_g = \int_{\mathcal{M}_g} \langle \underbrace{G_L^{-} \cdots G_L^{-}}_{3g-3} \underbrace{G_R^{-} \cdots G_R^{-}}_{3g-3} \rangle$$

FERMION ZERO MODES $= \hat{c}(1-g)$

$\mathcal{F}_g$ IS DEFINED WHEN $\hat{c} = 6$.

A-MODEL DEPENDS ON KÄHLER MODULI

B-MODEL DEPENDS ON COMPLEX MODULI

SIMILARLY, FOR D BRANES WRAPPING ON

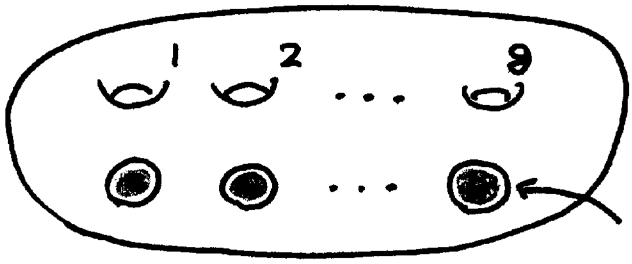
- LAGRANGIAN 3 CYCLES IN A-MODEL : A-CYCLES
- HOLOMORPHIC 2 CYCLES IN B-MODEL : B-CYCLES

KONTSEVICH ALG-GEOM/9411018

OOGURI - OZ - YIN HEP-TH/9606112

WE CAN DEFINE TOPOLOGICAL STRING AMPLITUDES

$$F_{g,n} =$$



F_g FOR CLOSED STRING

$F_{g,n}$ FOR OPEN STRING

COMPUTE VARIOUS SUPERPOTENTIAL TERMS FOR TYPE II STRING ON CY_3

BERSHADSKY - CECOTTI - OOGURI - VAFA HEPTH/9309140,

WHERE WE ALSO SHOWED HOW TO COMPUTE F_g USING THE HOLOMORPHIC ANOMALY.

CLOSED STRING

$$F_g \cdot (W_{\alpha\beta} W^{\alpha\beta})^{2g}$$

SUPERPOTENTIAL FOR $W_{\alpha\beta}$: $N=2$, 4d SUPERGRAVITY CHIRAL FIELD STRENGTH

OPEN STRING

$$F_g = \sum_n F_{g,n} S^n$$

$$\Downarrow$$

$$N \frac{\partial F_{g=0}}{\partial S} - 2\pi i S$$

SUPERPOTENTIAL FOR $S = \frac{1}{32\pi^2} \operatorname{tr} W_\alpha W^\alpha$

W_α : GAUGINO SUPERFIELD

FOR $N=1$, 4d GAUGE THEORY

ON

$$\begin{cases} N \ D6 \text{ BRANES ON } A \text{ CYCLE} \times R^4 \\ N \ D5 \text{ BRANES ON } B \text{ CYCLE} \times R^4 \end{cases}$$

SIMILAR INTERPRETATION FOR F_g

$$F_g(S = N\lambda_s) = \sum_n F_{g,n} \times (N\lambda_s)^n$$

CAN BE COMPUTED AS A PARTITION FUNCTION

OF
- CHERN-SIMONS GAUGE THEORY ON A CYCLE
- HOLOMORPHIC CHERN-SIMONS THEORY ON B CYCLE

WITTEN HEP-TH/9207094

MATRIX MODEL, WHEN B-CYCLE = S^2

DIJKGRAAF + VAFA, HEP-TH/0206255

$\Downarrow$

SUPERPOTENTIALS FOR THESE $N=1$ GAUGE THEORIES CAN BE COMPUTED BY PLANAR DIAGRAMS OF THE MATRIX MODELS.

THIS, BY ITSELF, IS A <u>KINEMATIC</u> FACT.

(THIS WAS EXPLICITLY VERIFIED BY DIJKGRAAF-GRISARU-LAM-VAFA-ZANON HEP-TH/0211017)

THE LARGE N DUALITY I AM GOING TO PROVE SHOWS THAT THESE GAUGE THEORY SUPERPOTENTIALS ARE RELATED TO $F_{g=0}$ OF <u>CLOSED</u> STRING, WHICH FOR B-MODELS CAN BE EVALUATED BY CLASSICAL GEOMETRY (PERIOD INTEGRALS).

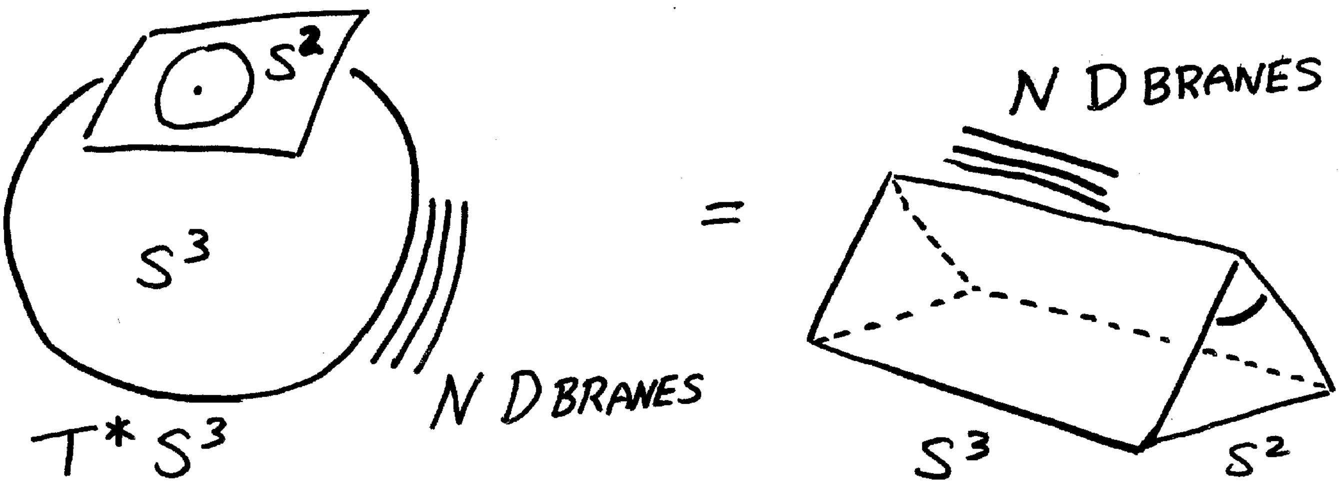
$F_{g\geq 1}$ FROM QUANTIZATION OF KODAIRA-SPENCER THEORY.

LARGE N DUALITY (A-MODEL VERSION)

OPEN TOPOLOGICAL STRING = GAUGE THEORY

U(N) GAUGE GROUP

$\lambda_s = g_{YM}^2$: STRING COUPLING CONSTANT

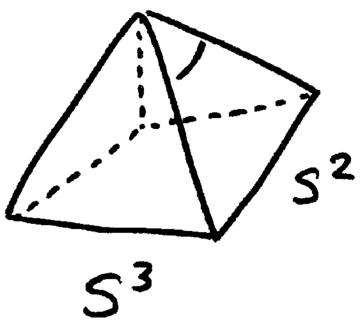


$\Updownarrow$

CLOSED TOPOLOGICAL STRING

ON CONIFOLD

λ_s : STRING COUPLING CONSTANT



$$t = N\lambda_s$$
$$= i\int_{S^2} B$$

't HOOFT COUPLING TURNS INTO THE KÄHLER MODULI.

PROOF

- START WITH THE SIGMA-MODEL WHOSE TARGET SPACE IS THE CONIFOLD.

$$i \int_{S^2} B = t = \frac{iN}{k+N} = g_{YM}^2 N$$

- STUDY THE $t \to 0$ LIMIT AND SEE HOW THE HOLES EMERGE.

WE NEED A GOOD DESCRIPTION AT $t=0$

IN MANY QUANTUM FIELD THEORIES, SINGULARITIES AT SOME VALUES OF PARAMETERS INDICATE NEW LIGHT DEGREES OF FREEDOM.

IN SUCH CASES, GOOD DESCRIPTIONS SHOULD TREAT THESE NEW DEGREES OF FREEDOM PROPERLY.

FOR THE CONIFOLD, THE GOOD DESCRIPTION IS IN TERMS OF THE LINEAR SIGMA-MODEL

WITTEN / 9301042

V : VECTOR MULTIPLET

A_i, B_i $(i=1,2)$: CHIRAL MULTIPLETS OF OPPOSITE CHARGES

$$\text{POTENTIAL} = |\sigma|^2 \left(|a_1|^2 + |a_2|^2 + |b_1|^2 + |b_2|^2 \right) + e^2 \left(|a_1|^2 + |a_2|^2 - |b_1|^2 - |b_2|^2 \right)$$

$$\sigma \in V, \quad a_i \in A_i, \quad b_i \in B_i$$

e : ELECTRIC CHARGE $\longrightarrow \infty$ IN THE INFRARED.

$t = g_{YM}^2 N$ APPEARS IN THE THETA TERM :

$$\frac{t}{2\pi} F_{12} = \int d\theta \, W$$

$$W = t\,\Sigma \quad : \text{SUPERPOTENTIAL}$$

$$\Sigma = \bar{D}_+ D_- V = \sigma + \cdots$$

TWISTED CHIRAL SUPERFIELD

CLASSICALLY

HIGGS BRANCH : $\sigma = 0$; $a_i, b_i \neq 0$

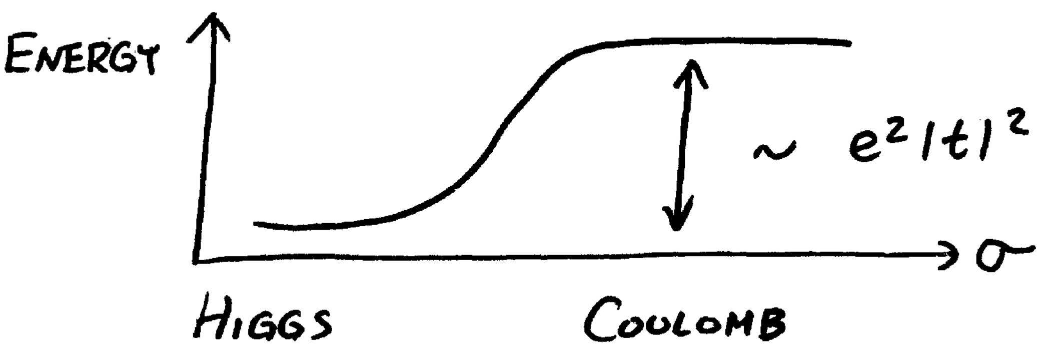
$\Rightarrow$ TARGET SPACE GEOMETRY

COULOMB BRANCH : $\sigma \neq 0$; $a_i, b_i = 0$

$\Rightarrow$ NEW BRANCH ATTACHED AT THE CONIFOLD SINGULARITY.

IF $t \neq 0$, THE COULOMB BRANCH HAS NON-ZERO ENERGY.

($\because$) THE THETA TERM $t F_{12}$ INDUCES A CONSTANT ELECTRIC FIELD.)

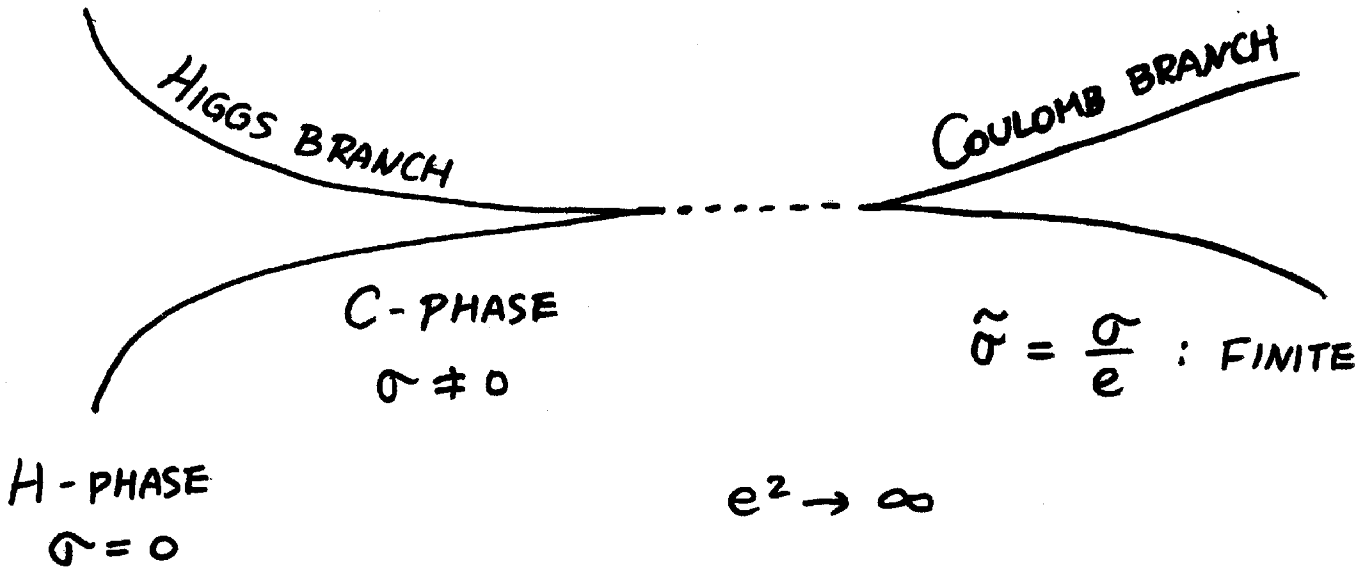


AS $t \to 0$, THE COULOMB BRANCH EMERGES AS A NEW PHASE.

For cognoscenti:

IN THIS MODEL, IT IS KNOWN THAT THE HIGGS AND THE COULOMB BRANCHES DECOUPLE IN THE INFRARED $e^2 \to \infty$.

WHAT WE CALLED THE NEW PHASE IS A PART OF THE HIGGS BRANCH THEORY.

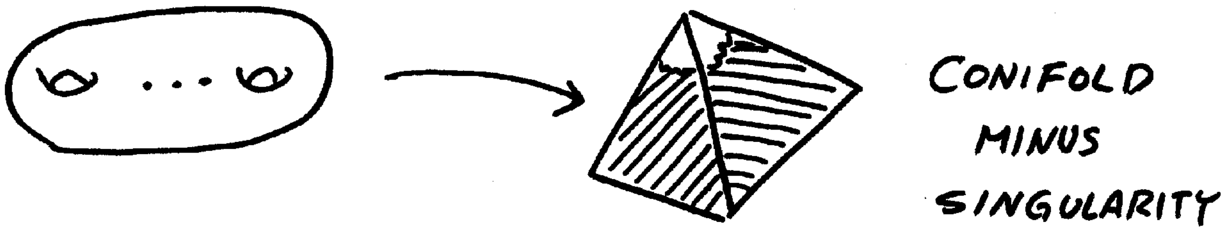


To summarize:

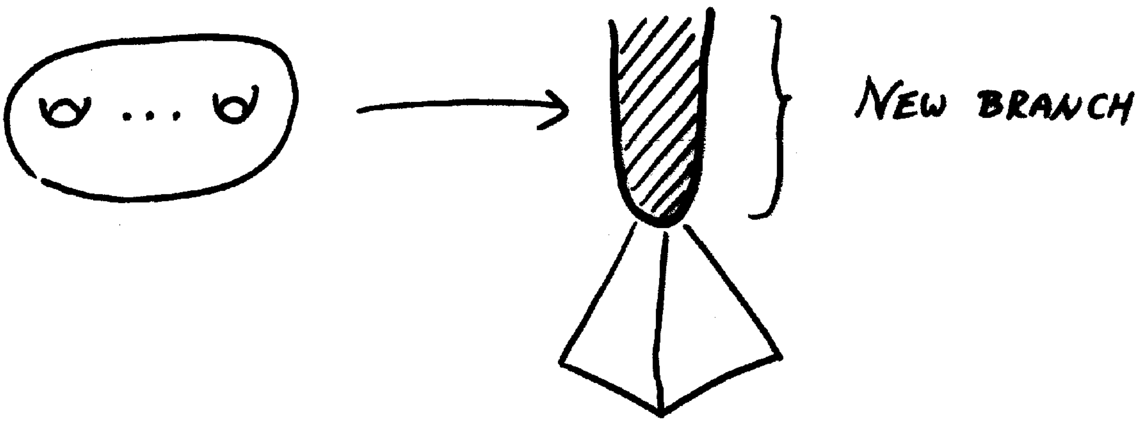
The worldsheet theory is the linear sigma-model

$$\text{Potential} = |\sigma|^2 \left(|a_1|^2 + |a_2|^2 + |b_1|^2 + |b_2|^2 \right) + e^2 \left(|a_1|^2 + |a_2|^2 - |b_1|^2 - |b_2|^2 \right)$$

H-branch: $|\sigma| < \sigma_*$, $a_i, b_i \neq 0$



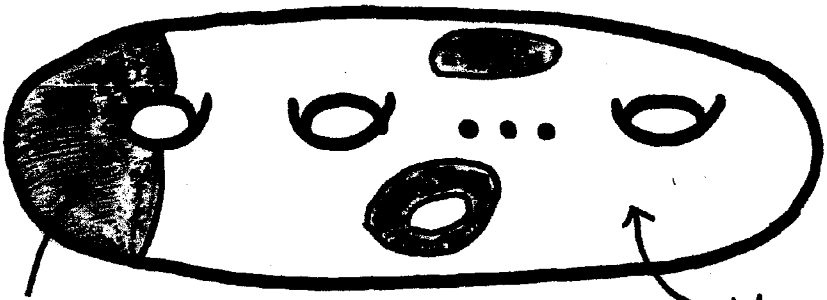
C-branch: $|\sigma| > \sigma_*$, $a_i, b_i = 0$



Superpotential $W = t\,\Sigma$

$$\Sigma = \sigma + \cdots$$

WE CAN SEPARATE THE WORLDSHEET INTO THE TWO PHASES.



σ_* : CUTOFF

H PHASE ; $|\sigma| < \sigma_*$

C PHASE : $|\sigma| > \sigma_*$

A PART OF THE FUNCTIONAL INTEGRAL OVER σ CAN BE TURNED INTO A SUM OVER <u>SHAPES AND SIZES OF THE C DOMAIN</u>.

COLLECTIVE COORDINATES OF σ

WE NEED TO SHOW :

- CONTRIBUTIONS FROM C DOMAINS SUCH AS :

VANISH.

NAMELY, EVERY C DOMAIN HAS THE TOPOLOGY OF THE DISK.

- EACH DISK IN C PHASE GIVES THE FACTOR OF t.

 $= t = g_{YM}^2 N$

OUR TASK IS SIMPLIFIED
BY THE LOCALIZATION OF σ.

WITTEN/9207094
BCOV/9309140

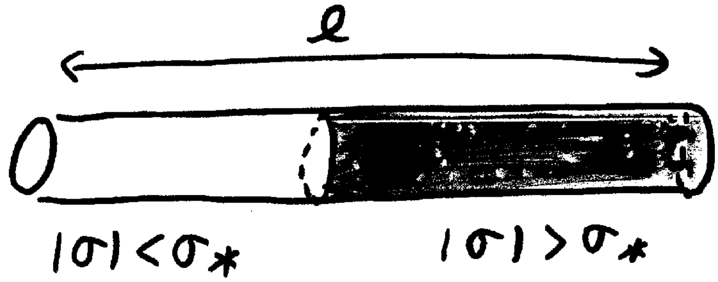
BECAUSE OF THE TOPOLOGICAL TWISTING,
σ = CONST ON A <u>FIXED</u> RIEMANN SURFACE.

$\Rightarrow$ THE WORLDSHEET IS EITHER IN THE PURE C-PHASE OR IN THE PURE H-PHASE.

C AND H DO NOT CO-EXIST.

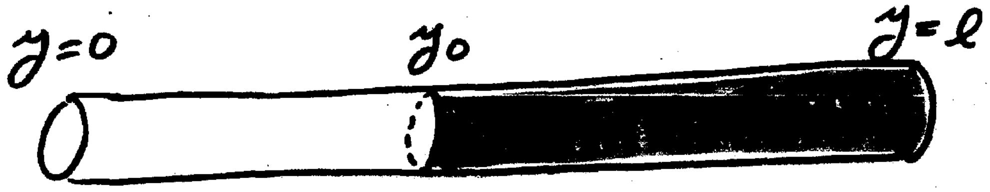
IN THE TOPOLOGICAL STRING, WE INTEGRATE OVER $\mathcal{M}_g$.

σ CAN BE NONCONSTANT, IF THERE IS A LONG CYLINDER.



$\ell \to \infty$

$\Rightarrow$ REDUCTION TO A ONE-DIMENSIONAL QUANTUM MECHANICS.



$$\sigma(y_0) = \sigma_0 , \qquad |\sigma_0| = \sigma_*$$

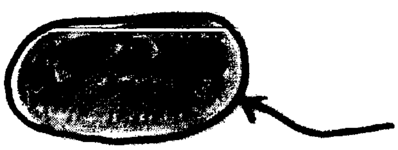
CHANGE OF FUNCTIONAL INTEGRAL VARIABLES :

$\sigma(y_0)$, $\bar{\sigma}(y_0)$ $\longrightarrow$ y_0 AND PHASE OF σ_0.

$$d\sigma(y_0)\, d\bar{\sigma}(y_0) = dy_0 \oint d\sigma_0 \frac{\partial}{\partial \sigma_0}$$

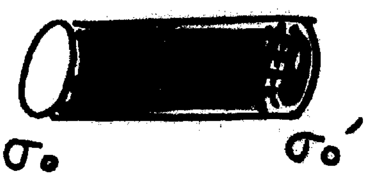
- EACH C-DOMAIN CONTRIBUTES AS

$$\oint d\sigma_0 \frac{\partial}{\partial \sigma_0} \underbrace{\mathcal{Z}^{(C)}(\sigma_0)}$$



PARTITION FUNCTION
OF THE C-DOMAIN
WITH THE BOUNDARY CONDITION
$\sigma = \sigma_0$.

- IF THE C-DOMAIN HAS MORE THAN ONE BOUNDARIE
ACT $\oint d\sigma_0 \frac{\partial}{\partial \sigma_0}$ on EACH BOUNDARY.



- y_0 BECOMES A MODULUS OF THE SURFACE
WITH THE BOUNDARY.

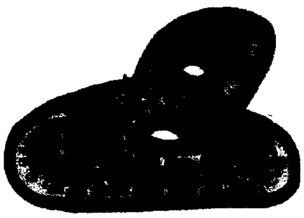
BY THE TOPOLOGICAL BRST SYMMETRY,

THE AMPLITUDE $\mathcal{F}^{(C)}(\sigma_0)$ OF EACH C DOMAIN

IS A HOLOMORPHIC FUNCTION OF σ_0.

$$\Rightarrow \quad \oint d\sigma_0 \frac{\partial}{\partial \sigma_0} \mathcal{F}^{(C)}(\sigma_0) = 0 \quad \text{IF } \mathcal{F}^{(C)}(\sigma_0) \text{ IS SINGLE-VALUED}$$

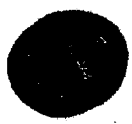
THIS IS THE CASE FOR

 ,  , ...

THE ONLY NON-ZERO CONTRIBUTION COMES

FROM THE CASE WHEN THE C DOMAIN IS A DISK.

THE AMPLITUDE CAN BE COMPUTED EXACTLY.

$$\oint d\sigma_0 \frac{\partial}{\partial \sigma_0}$$  $$= \oint \frac{d\sigma_0}{\sigma_0^2} e^{t\sigma_0}$$

DISK

$$= t = g_{YM}^2 N$$

THIS IS WHAT WE WANTED

TO SHOW.

PURE H - PHASE

NO HOLES.

PURELY CLOSED STRING CONTRIBUTION
IN THE PRESENCE OF D BRANES

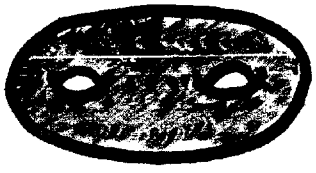
- TURNS OUT TO BE TRIVIAL IN OUR CASE.
- RELEVANT IF ONE WANTS TO PROVE

  D BRANES $\leftrightarrow$ CLOSED STRING BACKGROUND

  BEFORE TAKING THE LOW ENERGY LIMIT.

$g=0$: $\frac{1}{2} t^2 \log t$

$g=1$: $-\frac{1}{12} \log t$

$$\int_{\mathcal{M}_g}$$  $$= \chi(\mathcal{M}_g) \times \frac{1}{t^{2g-2}}$$

SINGULAR AS $t \to 0$.

THIS DOES NOT CORRESPOND TO ANY TERM IN 't HOOFT EXPANSION.

WHAT IS IT IN THE GAUGE THEORY?

$$\sum_g (g_{YM}^2)^{2g-2} \times \chi(\mathcal{M}_g) \times \frac{1}{(g_{YM}^2 N)^{2g-2}}$$

$$= \sum_g \frac{B_{2g}}{2g(2g-2)\, N^{2g-2}} \simeq -\log \mathrm{VOL}(U(N))$$

$$\mathrm{VOL}(U(N)) = \frac{(2\pi)^{\frac{1}{2}N(N+1)}}{(N-1)!\,(N-2)! \cdots 3!\,2!\,1!}$$

$$(\because \; U(N) \sim S^1 \times S^3 \times \cdots \times S^{2N-1})$$

VOL$(U(N))$ IS THE MEASURE FACTOR IN

$$\int [dA]\, e^{\frac{ik}{4\pi}\int \mathrm{tr}\,(AdA + \frac{2}{3}A^3)}$$

$$g=0 : \frac{\partial F_0(S)}{\partial S} = S \log S$$

THE CLOSED STRING THEORY KNOWS

ABOUT THE GAUGE THEORY

BEYOND THE 't HOOFT EXPANSION.

SO(N), Sp(N) CASES

... CONIFOLD + ORIENTIFOLD.

log VOL (SO(2m+1))

$$= -\frac{1}{2} \sum_g \left\{ \frac{\chi(\mathcal{M}_g)}{(2m)^{2g-2}} + \frac{\chi(\mathcal{M}_g^1)}{(2m)^{2g-1}} \right\}$$

log VOL (Sp(2m-1))

$$= -\frac{1}{2} \sum_g \left\{ \frac{\chi(\mathcal{M}_g)}{(2m)^{2g-2}} - \frac{\chi(\mathcal{M}_g^1)}{(2m)^{2g-1}} \right\}$$

$$\chi(\mathcal{M}_g) = \frac{B_{2g}}{2g(2g-2)}$$

$$\chi(\mathcal{M}_g^1) = \frac{(2^{2g-2} - 2^{-1})}{2g(2g-1)} B_{2g}$$

ONE CROSSCAP

GOULDEN, HARER + JACKSON (2001)

matrix model plan → Γ_{eff}. what non-planar? non-perturbative?

C-deformation and non-planar diagrams.

OV/0302109
0303063

$\mathcal{N}=1$ SUSY $\quad Q_\alpha,\ Q_{\dot\alpha} \qquad \alpha = 1.2$

$\{Q_{\dot\alpha}, Q_{\dot\beta}\} = 0 \Rightarrow$ cohomology ... chiral ring

vector multiplet : $A_{\alpha\dot\alpha}$, ψ_α

$$[Q_{\dot\alpha}, A_{\beta\dot\beta}] = \epsilon_{\dot\alpha\dot\beta}\psi_\beta$$

$$\{Q_{\dot\alpha}, \psi_\beta\} = 0$$

$\mathrm{tr}\,\psi_{\alpha_1}\cdots\psi_{\alpha_n}$ can be a chiral primary.

However :

$$\{\psi_\alpha, \psi_\beta\} = \frac{1}{2}\{Q^{\dot\alpha}, D_{\alpha\dot\alpha}\psi_\beta\} \simeq 0$$

$D_{\alpha\dot\alpha} = \partial_{\alpha\dot\alpha} + A_{\alpha\dot\alpha}$

(Note : $\psi_\alpha = \frac{1}{2}[Q^{\dot\alpha}, A_{\alpha\dot\alpha}] \not\simeq 0$ since $A_{\alpha\dot\alpha}$ is not gauge covariant.)

Thus

$\mathrm{tr}\,\psi_\alpha$, $\mathrm{tr}\,\psi_\alpha\psi_\beta$ are chiral primaries,

but $\mathrm{tr}\,\psi_\alpha\psi_\beta\psi_\gamma$ is not.

⇒ important in classification of chiral primaries in particular in AdS/CFT.

(Note: This is different from ψ's being Grassmannian)

ψ_α is the lowest comp of $W_\alpha = \psi_\alpha + \cdots$
"gluino chiral superfield".

$\{W_\alpha, W_\beta\} \simeq 0$ as chiral superfield.

This is one of the important assumptions in the proof that the superpotential of the glueball superfield $S = \mathrm{tr}\, W_\alpha W^\alpha$ receives corrections only from planar diagrams of the $N=1$ gauge theory and that they can be computed in the planar limit of the associated matrix model.

In our recent paper, we pointed out that this is modified if we turn on the $N=1$ supergravity background

$W_{\alpha\beta\gamma}$: totally sym gravitino superfield.

$$W_{\alpha\beta\gamma} = (\text{gravitino}) + \theta \cdot R^2 + \cdots$$

Defined as a torsion $[D_{\alpha\dot\alpha}, D_{\beta\dot\beta}] = \epsilon_{\dot\alpha\dot\beta} W_{\alpha\beta\gamma} D^\gamma + \cdots$

Bianchi identity

$$\Rightarrow \{W_\alpha, W_\beta\} = W_{\alpha\beta\gamma} W^\gamma$$ as chiral ~~superfield~~ superfield.

deforming the chiral ring relation.

This can be further deformed as

$$\{ W_\alpha , W_\beta \} = F_{\alpha\beta} + W_{\alpha\beta\gamma} W^\gamma \qquad (*)$$

In the string theory context.

type II string on $CY_3 \times \mathbb{R}^4$: $\mathcal{N}=2$ SUSY on $\mathbb{R}^4$

$\mathcal{N}=2$ SUGRA $\ni$ $U(1)$ gauge field "graviphoton"

Introduce D branes extended along $\mathbb{R}^4$

$\Rightarrow$ $\mathcal{N}=1$ SUSY

$F_{\mu\nu}$: field strength of the graviphoton

— parameter of $\mathcal{N}=1$ gauge theory

Lorentz violating

$$F_{\alpha\dot{\alpha}\beta\dot{\beta}} = \underbrace{\epsilon_{\dot{\alpha}\dot{\beta}} F_{\alpha\beta}}_{\text{self-dual part.}} + \epsilon_{\alpha\beta} F_{\dot{\alpha}\dot{\beta}}$$

If we postulate $(*)$,

non-planar diagrams contribute to the superpotential

of $S = \mathrm{tr}\, W_\alpha W_\beta \epsilon^{\alpha\beta}$

and ~~those~~ are related to the full partition function (all genus) of the associated matrix model.

Note : The relation (*) modifies the standard assumption that the gluino field W_α is in the adjoint representation of the gauge group, taking values in Grassmannian variables :

$$W_\alpha = W_\alpha{}^A \, T^A \qquad T^A : \text{Lie alg. generator}$$

$$\{ W_\alpha , W_\beta \} = W_\alpha{}^A \, W_\beta{}^B \, \tfrac{1}{2} f^{ABC} \, T^C$$

$$\text{if} \quad \{ W_\alpha{}^A , W_\beta{}^B \} = 0$$

With $F_{\alpha\beta} \neq 0$, this is clearly not compatible with (*).

[$f^{ABC} = 0$ for $U(1)_{\text{diag}}$]

We call (*) as the C-deformation of the gluino

C = Clifford, Chiral, or Classical

Motivation :

~~The matrix model~~
There are non-planar diagrams in the matrix model.
What do they mean in the context of the $N=1$ gauge theory in four dimensions ?

Topological string theory has already had an answer to this (BCOV '93).

We want to understand this answer in the language of the $\mathcal{N}=1$ gauge theory. ... (*).

Many supersymmetric gauge theories in 4d are realized as low energy limits of type II string on $CY_6 \times \mathbb{R}^4$ with/without branes, with/without fluxes.

Their superpotentials are computable by the method of topological string.

e.g.

$\mathrm{II}B$ on $\mathbb{R}^4 \times CY_3 \Rightarrow \mathcal{N}=2$ SUSY on $\mathbb{R}^4$

F-terms

$$\Gamma = \int d^4x \, d^2\theta \, d^2\bar\theta \sum_{g=0}^{\infty} (W_{\alpha\beta} W^{\alpha\beta})^g F_g(T)$$

$$W_{\alpha\beta} = F_{\alpha\beta} + W_{\alpha\beta\gamma}(\theta^\gamma - \bar\theta^\gamma) + \cdots$$

T : chiral superfield whose lowest comp is a complex moduli of CY_3 in the special coords.

$T \longleftrightarrow$ a 3 cycle in CY_3.

Turn on an N unit of the RR 3-form flux through the 3-cycle.

$$\left\{ \begin{array}{l} \bullet\ T \to T + N(\theta_\alpha - \bar{\theta}_\alpha)(\theta^\alpha - \bar{\theta}^\alpha) \\ \\ \bullet\ \text{Breaks } N=2 \to N=1. \end{array} \right.$$

Write Γ as $N=1$ superpotential (integrate out $(\theta - \bar{\theta})$)

$$\Gamma = \Gamma_1 + \Gamma_2 + \cdots$$

$$\Gamma_1 = \int d^4x\, d^2\theta \sum_{g=1}^{\infty} g\, W_{\alpha\beta\gamma} W^{\alpha\beta\gamma} (F_{\alpha\beta} F^{\alpha\beta})^{g-1} F_g(T)$$

$$\Gamma_2 = \int d^4x\, d^2\theta \sum_{g=1}^{\infty} (F_{\alpha\beta} F^{\alpha\beta})^g\, N_i \frac{\partial F_g}{\partial T_i}(T)$$

This is about $N=1$ theories which arises from IIB on CY_3 with 3 fluxes.

Another way to get $N=1$ theories is to put IIB on CY_3 with $D5$ branes wrapping on 2 cycles in CY_3.

In fact they are dual to each other in many cases.

In the context of topological string,

deformed conifold with complex structure T

$\Updownarrow$

resolved conifold with branes on S^2

with T as 't Hooft loop counting parameter.

open / closed top. string duality $\left\{ \begin{array}{l} \text{conjectured by Gopakumar + Vafa} \\ \text{proven by Ooguri + Vafa.} \end{array} \right.$

Embedded in the type II string context. Vafa

Γ_1, Γ_2 compute the F-terms of $N=1$ gauge theory

with $T_i = S_i = \underbrace{tr_i}_{} W_\alpha W^\alpha$

tr over the i-th gauge group.

This means that, if $F_{\alpha\beta} = 0$, $W_{\alpha\beta\gamma} = 0$,

$$\Gamma_1 = 0$$

$$\Gamma_2 = \int d^4x\, d^2\theta\; N_i \frac{\partial F_{g=0}}{\partial S_i}(S)$$ ← only planar diagrams contribute

If we turn on $W_{\alpha\beta\gamma}$, genus one diagrams contribute to Γ_1

giving rise to R^2 corrections.

If $F_{\alpha\beta} \neq 0$, all genus contribute.

If $F_{\alpha\beta} = 0$, $W_{\alpha\beta\gamma} = 0$,

why only planar diagram contribute to the superpotential?

Again it is ~~...~~ easiest to use the string theory realization of the gauge theory.

II B on $CY_3 \times \mathbb{R}^4$

Berkovits formalism

$$\mathcal{L} = \frac{1}{2} \partial X_\mu \bar{\partial} X^\mu + p_\alpha \bar{\partial} \theta^\alpha + p_{\dot{\alpha}} \bar{\partial} \theta^{\dot{\alpha}}$$
$$+ \bar{p}_\alpha \partial \bar{\theta}^\alpha + \bar{p}_{\dot{\alpha}} \partial \bar{\theta}^{\dot{\alpha}}$$

+ chiral boson

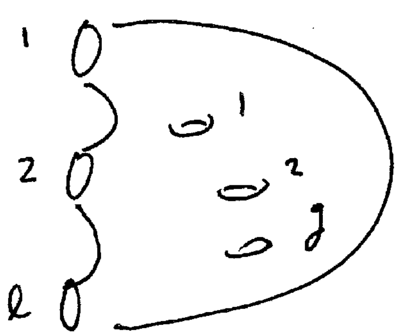
+ $N=2$ sigma model on CY_3, $\Rightarrow$ choice of gauge theory.

wrap D branes ~~on~~ ~~...~~

Boundary condition $p_\alpha = \bar{p}_\alpha$, $\theta_\alpha = \bar{\theta}_\alpha$.

$\uparrow$

(1.0) form



$l = 2g + h - 1$ zero modes

for p_α $\times 2$ $(\alpha = 1, 2)$

The gluino W_α couples to the string worldsheet

as $\displaystyle\sum_{i=1}^{h} \oint_{\gamma_i} W^\alpha p_\alpha$:

We assume W^α : const since we are computing its potential

If $F_{\alpha\beta}=0$, $W_{\alpha\beta\gamma}=0 \Rightarrow \{W_\alpha, W_\beta\}=0$

The ordering of W_α does not matter.

$\Downarrow$

We can use $\oint_{\gamma_i} p_\alpha$ to absorb the p_α zero modes.

But we have the constraint

$$\sum_{i=1}^{h} \oint_{\gamma_i} p_\alpha = 0$$

So there are only $(h-1)$ linearly indep p_α that couple to W_α.

Since there are $l = 2g+h-1$ zero modes.

we have to have $g=0$ to absorb all the zero mode

The proof by Dijkgraaf, Grisaru, Lam, Vafa and Zanon is a direct field theory limit of this statement.

This is modified if we turn on $F_{\alpha\beta}$ and $W_{\alpha\beta\gamma}$.

For example, if $\{W_\alpha, W_\beta\} = F_{\alpha\beta}$.

$$\mathcal{P}\exp\left(\oint W^\alpha p_\alpha\right) = \exp\left(F^{\alpha\beta}\underbrace{\oint \underset{d\tau}{p_\alpha(\tau)} \int^\tau p_\beta(\tau')\,d\tau'}_{} + \cdots\right)$$

We can take the field theory limit and showed that Γ_1 and Γ_2 can be obtained by standard Feynman diagram computation for the C-deformed gluino ··· read our papers!

This can absorb the extra $2g$ zero modes for $g \geq 1$.

How did we found out about the C-deformation?

(For $W_{\alpha\beta\gamma}W^\gamma$, we found out about the derivation based on the SUGRA tensor calculus after the fact.)

We started by looking at how $F_{\alpha\beta}$, $W_{\alpha\beta\gamma}$ couple to the string world sheet

$$F^{\alpha\beta}\iint p_\alpha \cdot \bar{p}_\beta$$

Similarly for $W_{\alpha\beta\gamma}$.

But we were puzzled since $p_\alpha = \bar{p}_\alpha$ in the field theory limit and this coupling vanishes in that limit!

So the topological string computation with the large N duality predict that

$F_{\alpha\beta}$ modifies the F terms of the gauge theory (by non-planar diagrams) even in the field theory limit

but the vertex operator for $F_{\alpha\beta}$ seemed to vanish!

Then we noticed that $Q_{\dot\alpha}$ symmetry is closely related to the top BRST symmetry on the string worldsheet.

In fact it is why the F terms (inv under $Q_{\dot\alpha}$) is computable using the topological string.

So we then tried to see if $F^{\alpha\beta} \iint p_\alpha \bar{p}_\beta$ preserves $Q_{\dot\alpha}$ symmetry.

From the FT point of view, it should since

$F^{\alpha\beta}$ is inv under $Q_{\dot\alpha}$ (so is W_α).

But, actually

$$[\,\epsilon^{\dot\alpha} Q_{\dot\alpha}\,,\ F^{\alpha\beta} \iint p_\alpha \bar{p}_\beta\,]$$

$$= \epsilon^{\dot\alpha} F^{\alpha\beta} \iint d\left[(X_{\alpha\dot\alpha} + \cdots)(p_\beta + \bar{p}_\beta)\right]$$

$$= \epsilon^{\dot\alpha} F^{\alpha\beta} \sum_{i=1}^{h} \oint_{\gamma_i} (X_{\alpha\dot\alpha} + \cdots)\, p_\beta$$

$$(p_\beta = \bar{p}_\beta \text{ on } \gamma_i)$$

This would have been zero
if there were no boundaries.

On the other hand

$$\{\,\epsilon^{\dot\alpha} Q_{\dot\alpha}\,,\ \oint W_\alpha{}^{\alpha}\, p_\alpha\,\}$$

$$= \epsilon^{\dot\alpha} \oint W^\alpha\, d\,(X_{\alpha\dot\alpha} + \cdots)$$

They cancel with each other if

$$\{W_\alpha, W_\beta\} = F_{\alpha\beta}.$$