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SMR.1498 - 2

SPRING SCHOOL ON SUPERSTRING THEORY AND RELATED TOPICS

31 March - 8 April 2003

MIRROR SYMMETRY AND N=1 SUPERSYMMETRY

Lectures 1 and 2

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Mirror Symmetry and N=1 Supersymmetry

Part 1 W.Lerche, Trieste Spring School 2003

- Exactly computable quantities are typically “BPS”: holomorphic objects protected by SUSY

N=2: prepotential $\mathcal{F}$ (+ infin sequence $\mathcal{F}_g$)

- Recent progress:

N=1: superpotential $\mathcal{W}$, gauge coupling τ
(+ infin sequence $\mathcal{F}_{g,h}$)

Non-pert. exact results for string and YM theories !
(matrix theory, Chern-Simons, mirror symmetry....)

Reminds of the well-known computation of F for N=2 SUSY:
“special geometry”, “TFT”, “geometric engineering”

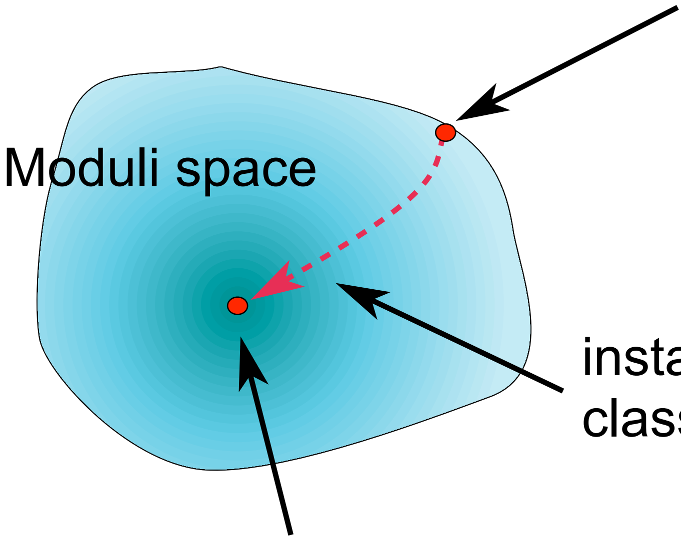
- We will show how to put the computation of N=1 superpotentials on an analogous footing:

N=1 Special Geometry

(main new ingredient: D-branes)

Motivation: Quantum Geometry of D-branes

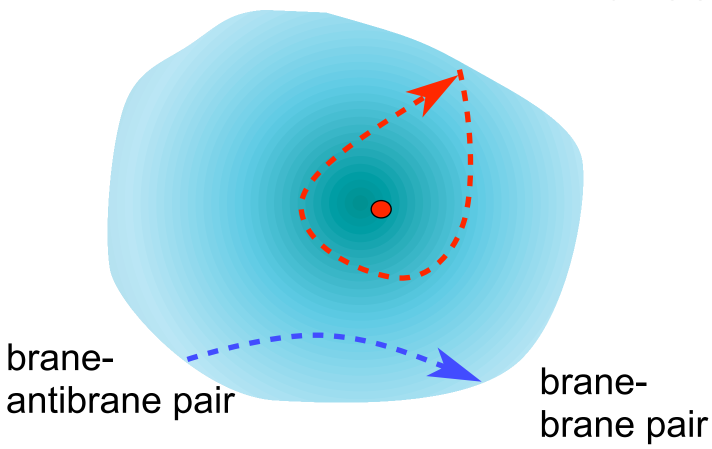
- Notions of classical geometry (eg., “branes wrapping p-cycles, with gauge bundles on top”) make sense only at large radius/weak coupling



instanton corrections wipe out classical geometry

“Gepner point”: rational CFT description

- Monodromy:
  brane configuration maps into “different” one
  involving “other” branes and fluxes



- Stability:
  brane configuration may become unstable

- To address such problems, we need to have full analytical control of F,W, over the full parameter space ....which is more than just a series expansion at weak coupling !

- Note however:

  N=2 SUSY: moduli space

  N=1 SUSY: W = obstruction to moduli space

Overview

- **Part 1**
  Recap: N=2 special geometry and mirror symmetry
  - Type II strings on Calabi-Yau manifolds
  - Mirror map
  - Topological field theory
  - Hodge variation and DEQ for period integrals

- **Part 2**
  Fluxes and D-branes on Calabi-Yau manifolds
  - Superpotentials from fluxes
  - Mirror symmetry and D-branes
  - Quantum D-geometry

- **Part 3**
  N=1 SUSY and open string mirror symmetry
  - Superpotentials from D-branes
  - Relative cohomology and mixed Hodge variations
  - Differential equations for exact superpotentials

Recap: Type II Strings on Calabi-Yau 3-folds

- For preserving N=2 SUSY in d=4, the compact 6dim manifold X should be Kahler and moreover, a

Calabi-Yau manifold $\begin{cases} c_1(R) = 0 \\ \text{Holonomy group SU(3)} \\ \text{global holom 3-form } \Omega^{(3,0)} \end{cases}$

metric $g_{i\bar{j}} = \partial_i \bar{\partial}_j K$ Kahler potential

Kahler (1,1) form $J^{(1,1)} = i g_{i\bar{j}} dz^i d\bar{z}^{\bar{j}}$

- The string compactification is described by a 2dim N=(2,2) superconformal sigma model on X with c=9, plus a free space-time sector

- The induced N=2 SUSY effective action in d=4 contains massless fields, including hyper- and vector supermultiplets

"decoupling":

Its bosonic sector gives a sigma model with target space

$$\mathcal{M} = \mathcal{M}_V \times \mathcal{M}_H$$

(special Kahler) (quaternionic)

- These massless scalar fields correspond to deformation parameters (moduli) of the CY, X.

  These are associated with (p,q) differential forms

$$\omega^{(p,q)} \equiv \omega_{i_1,\ldots,i_p,\bar{j}_1,\ldots,\bar{j}_q} dz^{i_1} \wedge \ldots dz^{i_p} \wedge d\bar{z}^{\bar{j}_1} \wedge \ldots d\bar{z}^{\bar{j}_q}$$

  which are closed but not exact, ie., are non-trivial elements of the cohomology groups

$$H^{p,q}_{\bar{\partial}}(X,C) \equiv \frac{\{\omega^{(p,q)} | \bar{\partial}\omega^{(p,q)} = 0\}}{\{\eta^{(p,q)} | \eta^{(p,q)} = \bar{\partial}\rho^{(p,q-1)}\}}$$

  These give zero modes of Laplacian: $\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^{\dagger} + \bar{\partial}^{\dagger}\bar{\partial}$
  (massless fields in 4d)

- There are two sorts of moduli:

  Kahler moduli (size parameters)

$$t_i \sim \omega_i^{(1,1)}, \qquad i = 1, \ldots, h^{1,1} \equiv dim H^{1,1}$$

  Complex structure moduli (shape parameters)

$$z_a \sim \omega_a^{(2,1)}, \qquad a = 1, \ldots, h^{2,1} \equiv dim H^{2,1}$$

  ($h^{p,q}$ = “Hodge numbers”)

- How do the moduli map to the fields in the effective Lagrangian ?

Mirror Symmetry of CY threefolds

- For "every" Calabi-Yau X, there exists a mirror $\widehat{X}$ such that the Kahler and complex structure sectors are exchanged:

$$H^{1,1}(X) \cong H^{2,1}(\widehat{X})$$
$$H^{2,1}(X) \cong H^{1,1}(\widehat{X})$$

i.e., $h^{p,q}(X) = h^{3-p,q}(\widehat{X})$

- The physical meaning is:

Type IIA strings compactified on X are indistinguishable from Type II strings compactified on the mirror of $\widehat{X}$

$$IIA\,/\,X \quad \longleftrightarrow \quad IIB\,/\,\widehat{X}$$

(-dilaton)

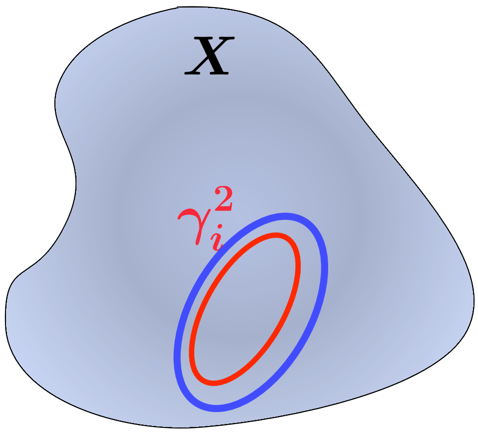
$$\mathcal{M}_H = \mathcal{M}_{CS}(X) = \mathcal{M}_{KS}(\widehat{X})$$
$$\mathcal{M}_V = \mathcal{M}_{KS}(X) = \mathcal{M}_{CS}(\widehat{X})$$

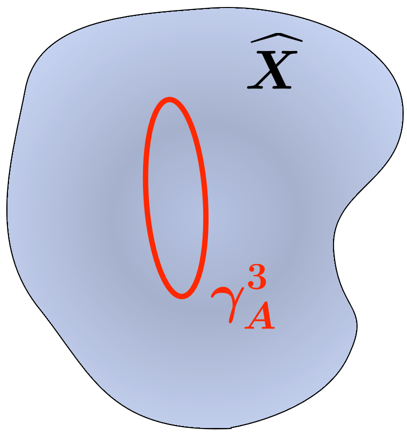
(We will consider here only the vector supermultiplet moduli space)

Why is mirror symmetry useful ?

... basic idea:

Type IIA $\longleftrightarrow$ Type IIB





size of 2-cycles γ_i^2 governed by Kahler param t_i

size of 3-cycles γ_A^3 governed by CS param z_a

Important quantities: quantum volumes ("periods") Π_A

$$\int_{\gamma^{2k}} (\wedge J^{(1,1)})^k + \ldots \quad = \quad \Pi_A \quad = \quad \int_{\gamma^3} \Omega^{(3,0)}$$

$$\sim \; t^k + \mathcal{O}(e^{-t}) \qquad\qquad \sim \; \ln(z)^k + \mathcal{O}(z)$$

world sheet instantons wrapping γ_i^2

no instantons can wrap !

"A-model": corrected

"B-model": exact !

- Significance:
  The periods are the building blocks of the prepotential.

Pick an integral basis of homology 3-cycles with

intersection metric $\Sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $Sp(2h^{2,1}+2, Z)$ structure

Thus one can split: $\{\gamma_A\} \;\rightarrow\; \{\gamma_a, \gamma_b\}$
and write:

$$\Pi_A(z) \;=\; \left(X_a,\, \mathcal{F}^b\right) \;\equiv\; \left(\int_{\gamma_a^3} \Omega^{(3,0)},\, \int_{\gamma_b^3} \Omega^{(3,0)}\right)(z)$$

In terms of these ''symplectic sections", one has for the prepotential:

$$\boxed{\mathcal{F}(z) \;=\; \frac{1}{2} X_a \mathcal{F}^a(z)}$$

What remains to do is to insert the mirror map:

$$t_i(z) \;=\; -\ln(z_a) + \ldots \;\rightarrow\; z_a = q_i(1+\mathcal{O}(q))$$

$$(q \equiv e^{-t})$$

which gives:

$$\boxed{\mathcal{F}(t) \;=\; \underbrace{\frac{1}{3!} c^0_{ijk} t^i t^j t^k}_{\text{classical}} + \sum_{n_1 \ldots n_r} N_{n_1 \ldots n_r} \underbrace{Li_3({q_1}^{n_1} \ldots {q_r}^{n_r})}_{\text{instanton corrections}}}$$

Integers counting maps
$P^1 \rightarrow X$

$$Li_s(q) \equiv \sum_k \frac{q^k}{k^s}$$

Special Geometry of the N=2 Vector-Moduli space

The prepotential F can be understood from three inter-related viewpoints:

A) **as 4d N=2 space-time effective Lagrangian of vector supermultiplets**

gauge couplings	$\tau_{ij}(t) = \partial_i \partial_j \mathcal{F}(t)$
"Yukawa" couplings	$c_{ijk}(t) = \partial_i \partial_j \partial_k \mathcal{F}(t)$
Kahler potential	$K(t,\bar{t}) = -\ln[\bar{X}_a \mathcal{F}^a - X_a \bar{\mathcal{F}}^a]$

B) **2d world-sheet topological field theory**

F = generating function of TFT correlators

$$c_{ijk}(t) \equiv \langle O_i O_j O_k \rangle \;=\; \partial_i \partial_j \partial_k \mathcal{F}(t)$$

OPE: $$\boxed{O_i \cdot O_j \;=\; \sum_k c_{ij}{}^k(t) O_k \qquad \text{"chiral ring" } \mathcal{R}}$$

chiral, primary chiral fields:

$$G^{+}_{-1/2} O_i|0\rangle_{NS} \;=\; G^{\pm}_{+1/2} O_i|0\rangle_{NS} \;=\; 0$$

From N=2 algebra follows:

$$\{G^{+}_{-1/2}, G^{-}_{1/2}\} O_i|0\rangle_{NS} \;=\; (2L_0 - J_0) O_i|0\rangle_{NS} \;=\; 0$$

Thus: $h(O_i) = 1/2|q(O_i)|$ (no pole in OPE)

- However: need consider pairing of left-, right-moving sectors .... (c,c) and (a,c) rings

Topological Sigma-Model on Calabi-Yau Manifold

$$S = \frac{1}{4\pi\alpha'}\int_\Sigma d^2z\,\big[1/2 g_{mn}\partial X^m \bar{\partial} X^n +$$
$$+ i\, g_{\bar{i}j}\lambda^{\bar{i}} D_z \lambda^j + i\, g_{\bar{i}j}\psi^{\bar{i}} D_{\bar{z}}\psi^j + R_{i\bar{i}j\bar{j}}\psi^i\psi^{\bar{i}}\lambda^j\lambda^{\bar{j}}$$

N=(2,2) supercharges:

$$Q_+ = \oint g_{\bar{i}j}\psi^{\bar{i}}\partial X^j \qquad Q_- = \oint g_{i\bar{j}}\psi^i \partial X^{\bar{j}}$$
$$\bar{Q}_+ = \oint g_{\bar{i}j}\lambda^{\bar{i}}\bar{\partial} X^j \qquad \bar{Q}_- = \oint g_{i\bar{j}}\lambda^i \bar{\partial} X^{\bar{j}}$$

Topological twist:
Redefine spins such that two of these supercharges become scalars to serve as BRST operator with

$$Q_{BRST}{}^2 = 0$$

This condition projects to a finite number of physical states in the TFT

- Idea: the physical spectrum corresponds to the non-trivial cohomology elements on X, via

$$Q_{BRST} \leftrightarrow d = \partial + \bar{\partial}$$

Ambiguity in choosing which supercharges correspond to $\partial,\ \bar{\partial}$!

There are 2 inequivalent possibilities:

$$\boxed{\begin{array}{ll} \text{“A-model”:} & Q_{BRST} = Q_+ + \bar{Q}_- \\ \text{“B-model”:} & Q_{BRST} = Q_+ + \bar{Q}_+ \end{array}}$$

A-Model: $Q_{BRST} = Q_+ + \bar{Q}_-$

Observables: $O_A^{(p,q)} = \omega^{(p,q)}_{i_1 \ldots i_p \bar{j}_1 \ldots \bar{j}_q} \lambda^{i_1} \ldots \lambda^{i_p} \psi^{\bar{j}_1} \ldots \psi^{\bar{j}_q}$

correspond to differential forms on X via:

$$\lambda^i \leftrightarrow dz^i, \quad \psi^{\bar{j}} \leftrightarrow d\bar{z}^{\bar{j}}$$

BRST non-trivial operators $O_A^{(p,q)}$ correspond to cohomology classes $H^{0,q}_{\bar{\partial}}(\wedge^p T^*) \cong H^{p,q}_{\bar{\partial}}(X)$

The Kahler moduli correspond to

$$O_A^{(1,1)} = \omega^{(1,1)}_{i\bar{j}} \lambda^i \psi^{\bar{j}} \in H^{1,1}$$

and generate the (c,c) chiral ring via the OPE:

$$\mathcal{R}^{(c,c)}: \quad O^{(1,1)}_{A,i} \cdot O^{(1,1)}_{A,j} = \sum_k c_{ij}{}^k \, O^{(2,2)}_{A,k}$$

The 3-point correlators look:

$$c_{ijk}(t) = \langle O^{(1,1)}_{A,i} O^{(1,1)}_{A,j} O^{(-2,-2)}_{A,k} \rangle$$

$$= \int_X \omega_i^{(1,1)} \wedge \omega_j^{(1,1)} \wedge \omega_k^{(1,1)}$$ classical "intersection"

$$+ \sum_{\{u\}} e^{-\int u^* J} \int u^* \omega_i^{(1,1)} \int u^* \omega_j^{(1,1)} \int u^* \omega_k^{(1,1)}$$

Instanton corrections

{u} = holomorphic rational maps $P^1 \longrightarrow X$

B-Model: $Q_{BRST} = Q_+ + \bar{Q}_+$

Observables: $O_B^{(p,q)} = \omega^{(p,q)}{}^{i_1 \dots i_p}_{\bar{j}_1 \dots \bar{j}_q} \lambda_{i_1} \dots \lambda_{i_p} \psi^{\bar{j}_1} \dots \psi^{\bar{j}_q}$

correspond to differential forms on X via:

$$\lambda_i \equiv g_{i\bar{j}} \lambda^{\bar{j}} \leftrightarrow d/dz^i, \quad \psi^{\bar{j}} \leftrightarrow d\bar{z}^{\bar{j}}$$

BRST non-trivial operators $O_B^{(p,q)}$ correspond to cohomology classes $H_{\bar{\partial}}^{0,q}(\wedge^p T) \cong H_{\bar{\partial}}^{-p,q}(X)$

[Note: a negative degree can be converted to a positive via contraction with the holom 3-form:]

$$\Omega^{(3,0)} : \omega^{(-p,q)} \longrightarrow \omega^{(3-p,q)}$$

The complex structure moduli correspond to

$$O_B^{(-1,1)} = \omega^{(-1,1)}{}^{i}_{\bar{j}} \lambda_i \psi^{\bar{j}} \in H^{-1,1} \cong H^{2,1}$$

and generate the (a,c) chiral ring via the OPE:

$$\mathcal{R}^{(a,c)} : \quad O_{B,a}^{(-1,1)} \cdot O_{B,b}^{(-1,1)} = \sum_c c_{ab}{}^c \, O_{B,c}^{(-2,2)}$$

The 3-point correlators look:

$$c_{abc}(z) = \langle O_{B,a}^{(-1,1)} O_{B,b}^{(-1,1)} O_{B,c}^{(2,-2)} \rangle$$

$$= \int_X (\Omega^{(3,0)} \omega_a^{(-1,1)} \wedge \omega_b^{(-1,1)} \wedge \omega_c^{(-1,1)}) \wedge \Omega^{(3,0)}$$

This is an exact, classical result !
(constant maps only)

Recap: Classical and quantum cohomology rings

- B-Model**:** (complex structure moduli)

(a,c) chiral ring $O_{B,a}^{(-1,1)} \cdot O_{B,b}^{(-1,1)} = \sum_c c_{ab}{}^c \, O_{B,c}^{(-2,2)}$

is isomorphic to the classical cohomology ring

$$H^{2,1}(X) \cup H^{2,1}(X) \rightarrow H^{1,2}(X)$$

- A-Model**:** (Kahler moduli)

(c,c) chiral ring $O_{A,i}^{(1,1)} \cdot O_{A,j}^{(1,1)} = \sum_k c_{ij}{}^k \, O_{A,k}^{(2,2)}$

is isomorphic to a **quantum deformation** of the cohomology ring

$$H^{1,1}(X) \cup H^{1,1}(X) \rightarrow H^{2,2}(X)$$

because of the instanton corrections

- Mirror symmetry:

A model on X is equivalent to the B-model on $\widehat{X}$

$$\mathcal{R}^{(c,c)}(X) \cong \mathcal{R}^{(a,c)}(\widehat{X}) \cong H^3_{\bar\partial}(\widehat{X})$$

quantum corrected

classical

$$c_{ijk}^{(A)}(t) = \sum \frac{\partial z_a}{\partial t_i}\frac{\partial z_b}{\partial t_j}\frac{\partial z_c}{\partial t_k}\, c_{abc}^{(B)}(z(t))$$

$$\Vert$$

$$\partial_i \partial_j \partial_k \mathcal{F}(t)$$

C) Viewpoint of variation of Hodge structures

Consider in B-model the variation of the holomorphic 3-form under deformations of the complex structure:

$$
\begin{aligned}
\Omega^{(3,0)}(z) &\in H^{(3,0)} \\
\delta_z\Omega^{(3,0)}(z) &\in H^{(3,0)} \oplus H^{(2,1)} \\
(\delta_z)^2\Omega^{(3,0)}(z) &\in H^{(3,0)} \oplus H^{(2,1)} \oplus H^{(1,2)} \\
(\delta_z)^3\Omega^{(3,0)}(z) &\in H^{(3,0)} \oplus H^{(2,1)} \oplus H^{(1,2)} \oplus H^{(0,3)}
\end{aligned}
$$

(notion of complex structure changes)

Sequence terminates when H^3 is exhausted, so higher derivatives are not independent

Fixing a basis of H^3, we can thus write a matrix DEQ:

(true modulo exact pieces)

$$
\nabla_a \varpi \equiv \Big[\partial_{z_a} - A_a(z)\Big] \cdot \varpi = 0 \qquad \varpi \equiv \begin{pmatrix} \Omega^{(3,0)} \\ \omega_a^{(2,1)} \\ \omega_a^{(1,2)} \\ \Omega^{(0,3)} \end{pmatrix}
$$

Recursive elimination of the higher components gives a set of higher order "**Picard-Fuchs**" operators" acting on integrals of the holom 3-form:

$$
\mathcal{L}_a \cdot \int_{\gamma_A^3} \Omega^{(3,0)} \equiv \mathcal{L}_a \Pi_A = 0
$$

The solutions are thus nothing but the periods we were looking for !

- Flatness of moduli space:

The matrix first oder operator can be decomposed:

$$\nabla_a \equiv \partial_{z_a} - A_a(z) = \partial_{z_a} - \Gamma_a - C_a$$

$$\Gamma_a = \begin{pmatrix} * & & & \\ * & * & & \\ * & * & * & \\ * & * & * & * \end{pmatrix} \qquad C_a = \begin{pmatrix} & 1 & & \\ & & (c_a)_{bc} & \\ & & & 1 \\ & & & \end{pmatrix}$$

"Gauss-Manin"-connection

chiral ring structure constants

One can show that $\left[\nabla_a, \nabla_b\right] = 0$

which means that there are "flat" coordinates, for which the connection vanishes, $\Gamma_a = 0$

These flat coordinates are precisely the Kahler parameter of the associated A-model, $t_i(z_a)$!

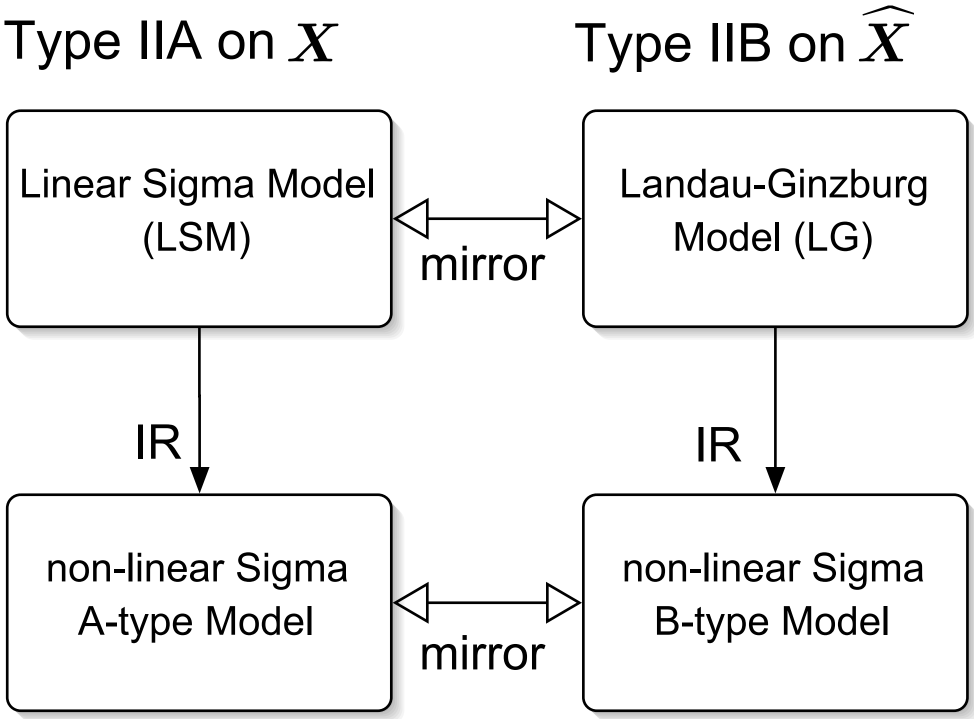
For these coordinates one has:

$$\begin{aligned} \Pi_A(z(t)) &= \left(X_0, X_i, \mathcal{F}^i, \mathcal{F}^0\right)(z(t)) \\ &= \left(1,\, t_i,\, \partial_i\mathcal{F},\, 2\mathcal{F} - t^j\partial_j\mathcal{F}\right) \\ &\sim \left(1,\, t,\, t^2 + \mathcal{O}(e^{-t}),\, t^3 + \mathcal{O}(e^{-t})\right) \end{aligned}$$

so indeed: $\mathcal{F}(t) = \dfrac{1}{2} X_a \mathcal{F}^a(z(t))$

Periods and DEQs for toric Calabi-Yau manifolds

Idea: describe 2d superconformal **non-linear** sigma-models as IR limits of a **linear** sigma model (A) or Landau-Ginzburg model (B)



- A-Model on X:

LSM = 2d U(1) gauge theory with fields ϕ_n, charges q_n^i

D-term potential: $V = D^2$,

$$D = \sum_n q_n^i |\phi_n|^2 - t_i = 0$$

Fayet-Iliopoulos parameters = Kahler moduli of X

$(i = 1, ..., h^{1,1}(X))$

The charge vectors q are the most basic data of "toric" Calabi-Yau's X: LSM formulation is canonical

B-Model on $\widehat{X}$:

Mirror geometry is described by IR limit of a 2d **Landau-Ginzbug** (LG) model, which is defined entirely in terms of the charge vectors q_n^i of the A-model !

LG superpotential: $$W_{LG} = \sum_n a_n y_n$$

with constraint: $$\prod_n y_n{}^{q_n^a} = 1$$

The $\{a_n\}$ parametrize the complex structure deformations of $\widehat{X}$ via

$$\prod_n a_n{}^{q_n^a} = z_a \qquad (a = 1, \ldots, h^{2,1}(\widehat{X}) \equiv h^{1,1}(X))$$

$$z_a \sim e^{-t_a} + \ldots \quad \text{(mirror map)}$$

- Note: $y_n \in \begin{cases} C & \text{if } \widehat{X} \text{ compact} \\ C^* & \text{if } \widehat{X} \text{non-compact } (y_n = e^{-\varphi_n}) \end{cases}$

We will consider only non-compact CY in the following

- holomorphic 3-form $\Omega^{(3,0)}(a(z)) = \prod_n \frac{dy_n}{y_n} e^{-W_{LG}(y,a)}$

satisfies Picard-Fuchs equation:

$$\boxed{\mathcal{L}_a \, \Omega^{(3,0)} \equiv \left[\prod_{n|q_n^a>0} \left(\frac{\partial}{\partial a_n}\right)^{q_n^a} - \prod_{n|q_n^a<0} \left(\frac{\partial}{\partial a_n}\right)^{q_n^a} \right] \Omega^{(3,0)} = 0}$$

All what remains to do is to change variables a -> z(a)

PF equations immediate once the defining toric data (charge vectors q) of the Calabi-Yau are given !

Example: normal bundle on P^2

- linear sigma model on P^2: $q_n^1 = (1,1,1)$

  linear sigma model on $O(-3)P^2$: $q_n^1 = (-3,1,1,1)$

  add extra non-compact coo to get CY $c_1 \sim \sum q_n = 0$

- B-model LG potential:

$$W_{LG} = a_0 y_0 + a_1 y_1 + a_2 y_2 + a_3 \frac{{y_0}^3}{y_1 y_2}$$

have used constraint $\frac{y_1 y_2 y_3}{{y_0}^3} = 1$

- PF operator: $\mathcal{L}_1 = \frac{\partial}{\partial a_1}\frac{\partial}{\partial a_2}\frac{\partial}{\partial a_2} - \left(\frac{\partial}{\partial a_0}\right)^3$

  rewriting in terms of $z = \frac{a_1 a_2 a_3}{{a_0}^3}$ gives:

$$\mathcal{L}_1(z) = \theta^3 + 3z\theta(1+3\theta)(2+3\theta)$$

...is of generalized hypergeometric type $(\theta \equiv z\partial/\partial z)$

- Solutions for the periods:

$$t(z) \sim \ln(z) + 3\sum(-)^n(3n-1)!(n!)^{-3}z^n$$

$$\partial_t F(z) \sim G^{3,1}_{3,3}(-z||1/3) + G^{3,1}_{3,3}(-z||2/3) \sim \ln(z)^2 + \ldots$$

invert t(z) and insert, integrate:

$$\mathcal{F}(t) = -1/18t^3 + \sum_n N_n Li_3(e^{-nt})$$

indeed integers... counting world-sheet instantons in P^2

Recap: N=2 Special Geometry and Mirror Symmetry

- Quantity of interest: N=2 prepotential of type II compactifications on CY threefolds

$$\mathcal{F}(t) \;=\; \frac{1}{2} X_a \mathcal{F}^a(z(t))$$

- Building blocks: periods

$$\Pi_A(z) \;\equiv\; \left(X_a, \mathcal{F}^b\right) \;=\; \int_{\gamma_A^3} \Omega^{(3,0)}(z)$$

in practice obtained as solution of PF diff eqs; these are obtained directly from the toric CY data

- (A-model)

$$\partial_i \partial_j \partial_k \mathcal{F}(t) \;=\; c_{ijk}(t) \;=$$

$$=\; c_{ijk}^{(0)} + \sum_{n_l} N_{n_i n_j n_k} n_i n_j n_k \frac{\prod_m {q_m}^{n_m}}{1 - \prod_m {q_m}^{n_m}}$$

(classical) (instanton corrections)

~ deformed chiral ring structure constants

$$\mathcal{R}^{(c,c)} : \;\; O_i \cdot O_j \;=\; \sum_k {c_{ij}}^k(t) O_k$$

- Mirror symmetry implies

$$\mathcal{R}^{(c,c)}(X) \;\cong\; \mathcal{R}^{(a,c)}(\widehat{X}) \;\cong\; H^3_{\bar\partial}(\widehat{X})$$

Recap: N=2 Special Geometry and Mirror Symmetry

- Quantity of interest: N=2 prepotential of type II compactifications on CY threefolds

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in practice obtained as solution of PF diff eqs; these are obtained directly from the toric CY data

- 3-point correlators:

$$\partial_i \partial_j \partial_k \mathcal{F}(t) \;=\; c_{ijk}(t) \;=$$

$$= \; c_{ijk}^{(0)} + \sum_{n_l} N_{n_i n_j n_k} n_i n_j n_k \frac{\prod_m {q_m}^{n_m}}{1 - \prod_m {q_m}^{n_m}}$$

instanton corrections

~ deformed chiral ring structure constants

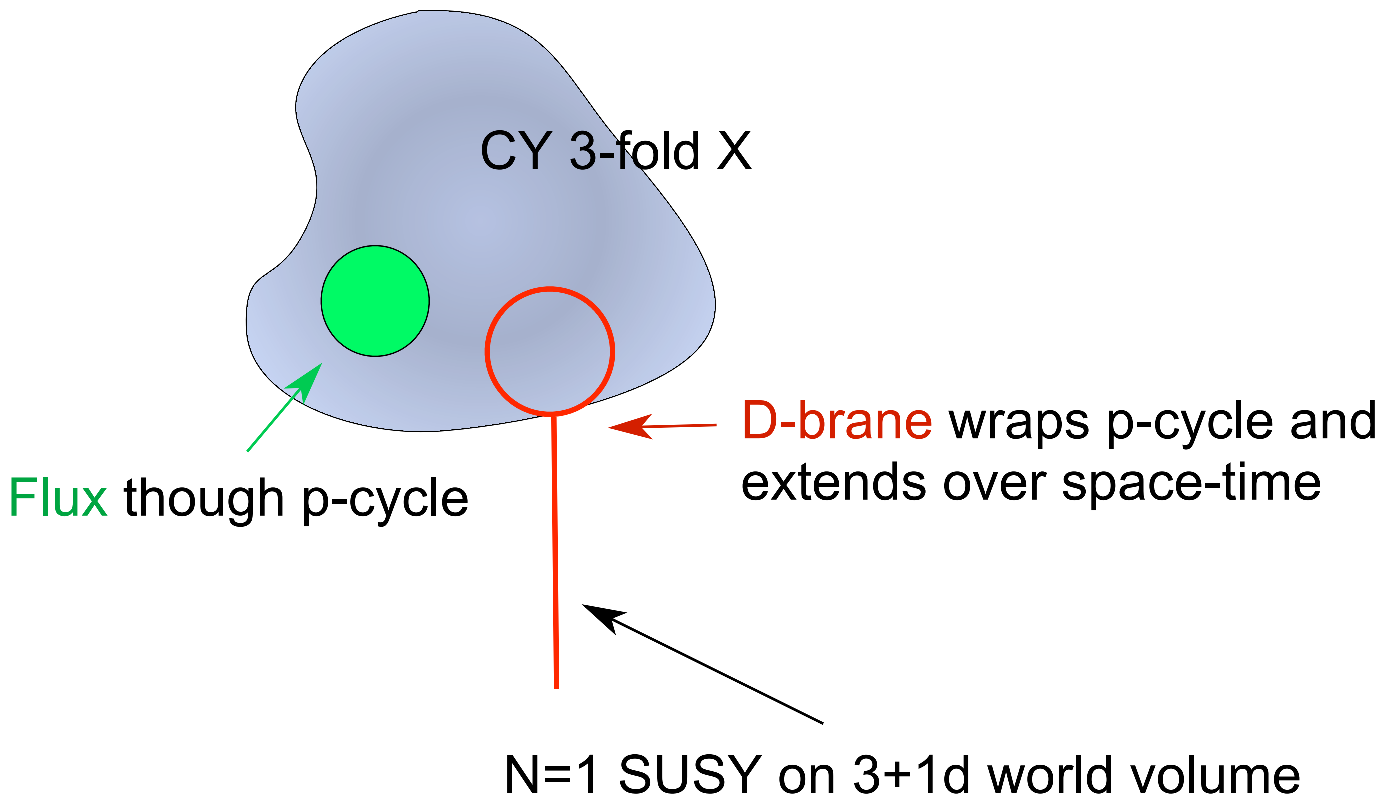
$$\mathcal{R} : \quad O_i \cdot O_j \;=\; \sum_k {c_{ij}}^k(t) O_k$$

- Mirror symmetry implies

$$\mathcal{R}^{(c,c)}(X) \;\cong\; \mathcal{R}^{(a,c)}(\widehat{X}) \;\cong\; H^3_{\bar\partial}(\widehat{X})$$

Fluxes and D-branes on Calabi-Yau manifolds

- Physical motivation:
  reduce SUSY from N=2 to N=1



What are the effective superpotential W,
and the effective gauge couplings ?

- New feature: open string instantons

Turning on fluxes

- The 10d Type II strings have various massless antisymmetric, (p-1)-form tensor fields $C^{(p-1)}$, coupling to (p-2)-branes.

  Field strengths: $H^{(p)} = dC^{(p-1)}$

	$H^{(p)}_{NSNS}$	$H^{(p)}_{RR}$
Type IIA: p=	3,7	2,4,6,8
Type IIB: p=	1,3,7	1,3,5,7,9

- In a CY compactification, various H's can be “turned on”, ie, the H-flux through a p-cycle is non-zero:

$$\int_{\gamma^p} H^{(p)} \neq 0$$

  We will mainly consider only (quantized) RR-fluxes, corresponding to D-branes

- 10d action: non-vanishing flux will typically induce non-zero potentials and SUSY breaking

$$S \sim \int H^{(p)} \wedge {}^{*}H^{(p)}$$

Type IIB string on three-fold $\widehat{X}$ with 3-form flux

- It can be shown that upon turning on $H^{(3)}$ flux, N=2 SUSY is broken to N=1 SUSY with superpotential:

$$\mathcal{W}_{IIB/\widehat{X}} = \int_{\widehat{X}} \Omega^{(3,0)} \wedge \tilde{H}^{(3)}$$

$$\tilde{H}^{(3)} \equiv \tau\, H^{(3)}_{NSNS} + H^{(3)}_{RR}$$

Type IIB coupling: $\tau \equiv C^{(0)} + i\, e^{-\varphi}$

set in the following $H^{(3)}_{NSNS} \to 0$

- Denote 3-cycle dual to flux $H^{(3)}$ by Γ^3 and expand in integral symplectic basis of 3-cycles:

$$\Gamma^3 = N^a \gamma_a^3 + N^b \gamma_b^3 \qquad N^a \in Z$$

Then

$$\mathcal{W}_{IIB/\widehat{X}}(z) = \int_{\Gamma^3} \Omega^{(3,0)}(z)$$

$$= N^a X_a + N_b \mathcal{F}^b \equiv N^A \Pi_A(z)$$

where $\Pi_A = (X_a, \mathcal{F}^b)$ are nothing but the period integrals !

Type IIA string on three-fold X with fluxes

- Rule: replace period by volume integrals
  ... will be corrected by world-sheet instantons

$$\mathcal{W}_{IIA/X}(t) = \int_X \sum_{k=1}^{3} H_{RR}^{(2k)} \left(\wedge J^{(1,1)} \right)^{3-k} + \ldots$$
$$= N^{(6)} + N^{(4)} t + N^{(2)} t^2 + N^{(0)} t^3 + \mathcal{O}(e^{-t})$$

flux numbers

- A priori, it would be hard to compute the instanton corrections, but mirror symmetry predicts

$$\boxed{\mathcal{W}_{IIA/X}(t) = \mathcal{W}_{IIB/\widehat{X}}(z) = \sum N^A \Pi_A(z(t))}$$

$$\Pi_A(z(t)) = (X_a, \mathcal{F}^b) = \left(1, t_i, \partial_i \mathcal{F}, 2\mathcal{F} - t_i \partial_i \mathcal{F}\right)$$

- Thus, the superpotential is completely determined by the "bulk" geometry: spont. broken N=2 SUSY

  Note that flux appears as auxiliary field in N=2 eff action

$$\Phi = t + \theta^2 H^{(2)} + \ldots$$

  Thus, if $\langle H^{(2)} \rangle = N^{(2)} \neq 0$

  then $\int d^4\theta \mathcal{F}(\Phi) \rightarrow \int d^2\theta N^{(2)} \frac{\partial}{\partial \Phi} \mathcal{F}(\Phi) \equiv \mathcal{W}$

as above !

A first glimpse of Quantum Geometry: monodromy

Periods $\Pi_A = (X_a, \mathcal{F}^b)$: sections valued in $Sp(2h^{2,1}+2, Z)$

Non-trivial loops in the moduli space $\mathcal{M}_{CS}(\widehat{X})$ will thus induce monodromy

$$\Pi_A \;\longrightarrow\; \Pi_A \cdot R, \quad R \in Sp(2h^{2,1}+2, Z)$$

- Consider eg looping around $z \sim e^{2\pi i t} \longrightarrow 0$ in the semi-classical, large volume regime:

$$t \sim \tfrac{1}{2\pi i} \ln z \;\longrightarrow\; t+1$$

Thus

$$\begin{aligned} Z &= N^{(6)} + N^{(4)} t + \ldots \\ &\longrightarrow (N^{(6)} + N^{(4)} + \ldots) + (N^{(4)} + \ldots) t + \ldots \end{aligned}$$

- Looping generic (non-perturbative) singularities will typically mix all fluxes which each other:

$$N^A \;\longrightarrow\; R \cdot N^A$$

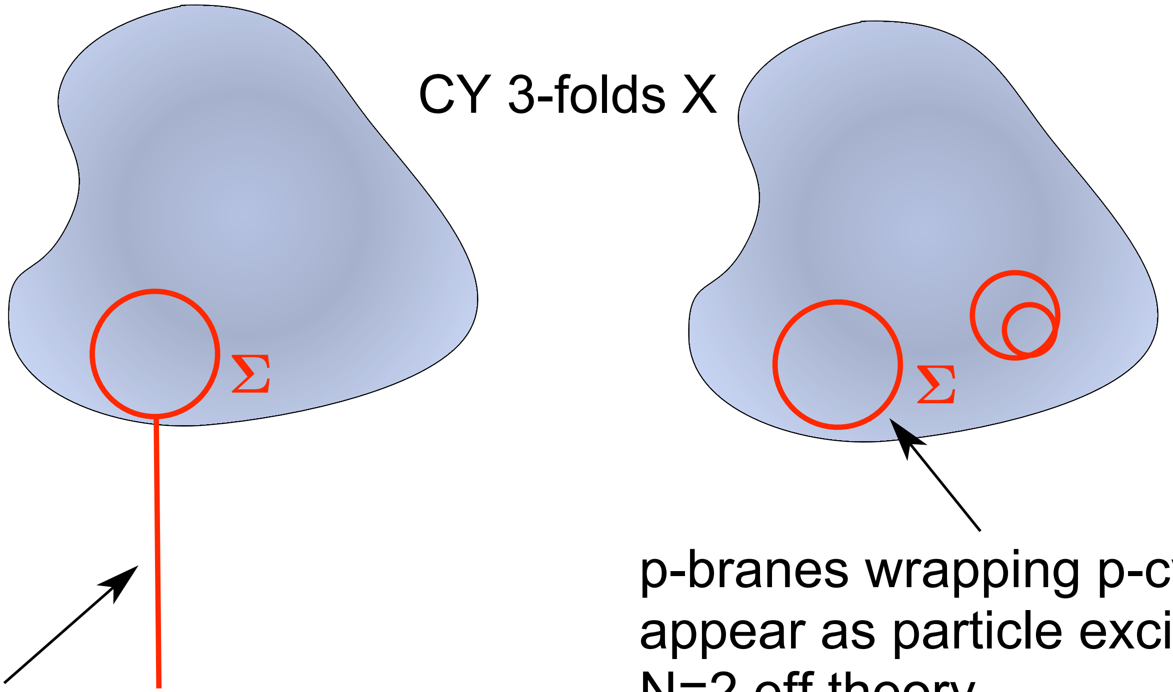
Since

$$N^A = \int_{\gamma^{p_A}} H^{(p_A)}$$

the dimensions of p-cycles loose their invariant meaning !

D-Branes on Calabi-Yau manifolds

Various manifestations:



- The eff space-time physics depends on the properties of the wrapped internal part of the brane

- We are interested in BPS configurations that break 1/2 of the SUSY (N=2 -> N=1)

  Condition for “SUSY p-cycles”:
  covariantly constant spinor η, with $(1-\Gamma)\eta = 0$

$$\Gamma \equiv \frac{1}{\sqrt{h}} \epsilon^{\alpha_1 \dots \alpha_{p+1}} \partial_{\alpha_1} X^{m_1} \dots \partial_{\alpha_{p+1}} X^{m_{p+1}} \Gamma_{m_1 \dots m_{p+1}}$$

induced metric — pull-back to world-vol — 10d Gamma matrices

- Two classes of solutions:

“A-type” branes: wrap special lagrangian cycles $\Sigma_A^{(p=3)}$
“B-type” branes: wrap holomorphic cycles $\Sigma_B^{(p=0,2,4,6)}$

A-type branes

- Wrap “special lagrangian” cycles Σ_A

$$dim(\Sigma_A) \;=\; 1/2 dim(X) \;=\; 3$$

- $f^* J^{(1,1)} \;=\; 0$

  Pull-back of Kahler form vanishes; $f : \Sigma_A \to X$

- $f^*(Im\, e^{i\theta}\Omega^{(3,0)}) \;=\; 0$

  Pull-back of holom 3-form vanishes

- $F = 0$

  U(1) gauge field on world-volume must be flat

- What are the moduli of the brane ?
  A priori:

$$dim_R(\mathcal{M}_{\Sigma_A}) \;=\; b_1(\Sigma_A)$$

which can be odd ...
but we need complex fields for SUSY reasons

→ Pair up with “Wilson line” moduli of the flat U(1) gauge connection to get complexified moduli fields:

$$\hat{t}_i, \quad i = 1, ..., dim_C(\mathcal{M}_{\Sigma_A}, WL) \;=\; b_1(\Sigma_A)$$

B-type D-branes

- Wrap holomorphic submanifolds: $\Sigma_B^{(p)}$, p=0,2,4,6

- Apart from the holomorphic embedding geometry, $f : \Sigma_B \longrightarrow X$, there is more structure: the gauge field configuration, "U(N) bundle V" (if N branes coincide)

  Eg for D6 branes (wrapping all of X), SUSY requires that the gauge bundle V is holomorphic:

$$F_{i\bar{j}} = 0$$

  (NB: further "stability" requirements)

- Important correspondence:

Gauge field configuration V	$\Longleftrightarrow$	brane bound states

...due to anomalous world-volume couplings:

$$S_{WZ} = \int_{\Sigma_B^{(p)} \times R} C \wedge Tr[e^F] \wedge \sqrt{\hat{A}(R)}\Big|_{p+1 form}$$

RR tensor fields

Dirac genus

$$C \equiv \bigoplus_k C^{(k)} \begin{cases} \text{Type IIA: k=odd} \\ \text{Type IIB: k=even} \end{cases}$$

$$\hat{A}(R) = 1 + 1/24 R^2 + \ldots$$

Chern character of V

- Example: D4-brane

$$S_{WZ} = \int_{\Sigma_B^{(4)} \times R} \tfrac{1}{2} C^{(1)} \wedge F \wedge F + \ldots$$

so if there is an instanton configuration V such that $\frac{1}{2}\int F \wedge F = n$
then there is an induced coupling

$$n \int C^{(1)}$$ = source term for n D0-branes !

- More generally:

  n gauge instantons on p-brane

  $\Longleftrightarrow$ bound state of the p- with n (p-4)-D-branes

- Even more generally:

  A brane configuration of r D6 branes on CY X
  is characterized by the "generalized Mukai" charge
  vector Q:

$$Q = Tr[e^F] \wedge \sqrt{\hat{A}(R)}$$

$$= \left(Tr1,\, TrF,\, \tfrac{1}{2}(TrF)^2 - TrF^2 + \tfrac{1}{24}TrR^2, \ldots\right)$$

Thus $\int_X Q =$

$$= \left(r(V), c_1(V), ch_2(V) + \tfrac{r}{24}c_2(T_{\Sigma_B}), ch_3(V) + \tfrac{r}{24}c_1(V)c_2(T_{\Sigma_B})\right)$$

$$= \left(M^{(6)},\, M^{(4)},\, M^{(2)},\, M^{(0)} \right)$$ D-brane RR charges

This gives direct translation between gauge bundle data
(Chern classes of V) and D-brane charge content

Mirror symmetry and D-branes

- Recap mirror map:

$$\mathit{Type\ IIA}/X \quad \longleftrightarrow \quad \mathit{Type\ IIB}/\widehat{X}$$

RR fields:

$$\{C^{(1)}, C^{(3)}, C^{(5)}, \ldots\} \longleftrightarrow \{C^{(0)}, C^{(2)}, C^{(4)}, \ldots\}$$

Dp(=even) branes Dp(=odd) branes

Equivalence of non-perturbative theories implies equivalence of

B-branes wrapped over holom. (0,2,4,6) cycles of X	$\Leftrightarrow$	A-branes wrapped over special lagrangian 3-cycles of $\widehat{X}$

- This is reflected in the 2d string world-sheet boundary conditions of the N=(2,2) superconformal currents:

B-type branes

$$\begin{aligned} J_L &= J_R \\ G_L^{\pm} &= \pm G_R^{\pm} \\ T_L &= T_R \end{aligned}$$

A-type branes

$$\begin{aligned} J_L &= -J_R \\ G_L^{\pm} &= \mp G_R^{\pm} \\ T_L &= T_R \end{aligned}$$

Mirror symmetry just switches $J_R \leftrightarrow -J_R$!

Tension of wrapped D-branes

(particles in 4d N=2 SUSY)

- Recall BPS mass formula: $m_{BPS} = |Z|$

  Central charge Z in N=2 SUSY algebra

  $$\{Q^+, Q^-\} = p\cdot\gamma + Z$$

  essentially given by volume of wrapped cycle

- Recall factorization of CY moduli space:

  $$\mathcal{M}_X = \mathcal{M}_{KS}(t) \times \mathcal{M}_{CS}(z)$$

  ~ ...even ...odd cycles

  The mass of wrapped B-branes depends only on the Kahler moduli t, while the mass of the A-branes depends only on the complex structure moduli z.

  - A-branes in Type IIB:

  $$\mathrm{Z}_{A/IIB}(z) = M^A \int_{\gamma_A^3} \Omega^{(3,0)}(z) = M^A \Pi_A(z)$$

  - B-branes in Type IIA:

  $$\mathrm{Z}_{B/IIA}(t) = \int_X e^{J^{(1,1)}} \wedge Q + \mathcal{O}(e^{-t}) \quad \text{(instanton corr)}$$

  $$= \mathrm{Q}_0 + \int J^{(1,1)} \wedge Q_2 + \tfrac{1}{2} \int J^{(1,1)} \wedge J^{(1,1)} \wedge Q_4 +$$

  $$= M^{(0)} + M^{(2)} t + M^{(4)} t^2 + M^{(6)} t^3 + \mathcal{O}(e^{-t})$$

- Mirror symmetry: $\boxed{Z_{B/IIA}(t) = Z_{B/IIA}(z(t))}$

$$= M^{(0)} + M^{(2)} t + M^{(4)} \partial_t F(t) + M^{(6)} (2\mathcal{F} - t\partial_t \mathcal{F})(t)$$

Quantum Volume

- Non-trivial identification:

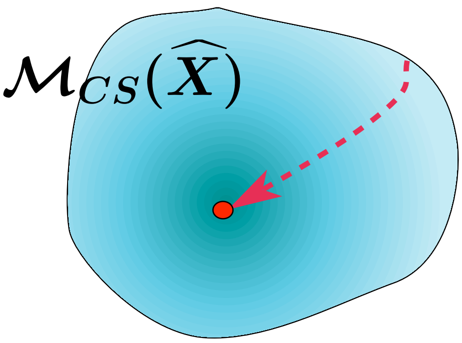
$$\mathrm{M}^{A} \int_{\gamma_A^3} \Omega^{(3,0)}(z) = M^{(0)} + M^{(2)} t + M^{(4)} \partial_t F(t) + M^{(6)} \mathcal{F}_0(t)$$

3-cycles on X on equal footing $\Longrightarrow$ 0,2,4,6-cycles on $\widehat{X}$ on equal footing too !

- Massless state in 4d:

$Z = 0 : \quad \Pi_A \longrightarrow 0$ for some A

Example:
conifold singularity (strong coupling region)



Type IIB: 3-cycle $\gamma_A^3 \longrightarrow 0$ $\Longrightarrow$ Type IIA: $\mathcal{F}_0(t) \longrightarrow 0$
6-cycle quantum volume (whole CY) X shrinks to nothing!

However, the "embedded" 0,2,4 cycles do not have vanishing quantum volume:

$(1, t, \partial_t F(t)) \not\to 0$

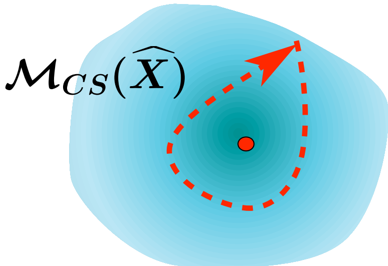
The classical geometric picture is swamped out by instanton corrections

Monodromy of RR charges

- Recall that when encircling singularities in $\mathcal{M}_{CS}(\widehat{X})$, monodromies will be induced on the periods:

$$\Pi_A \;\longrightarrow\; \Pi_A \cdot R, \quad R \in Sp(2h^{2,1}+2, Z)$$

Thus, just as before the flux numbers N^A, now the D-brane charges M^A will get mixed.



- Eg., encircling $z \sim e^{2\pi i t} \longrightarrow 0$ in $\mathcal{M}_{CS}(\widehat{X})$ induces t -> t+1, and

$$\begin{aligned} Z \;&=\; M^{(0)} + M^{(2)} t + ... \\ &\longrightarrow (M^{(0)} + M^{(2)} + ...) + (M^{(2)} + ...) t + ... \end{aligned}$$

ie., the D0 brane number jumps

roughly: "tensoring V by a line bundle": $Z \sim \int e^{J^{(1,1)}} \wedge e^F$

Again we see that the notion of p-cycles, and gauge bundle configurations V on top of them, has no good meaning away from the semi-classical large radius limit !

Central charge and domain walls

- We have seen that in type IIB compactifications, 3-fluxes $H^{(3)}$ induce an N=1 superpotential:

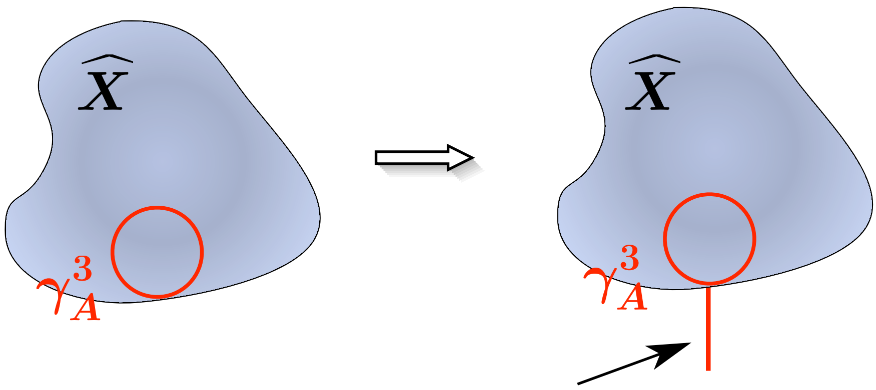
$$\mathcal{W}_{N=1}(z) \;=\; N^A \Pi_A(z)$$

However the same expression gave the central charge of a wrapped D3 A-type brane:

$$Z(z) \;=\; M^A \Pi_A(z)$$

What is the significance ?

- Replace fully wrapped D3 brane by a D5 brane:



Domain wall in 3+1d

Central charge of a DW is known to be $Z = \Delta\mathcal{W}_{N=1}$

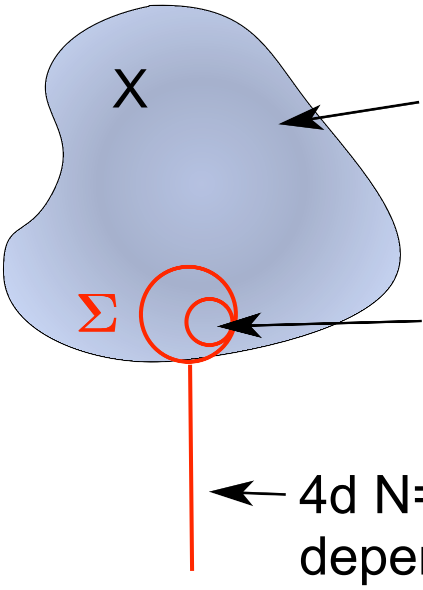
However, the D5 brane tension is still

$$Z = M^A \int_{\gamma_A^3} \Omega^{(3,0)} = M^A \Pi_A$$

and it generates M^A units of $H^{(3)}$ flux across the domain wall

Moduli of D-brane configurations

- Consider 1/2 BPS configurations breaking to N=1 SUSY:



CY moduli t, z

brane (and bundle) moduli $\hat{t}, \hat{z}$

4d N=1 SUSY eff action $S_{eff}(t, \hat{t}, z, \hat{z})$
depends how on moduli ?

- Focus on

  complex structure moduli:

  $z \sim \gamma_A^3$ sizes of 3-cycles

  $\hat{z} \sim \hat{\gamma}_N^3$ sizes of 3-chains

  Kahler moduli:

  $t \sim \gamma_i^2$ sizes of P^1's

  $\hat{t} \sim \hat{\gamma}_n^2$ sizes of disks ending on D-brane

- Decoupling theorems (from CFT):

B-branes $\begin{cases} W(z, \hat{z}),\ \tau(z, \hat{z}) & \text{holom. potentials} \\ D(t, t^*, \hat{t}, \hat{t}^*) & \text{FI D-term potential} \end{cases}$

A-branes $\begin{cases} W(t, \hat{t}),\ \tau(t, \hat{t}) & \text{holom. potentials} \\ D(z, z^*, \hat{z}, \hat{z}^*) & \text{FI D-term potential} \end{cases}$

Preview

- Next time: use mirror symmetry

$$W_{A/IIA}(t,\hat{t}) \;=\; W_{B/IIB}(z(t),\hat{z}(t,\hat{t}))$$

and set up math framework for systematically computing superpotentials for a large class of D-brane geometries