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SPRING SCHOOL ON SUPERSTRING THEORY AND RELATED TOPICS

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STRING INTERACTIONS IN PLANE WAVES

Lecture 3

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3 Light-Cone String Field Theory

Comments on the Neumann Coefficients

In the last lecture we wrote the cubic string vertex as a squeezed state in the three-string Fock space:

$$|V\rangle = \delta(p_1^+ + p_2^+ + p_3^+) f(p_1^+, p_2^+, p_3^+, \mu)(\det \Gamma)^{(2-D)/2} \exp\left[V(a_{(1)}^\dagger, a_{(2)}^\dagger, a_{(3)}^\dagger)\right] |0_{(1)}\rangle|0_{(2)}\rangle|0_{(3)}\rangle. \tag{1}$$

Here $f(p_1^+, p_2^+, p_3^+, \mu)$ is a measure factor which we have not yet determined and D is the dimensionality of space time. This enters the formula because one gets one factor of $(\det \Gamma)^{-1/2}$ for each dimension transverse to the light cone. Finally we have made the convenient definition

$$V(a_1, a_2, a_3) = \frac{1}{2} \sum_{r,s=1}^{3} \sum_{m,n=-\infty}^{\infty} \overline{N}_{mn}^{(rs)} a_{m(r)} a_{n(s)}, \tag{2}$$

where

$$\overline{N}_{mn}^{(rs)} = \delta^{rs}\delta_{mn} - 2\sqrt{\omega_{m(r)}\omega_{n(s)}}(X^{(r)\mathrm{T}}\Gamma^{-1}X^{(s)})_{mn}, \qquad \omega_{m(r)} = \sqrt{m^2 + (\alpha' p_{(r)}^+ \mu)^2}, \tag{3}$$

$$X_{mn}^{(1)} = \frac{1}{\pi}(-1)^{m+n+1}\frac{\sin(\pi m y)}{n - my}, \qquad X_{mn}^{(2)} = \frac{1}{\pi}(-1)^n \frac{\sin \pi m(1-y)}{n - m(1-y)}, \qquad X_{mn}^{(3)} = \delta_{mn}, \tag{4}$$

$$\Gamma_{mn} = \sum_{r=1}^{3} \sum_{p=-\infty}^{\infty} \omega_{p(r)} X_{mp}^{(r)} X_{np}^{(r)}. \tag{5}$$

The matrix element $\overline{N}_{mn}^{(rs)}$ expresses the coupling between mode m on string r and mode n on string s. These coefficients are called Neumann coefficients, for reasons that will become clear later in this lecture. Although the X matrices are independent of μ, the matrix Γ depends on μ (and the three p_+'s) in a highly nontrivial way. In the $\mu \to 0$ limit, it is rather easy to show that these Neumann coefficients reduce correctly to the flat space case, where explicit formulas are known for $\overline{N}_{mn}^{(rs)}$.

A huge technology has been developed towards obtaining explicit formulas for $\overline{N}_{mn}^{(rs)}$ as a function of μ and p_+^r. This material is too technical to present in detail, so I will just summarize the current state of the art. Recall that the dual BMN gauge theory is believed to be effectively perturbative in the parameter

$$\lambda' = \frac{1}{(\mu p^+ \alpha')^2}. \tag{6}$$

So, in order to make contact with perturbative gauge theory calculations, we are particularly interested in studying string interactions in the large μ limit. In this limit it can be shown that

$$\overline{N}_{mn}^{(13)} = \frac{1}{2\pi} \frac{(-1)^{m+n+1}\sin(\pi n y)}{p_3^+ \omega_{m(1)} + p_1^+ \omega_{n(3)}} \sqrt{\frac{y}{\omega_{m(1)}\omega_{n(3)}}}$$

$$\times \left[\sqrt{(\omega_{m(1)} + \mu p_1^+ \alpha')(\omega_{n(3)} + \mu p_3^+ \alpha')} + \sqrt{(\omega_{m(1)} - \mu p_1^+ \alpha')(\omega_{n(3)} - \mu p_3^+ \alpha')}\right]$$
$$+\mathcal{O}(e^{-2\pi\mu}, e^{-2\pi\mu y}, e^{-2\pi\mu(1-y)}). \tag{7}$$

The first term encodes all orders in a power series expansion in λ'. Specifically,

$$\overline{N}^{(13)}_{mn} = \left[\frac{(-1)^{m+n+1}}{\pi\sqrt{y}} \frac{\sin(\pi n y)}{n - m/y} + \mathcal{O}(\lambda') + \cdots\right] + \text{non} - \text{perturbative}. \tag{8}$$

It is intriguing that the nonperturbative corrections look like D-branes rather than instantons (i.e. they are $\mathcal{O}(e^{-1/g})$ rather than $\mathcal{O}(e^{-1/g^2})$).

We have only written the Neumann coefficient $\overline{N}^{(13)}$, but in fact it is easily shown that in the $\mu \to \infty$ limit,

$$\overline{N}^{(13)}_{mn} = \sqrt{y} X^{(1)\mathrm{T}}_{mn}, \qquad \overline{N}^{(23)}_{mn} = \sqrt{1-y} X^{(2)\mathrm{T}}_{mn}, \tag{9}$$

while all other components are zero. This fact actually has a very nice interpretation in the BMN gauge theory, so let me now present the following chart to explain how to think about the correspondence:

$$\begin{array}{ccc}
\text{`three-point' functions at } \lambda' = 0 & \longleftrightarrow & \text{matrix elements of } |V\rangle \text{ at } \mu = \infty \\
\nwarrow\searrow & & \nearrow\swarrow \\
& \text{matrix elements of } \Sigma, \text{ splitting-joining operator} &
\end{array}$$

I will not propose to make sense of this identification when $\lambda' \neq 0$. See next lecture.

The Consequence of Lorentz Invariance

Our vertex (1) still has an arbitrary function f of the light-cone momenta and μ, and a factor $(\det\Gamma)^{(2-D)/2}$, which is also a terribly complicated function of the light-cone momenta and μ. In flat space ($\mu = 0$), it was shown long ago that Lorentz-invariance of the vertex, and in particular, the covariance of S-matrix elements under J^{+-} Lorentz transformations, requires $D = 26$ and $f = (\det\Gamma)^{12}$.

This fact is nice for the oscillator representation since these factors then cancel and (1) can simply be written as

$$|V\rangle = \delta(p_1^+ + p_2^+ + p_3^+) \exp\left[V(a^\dagger_{(1)}, a^\dagger_{(2)}, a^\dagger_{(3)})\right] |0_{(1)}\rangle|0_{(2)}\rangle|0_{(3)}\rangle, \qquad \text{for } \mu = 0, \tag{10}$$

with no additional factors (except perhaps some innocent overall factors like 2π's which we have not carefully kept track of).

However, in the functional representation this fact is quite mysterious! It means that the correct, Lorentz-invariant string vertex in flat space,

$$H_3 = g_2 \int \delta(p_1^+ + p_2^+ + p_3^+)(\det \Gamma)^{12} \Delta[x_1(\sigma) + x_2(\sigma) - x_3(\sigma)] \prod_{r=1}^{3} \left(dp_r^+ Dx_r(\sigma) \; \Phi[p_r^+, x_r(\sigma)] \right), \tag{11}$$

has a very peculiar measure factor $(\det \Gamma)^{12}$ which would have been impossible to guess purely from the functional approach.

The plane wave background with $\mu > 0$ has fewer Lorentz symmetries than flat space. In particular, it does not have the J^{+-} or J^{-I} symmetries. This means that it is impossible to use Mandelstam's method to determine what the corresponding measure factor is when $\mu > 0$. Our vertex for string interactions in the plane-wave background remains ambiguous up to an overall (probably very complicated) function of p_1^+, p_2^+, p_3^+ and μ.

We determined the form of the vertex by requiring continuity of the string worldsheet, but evidently that is not enough to solve our problem. In the rest of this lecture we will learn why light-cone string field theory works, and what the physics is that does completely determine the light-cone vertex. To be precise: I should say that we will discuss the physics which *in principle* determines the light-cone vertex uniquely. The actual *calculation* of what this overall function is has not yet been performed, and is likely rather difficult.

The first step on this exciting journey into the why's and how's of string field theory will be a look at the four-particle scattering amplitude.

A Four-Particle Amplitude

We consider a $2 \to 2$ particle scattering process at tree level. This exercise will be useful for showing how to use the formalism of light-cone string field theory to do actual calculations. Without loss of generality we can choose to label the particles so that 1 and 2 are incoming (positive p^+) and 3 and 4 are outgoing (negative p^+) and furthermore $-p_4^+ > p_1^+ > p_2^+ > -p_3^+$.

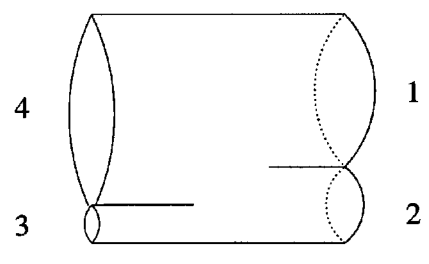

Figure 1: The s-channel, $1 + 2 \to 5 \to 3 + 4$.

The s-channel amplitude (see Fig (1)) is

$$\mathcal{A}_s = \int_0^\infty dT_5 \int_0^{p_5^+} d\sigma_5 \underbrace{\langle 0_{(5)} | V(a_4, a_3^\dagger, a_4^\dagger) | 0_{(3)} \rangle | 0_{(4)} \rangle}_{5 \to 3+4}$$

$$\times e^{-T_5(E_1+E_2-H_{(5)})+2\pi i\sigma_5(N_{(5)}-\widetilde{N}_{(5)})/p_5^+} \underbrace{\langle 0_{(1)}|\langle 0_{(2)}|V(a_1,a_2,a_5^\dagger)|0_{(5)}\rangle}_{1+2\to 5}. \quad (12)$$

Let us explain each ingredient. First of all, trivial overall p^+-momentum conserving delta functions are always understood but have not been written in order to save space. The processes $1 \to 2+5$ and $5 \to 3+4$ are as indicated, making use of our vertex function V. In between these two we have inserted the light-cone propagator for the intermediate string 5:

$$\frac{1}{E-H} = \int_0^\infty dT e^{-T(E-H)}. \quad (13)$$

By $H_{(5)}$ we mean of course the Hamiltonian for string 5:

$$H_{(5)} = \frac{1}{p_{(5)}^+} \sum_{n=-\infty}^{\infty} \omega_{n(5)} a_{n(5)}^\dagger a_{n(5)}. \quad (14)$$

Finally, the integral over σ_5 enforces the physical state condition on the intermediate string by projecting onto those states which satisfy

$$N_{(5)} - \widetilde{N}_{(5)} = \sum_{n=-\infty}^{\infty} n a_{n(5)}^\dagger a_{n(5)} = 0. \quad (15)$$

The full amplitude has two additional contributions. In the t-channel, we have first $1 \to 3+6$, and then $6+2 \to 4$:

$$\begin{aligned}\mathcal{A}_t = \int_{-\infty}^0 dT_6 \int_0^{p_6^+} \frac{d\sigma_6}{p_6^+} \langle 0_{2,6}|V(a_2,a_6,a_4^\dagger)|0_4\rangle \\ \times e^{-T_6(E_1-E_3-H_{(6)})+2\pi i\sigma_6(N_{(6)}-\widetilde{N}_{(6)})/p_6^+} \langle 0_1|V(a_1,a_3^\dagger,a_6^\dagger)|0_{3,6}\rangle.\end{aligned} \quad (16)$$

Finally in the u-channel, $2 \to 3+7$ and then $7+1 \to 4$:

$$\begin{aligned}\mathcal{A}_u = \int_{-\infty}^0 dT_7 \int_0^{p_7^+} \frac{d\sigma_7}{|p_7^+|} \langle 0_{1,7}|V(a_1,a_7,a_4^\dagger)|0_4\rangle \\ \times e^{-T_7(E_2-E_3-H_{(7)})+2\pi i\sigma_7(N_{(7)}-\widetilde{N}_{(7)})/p_7^+} \langle 0_2|V(a_2,a_3^\dagger,a_7^\dagger)|0_{3,7}\rangle.\end{aligned} \quad (17)$$

What are these things? Well, each $\mathcal{A}$ is just a state in $\mathcal{F}^4$, the fourth power of the string Fock space. If we want to know the amplitude for scattering four particular external states, then we just have to calculate

$$\langle 3|\langle 4|\mathcal{A}|1\rangle|2\rangle \quad (18)$$

(summed over channels) to get a number (well, a function of p_i^+ and μ), namely the scattering amplitude.

It should be emphasized that the harmonic oscillator algebra gives very complicated functions of T and σ which need to be integrated over. Actually performing this calculation is far outside the scope of these lectures, but I wanted to make one very important point about the general structure of this amplitude.

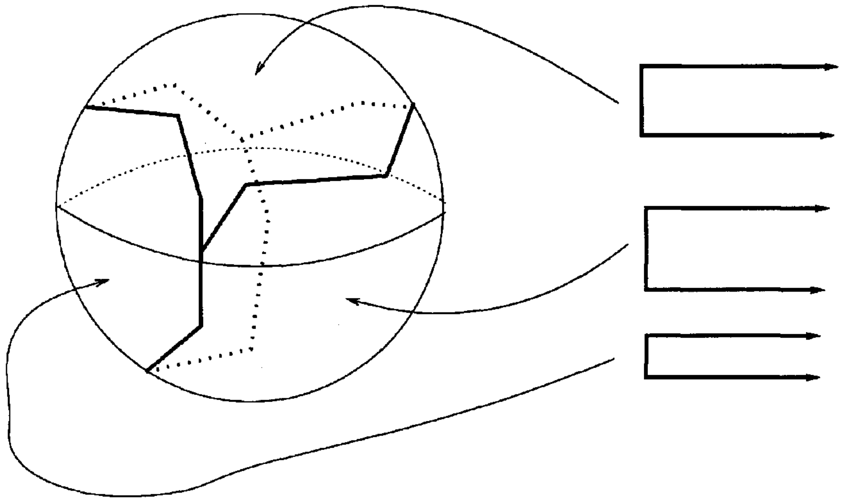

Figure 2: A schematic picture of the moduli map for the tree level 4-particle interaction.

Let us denote $\mathcal{R}_i = \{(T_r, \sigma_r) : T \in [0, \infty), \sigma_r \in [0, p_r^+]\}$, which are the two dimensional regions over which the quantities $\mathcal{A}_i$ must be integrated. There exists a particular map $z(T_r, \sigma_r)$[1] from the coordinates (T_r, σ_r) which patches together these three integration regions onto a sphere as shown in the above figure.

Let us define $\mathcal{A}(z)$ to be the image of the three individual $\mathcal{A}_s$, $\mathcal{A}_t$, and $\mathcal{A}_u$ on the sphere, patched together via the moduli map. It turns out that in flat space, precisely in the critical dimension $D = 26$, the function $\mathcal{A}(z)$ on the sphere is continuous along the boundaries between the images of $\mathcal{R}_i$ (that is, continuous along the dark lines in Figure (2)), which means that the amplitude is given completely by

$$\sum_{i=s,t,u} \int_{\mathcal{R}_i} dT_i d\sigma_i \mathcal{A}_i = \int_{S^2} d^2z \ \mathcal{A}(z). \tag{19}$$

[Note: From my expressions from $\mathcal{A}_i$ it appears that they do not depend on the dimension. I should explain why, in fact, they do.]

Why Light-Cone String Field Theory Works

The right hand side of (19) has a very familiar form. When most of us learned string theory, we learned that in order to calculate a four-particle amplitude at tree level in closed string theory, one inserts four vertex operators on the sphere. The positions of three vertex operators can be fixed using the conformal Killing vectors, and one is left with some amplitude (depending on the particular vertex operators inserted) which must be integrated over z, the position of the remaining vertex operator. The moduli space of a sphere with four marked points (the positions of vertex operators) is therefore the sphere itself.

[1] I am not aware that any name has been given to this map in the literature. I will call it the 'moduli map.' It should not be confused with the related moduli map, which is a conformal map from a light-cone diagram with **fixed** moduli into the complex plane.

In fact, equation (19) expresses the precise equivalence between amplitudes calculated in light-cone string field theory and amplitudes using the covariant Polyakov path integral. This equivalence relies on two important facts:

(1) Consider all of the light-cone diagrams which contribute to an amplitude with g closed string loops and n external particles. The diagrams will be labelled by $6g + 2n - 6$ parameters: g 'p^+-momentum fractions', $3g + n - 3$ twist angles, and $2g + n - 3$ interaction times. The first important fact is that the moduli map provides a one-to-one map between this $6g + 2n - 2$-dimensional parameter space and the moduli space of Riemann surfaces of genus g with n marked points (the locations of the vertex operators). A mathematical way of saying this is that the light-cone vertex provides a triangulation of the moduli space $\mathcal{M}_{g,n}$.

(2) The second important fact is that the integrand of the light-cone vertex, including all of the complicated structure involving the Neumann matrices and determinants thereof, maps under the moduli map to precisely the correct integration measure which arises from the Polyakov path integral!

The proof of these remarkable facts would take us too far afield, but I cannot stress enough the importance of these facts, which are deeply rooted in the underlying beautiful consistency of string theory. In fact, this equivalence can be used to prove the unitarity of the Polyakov path integral. Although the path integral is not manifestly unitary, it is equivalent to the light-cone formalism, which is manifestly unitary!

We are now in a position to answer some questions which may have been bothering some people since the last lecture: why is it sufficient to consider a cubic interaction between the string fields, and why is it sufficient to consider the simplest possible cubic interaction, with only a delta-functional (and, for example, no $\delta\Phi[x(\sigma)]/\delta x(\sigma)$)? Well, now we know that the cubic interaction is sufficient because (1) the iterated cubic interaction covers precisely one copy of moduli space and (2) the vertex we wrote down precisely reproduces the correct integration measure on this moduli space. We are not **allowed** to add any interactions besides the cubic one, because that would ruin our single cover of moduli space!

It is often said (even in papers that I have written) that the symmetry algebra (in particular, the supersymmetry algebra, for superstrings), 'uniquely' determines the interacting string Hamiltonian to all orders in the string coupling. This is a little bit misleading[2]. For example, in the supersymmetric theory one could take $Q =$ (anything) and then define $H = (\text{anything})^2$, and as long as (anything) commutes with rotations and translations, one would have a realization of the symmetry algebra! The symmetry argument, for example, provides no motivation for considering only a cubic interaction. The real physical principles at work are the ones we explained in this section: (1) correctly covering moduli space, and (2) getting the correct integration measure.

Contact Terms

Now, fact number (1), that the cubic delta-functional vertex covers moduli space precisely once, is essentially a mathematical theorem about a particular cell decomposition of $\mathcal{M}_{g,n}$ that holds quite generally. However, (2) can fail in subtle ways in certain circumstances.

[2]Those who are more direct may prefer the word 'wrong.'

In particular, it can happen that one or more of the $\mathcal{A}_i$'s has singularities in moduli space. A typical case might be for example that $\mathcal{A}_i(T,\sigma) \sim \frac{1}{T^3}$ near $T=0$, which is not integrable. This gives rise to divergences in string amplitudes, which need to be corrected by adding new string interactions to the Hamiltonian. However, these interaction terms are always delta-function supported on sets of measure zero ($T=0$ in this example) in moduli space, and therefore they do not spoil the beautiful triangulation that the cubic vertex provides. As long as we don't add any interaction with *finite* measure, we are OK.

Definition. I will define a **contact term** to be any additional interaction in the Hamiltonian, which necessarily lives on a set of measure zero in moduli space.

Corollary. All contact terms are divergent. Proof: If they were finite, and we integrate them over a set of measure zero, then they would give zero, so there would be no point to have them in the first place. :)

In flat space, it is known that the bosonic string requires no contact terms, while the IIB superstring is widely (though not universally) believed to require an infinite number of contact terms. The word 'believe' can be thought of in the following sense: since the purpose of contact terms is to eliminate divergences (and indeed we will see how they arise from short-distance singularities on the worldsheet), one can think of a contact term as a counterterm in the sense of renormalization. Now, there are infinitely many such counter terms that one can write down for the IIB string, and while some of them may have coefficients which are equal to zero, it is widely believed that infinitely many of them will have nonzero coefficients. We will see in tomorrow's lecture how these contact terms arise.

For strings in the plane wave background this question has not been addressed, mostly because we do not have the analogue of the covariant Polyakov formalism in which we can actually calculate anything. First we need to calculate this overall factor $f(p_1^+, p_2^+, p_3^+, \mu)$ and then see if there are any divergences which give rise to contact terms.

Any of the contact terms in IIB string theory in flat space will surely give rise to μ-dependent contact terms in the plane wave background. In principle there could be new contact terms introduced which go to zero in the limit $\mu \to 0$. Certainly I do not know how to disprove such a possibility, but I believe this is unlikely: contact terms may be thought of as coming from short-distance singularities on the string worldsheet, but the addition of a mass parameter μ on the worldsheet should not affect any of the short-distance behavior.

Superstrings

Let's go back to the beginning of lecture 2, but add fermions to the picture. We consider now a superparticle on the plane-wave solution of IIB supergravity. The physical degrees of freedom of the theory are encoded in a superfield $\Phi(x,\theta)$ which has an expansion of the form

$$\Phi(p^+,x,\theta) = (p^+)^2 A(x) + p^+\theta^a\psi_a(x) + \theta^{a_1}\theta^{a_2}p^+A_{a_1a_2}(x) + \cdots + \frac{1}{(p^+)^2}\theta^8 A^*(x), \qquad (20)$$

where θ^8 is short for eight powers of θ contracted with the fully antisymmetric tensor ϵ. Initially we allow all the component fields to be complex, but this gives too many components

(256 bosonic + 256 fermionic) so we impose the reality condition

$$\Phi(x,\theta) = (p^+)^4 \int d^8\theta^\dagger e^{i\theta\theta^\dagger/p^+} (\Phi(x,\theta))^\dagger \tag{21}$$

which cuts the number of components in half. Note, in particular, that this constraint correctly gives the self-duality condition for the five-form field strength in flat space.

When we second quantize, this hermiticity condition implies that the inner product on the string field theory Hilbert space $\mathcal{H}$ is **not** the inner product naïvely inherited from the single string Hilbert space. In particular,

$$\mathcal{A}_{|a\rangle}(p^+)^\dagger = \mathcal{A}_{|a'\rangle}(-p^+), \tag{22}$$

where the states $|a\rangle$ and $|a'\rangle$ differ by reversing the occupation of all of the fermionic zero modes, i.e. if $|a\rangle = |0\rangle$, then $|a'\rangle = \theta^8|0\rangle$, etc.

The action for the free superparticle is

$$S = \frac{1}{2}\int d^{10}x d^8\ \Phi(\nabla^2 - 2i\mu\partial_-\theta\Pi\partial_\theta)\Phi, \tag{23}$$

where $\Pi = \Gamma^1\Gamma^2\Gamma^3\Gamma^4$. The quantity in brackets is the quadratic Casimir of the plane wave superalgebra. It is straightforward to insert the superfield (21) into (23) and find the resulting spectrum.

The action (23) of course may also be obtained simply by linearizing the action for IIB supergravity around the plane wave background, and the spectrum may be obtained by linearizing the equations of motion around the background and finding the eigenmodes. This has been worked out in detail by Metsaev & Tseytlin, but we will use only one fact which emerges from this analysis. It turns out that there is a unique state with zero energy, which we will of course call $|0\rangle$. The corresponding spacetime field is a linear combination of the trace of the graviton over four of the eight transverse dimensions, h_{ii}, and the components of the four-form gauge potential a_{1234} in the first four directions. This field lives in the θ_{R}^4 component of the superfield, where we define left and right chirality with respect to Π (i.e., $\theta_{\mathrm{R,L}} = \frac{1}{2}(1 \pm \Pi)\theta$). The only important fact which you might want to keep in mind is that this spacetime field is **odd** under the Z_2 symmetry which exchanges the two SO(4)'s:

$$Z\mathcal{A}(p^+)_{|0\rangle} = -\mathcal{A}(p^+)_{|0\rangle}Z. \tag{24}$$

When we promote the superfield to string theory, it becomes a functional of the embedding of the string into superspace: $\Phi[p^+, x(\sigma), \theta(\sigma)]$. The cubic interaction term has a delta-functional for continuity of $x(\sigma)$, and also a delta-functional for the superspace coordinates:

$$\Delta[\theta_1(\sigma) + \theta_2(\sigma) - \theta_3(\sigma)]. \tag{25}$$

One can write this delta-functional in an oscillator representation as a squeezed state involving the fermionic creation operators. The 'fermionic' Neumann matrices are easily obtained from the bosonic Neumann matrices.