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SPRING SCHOOL ON SUPERSTRING THEORY AND RELATED TOPICS

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STRING INTERACTIONS IN PLANE WAVES

Lecture 4

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4 Parting Thoughts

The 'Prefactor'

The (relevant part of the) spacetime supersymmetry algebra is

$$\{Q^-, \overline{Q}^-\} = 2H, \qquad [Q^-, H] = 0, \qquad [\overline{Q}^-, H] = 0. \tag{1}$$

At the free level, a realization of this algebra is given by the free Hamiltonian H_2 we met before, and the free supercharges Q_2^-, which are given by

$$Q_2^- = \int dp^+ \; p^+ \Phi^\dagger q^- \Phi, \tag{2}$$

where the worldsheet supercharge is

$$q^- = \int d\sigma \left[4\pi e(p_r^+) p^I \gamma_I \lambda - \frac{i}{4\pi} \partial_\sigma x^I \gamma_I \theta - i\mu x^I \gamma \Pi \lambda \right]. \tag{3}$$

Here λ is the 'fermionic momentum' conjugate to θ (i.e., it is just $\overline{\theta}$)).

When we turn on an interaction H_3 in the Hamiltonian, we also need to turn on interactions Q_3, $\overline{Q}_3$ in the dynamical supercharges to ensure that the generators

$$H = H_2 + g_s H_3, \qquad Q^- = Q_2^- + g_2 Q_3^-, \qquad \overline{Q}^- = \overline{Q}_2^- + g_s \overline{Q}_3^- \tag{4}$$

provide a (non-linear) realization of the supersymmetry algebra (1).

Now there is a simple argument (see chapter 11 of Green, Schwarz & Witten) which shows that the choice $|H_3\rangle = |V\rangle$ would be incompatible with the supersymmetry algebra. Consider the relation $0 = [\overline{Q}^-, H]$ at first order in the string coupling. This gives (via the state-operator correspondence)

$$0 = \sum_{r=1}^{3} H_{(r)} |Q^-\rangle + \sum_{r=1}^{3} Q^-_{(r)} |V\rangle. \tag{5}$$

Now let us consider (for example) a matrix element of this relation where we sandwich three on-shell states on the left. Then $\sum_{r=1}^3 H_{(r)}$ acts to the left and gives zero, leaving us with only the second term. Now the state $|V\rangle$ indeed is annihilated by the constraints

$$\sum p_{(r)} |V\rangle = \sum \lambda_{(r)} |V\rangle = \sum e(p_r^+) x_{(r)} |V\rangle = \sum e(p_r^+) \theta_{(r)} |V\rangle = 0. \tag{6}$$

After looking at (3), the conditions (6) seem to imply that $Q^-_{(r)}|V\rangle = 0$.

However, one can check that the operators in (3) are actually singular near the interaction point. For example, we have $p(\sigma)\lambda(\sigma)|V\rangle \sim \partial_\sigma x(\sigma) \theta(\sigma) |V\rangle \sim \epsilon^{-1}|V\rangle$ near $\sigma = \sigma_{\rm I}$. Therefore, although $\sum_{r=1}^3 Q^-_{(r)}|V\rangle$ vanishes pointwise in σ (except at $\sigma = \sigma_{\rm I}$), the singular operators nevertheless give a finite contribution when integrated over σ. This contribution can be calculated by deforming the σ contour in an appropriate way and reading off the residue of the pole at $\sigma = \sigma_I$.

By calculating the residue of the pole, it can be shown that in order to supersymmetrize the vertex, it is necessary to introduce some operators (called 'prefactors') $\hat{h}$, $\hat{q}^-$, $\hat{\overline{q}}^-$ such that the interacting hamiltonian and supercharges are given by

$$|H_3\rangle = \hat{h}|V\rangle, \qquad |Q_3\rangle = \hat{q}|V\rangle, \qquad |\overline{Q}_3\rangle = \hat{\overline{q}}^-|V\rangle. \tag{7}$$

It turns out that $\hat{h}$ is a second-order polynomial in bosonic mode-creation operators (the $a^\dagger$'s) while $\hat{q}$ and $\hat{\overline{q}}^-$ are linear in bosonic creation operators. They also have a very complicated expansion in terms of fermionic modes, and I will not give the complete formula here.

It is essential to note, however, that the last term in (3) is non-singular when acting on $|V\rangle$. This makes sense, since the parameter μ introduces a scale in the worldsheet theory, but this should not affect the short distance physics. Therefore the functional form of the prefactor has essentially the same form as in flat space (there are subtleties in passing from the functional representation to the oscillator representation, though).

We have shown that in the functional representation, the cubic interaction between three string super-fields is **not** given simply by the delta-functionals

$$\Delta[x_1(\sigma) + x_2(\sigma) - x_3(\sigma)]\Delta[\theta_1(\sigma) + \theta_2(\sigma) - \theta_3(\sigma)] \prod_{r=1}^{3} \Phi[x_r(\sigma), \theta_r(\sigma)] \tag{8}$$

In addition, there is a complicated combination of functional derivatives acting on the Φ fields, inserted at the point σ_I where the strings split. We call this the 'interaction point operator.'

Contact Terms from the Interaction Point Operator

The prefactor $\hat{h}$ is an operator of weight $\frac{3}{2}$, which means that at short distances we have $\hat{h}(x)\hat{h}(y) \sim (x-y)^{-3}$. (This is not a conformal field theory, so by 'weight' I simply mean the strength of the coincident singularity.) A light-cone string diagram in which two (or more) of these prefactors come very close to each other will therefore be divergent. The simplest example occurs in the two-particle amplitude at one loop (i.e., a contribution to the one-loop mass renormalization), shown in the following diagram:

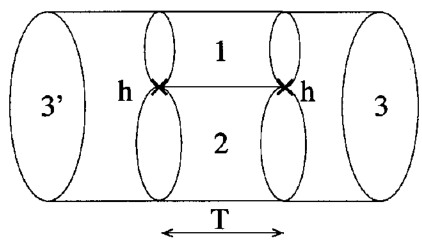

Figure 1: The one-loop mass renormalization of string 3.

This amplitude has an integral $\int_0^\infty dT$ over the Schwinger parameter giving the light-cone time between the splitting and joining, but the integrand is divergent like T^{-3} due to the colliding prefactors. It is clear that at higher order in the string coupling (and/or with more external states), we can draw diagrams which have arbitrarily many colliding prefactors. These divergent contributions to string amplitudes must be rendered finite by the introduction of (divergent) contact interactions as discussed in the previous lecture. The belief is that there is a unique set of contact interactions which preserves all the symmetries (Lorentz invariance, supersymmetry) and which renders all amplitudes finite. But these contact terms are very unwieldy, and almost impossible to calculate explicitly, so they haven't really been studied in very much detail.

We will have a little bit more to say about these contact terms below.

The S-matrix in the BMN Correspondence

In these lectures we have demonstrated how to determine the light-cone Hamiltonian of second-quantized IIB string theory in the plane wave background. The two remaining ambiguities are (1) that we have not determined some overall factor $f(p_1^+, p_2^+, p_3^+, \mu)$ that appears in the cubic coupling, and (2) that there are (probably) infinitely many contact counterterms which need to be added to the action. In principle (although probably not in practice), the light-cone string field theory approach allows one to calculate S-matrix elements to arbitrary order in the string coupling via a quite straighforward Hamiltonian approach: there is a (large) Hilbert space of states, with a light-cone Hamiltonian acting on it, and one can easily apply the rules of quantum mechanical perturbation theory to give the S-matrix

$$\langle 1|S|2\rangle = \langle 1|1 - 2\pi i\delta(E_1 - E_2)T(E + i\epsilon)|2\rangle, \tag{9}$$

where $H_2|E_i\rangle = E_i$ and $T(z)$ is the transition operator

$$T(z) = V + VG(z)V, \qquad V \equiv H - H_2, \qquad G(z) = (z - H)^{-1}. \tag{10}$$

Typically $T(z)$ is calculated via the Born series

$$T(z) = V + VG_0(z)V + VG_0(z)VG_0(z)V + \cdots, \tag{11}$$

where $G_0(z) = (z - H_2)^{-1}$ is the 'bare' propagator.

Typically, the S-matrix is the only good 'observable' of string theory. Local observables are not allowed in string theory since string theory is a theory of quantum gravity, and in particular has diffeomorphism invariance. The question is then, how is the string S-matrix encoded in the BMN limit of $\mathcal{N} = 4$ Yang-Mills theory? Note that I am not talking about the S-matrix of the gauge theory (which doesn't exist, since it is a CFT) — the question is about how the string theory S-matrix may be derived from the BMN gauge theory.

The Quantum Mechanics of BMN Operators

It has been emphasized by a number of authors that it can be useful to think of gauge theory in the BMN limit as a quantum mechanical system. There is a space of states (the BMN operators), an inner product (the gauge theory two-point function), and a Hamiltonian, given by $\Delta - J$. Perturbation theory in the gauge theory organizes itself into the two parameters

$$\lambda' = \frac{\lambda}{J^2} = \frac{g_{\rm YM}^2 N}{J^2}, \qquad g_2 = \frac{J^2}{N}, \tag{12}$$

which are respectively the effective 't Hooft coupling and the effective genus counting parameter, respectively.

Recall that the spectrum of BMN operators takes the following form

$$\Delta - J = \sum_{n=-\infty}^{\infty} N_n \sqrt{1+\lambda' n^2}, \tag{13}$$

where N_n is the number of impurities with phase n. To one-loop, the gauge theory Hamiltonian $H = \Delta - J$ takes the following form

$$H = \underbrace{\sum_{n=-\infty}^{\infty} N_n + \lambda' \sum_{n=-\infty}^{\infty} \frac{1}{2} n^2 N_n}_{H_0} + \underbrace{\lambda' g_2 (H_+ + H_-)}_{V}, \tag{14}$$

where H_+ and H_- respectively increase and decrease the number of traces. That is, if we act with H_+ on a k-trace operator, then it 'splits' one of the traces so that we get a $k+1$-trace operator. Similarly H_- 'joins' two traces.

The first two terms in (14) are clearly just the first two terms in (13), expanded to order λ', so we have labelled them H_0 — they constitute the 'free' Hamiltonian. The third term in (14) has the structure of a three-string vertex, and incorporates the string interactions, so we have labelled this term V.

The next step is to recall that we should keep in mind the basis transformation that Herman Verlinde mentioned during his lectures. In our first lecture we identified a precise correspondence between operators in the gauge theory and states in string theory. It is natural therefore to identify a double-trace operator in the gauge theory with the corresponding two-particle state in the string theory, etc. However this identification breaks down at $g_2 \neq 0$. One way to see this is to note that in string theory, a k particle state and an l particle state are necessarily orthogonal for $k \neq l$. However in the gauge theory, it is not hard to check that the gauge theory overlap (given by the two-point function in the free theory) is typically given by

$$\langle O_{\rm k-trace}(x) O_{\rm l-trace}(0)\rangle \sim g_2^{|k-l|}. \tag{15}$$

It has been conjectured (and motivated in Verlinde's talk) that one can write an exact formula, valid to all orders in g_2, for the inner product:

$$\langle 1|2\rangle = \left(e^{g_2\Sigma}\right)_{12}, \tag{16}$$

where Σ is the simple 'splitting-joining' operator of the bit model. The inner product (16) is diagonalized by the basis transformation $S^{1/2} = e^{g_2\Sigma/2}$. Therefore, we propose the following BMN identification at finite g_2:

$$|0;p^+\rangle \iff S^{-1/2}{\rm Tr}[Z^J],$$

$$\begin{aligned} (a_0^i)^\dagger|0;p^+\rangle &\Longleftrightarrow S^{-1/2}\mathrm{Tr}[\phi^i Z^J], \\ (a_n^i a_{-n}^j)^\dagger|0;p^+\rangle &\Longleftrightarrow S^{-1/2}\sum_{k=0}^{J} e^{2\pi i k n/J}\mathrm{Tr}[\phi^i Z^k \phi^j Z^{J-k}], \\ &\quad etc. \end{aligned} \tag{17}$$

Now the multi-particle states constructed from the operators on the right hand side of this correspondence will have the property that k-string states are orthogonal to l-string states for $k \neq l$. However, we have lost the identification of 'number of traces' with 'number of strings'. Instead, we have something of the form

$$\begin{aligned} k - \text{string state} &\Longleftrightarrow [k - \text{trace operator}] \\ &+g_2 \times [(k-1) - \text{trace operators} + (k+1) - \text{trace operators}] + \cdots \end{aligned} \tag{18}$$

It is convenient to perform this basis transformation on the operator H, to define what I will call the string Hamiltonian $\widetilde{H}$, given by

$$\widetilde{H} = S^{1/2} H S^{-1/2} \equiv H_0 + W, \tag{19}$$

for some new interaction W (which is easily calculated). Now $\widetilde{H}$ is simply a non-relativistic quantum mechanical Hamiltonian, and it is straightforward to derive from it an S-matrix. This S-matrix should be that of IIB string theory on the plane wave background.

Off-shell Matrix Elements

We have phrased the BMN duality in terms of S-matrix elements, but in fact there has been much success in the literature at matching off-shell matrix elements of the interaction Hamiltonian W defined above, to matrix elements of the corresponding interaction Hamiltonian H_3 in the string field theory. Currently there are some problems getting this to work out at higher loops in the gauge theory (higher powers of λ'), but it is intriguing to address this issue more closely.

As discussed above, matrix elements of H are not good observables of string theory. In previous lectures when we constructed the Hamiltonian of string theory, I should have said that we constructed **a** Hamiltonian of string theory. In general there can be infinitely many different Hamiltonians that all give rise to precisely the same S-matrix, and no one choice is better than any other.

However, one intriguing possibility is that the Hamiltonian $\widetilde{H}$ which we calculate from the gauge theory might agree precisely, matrix element by matrix element, with the Hamiltonian constructed in previous lectures, or possibly a suitable modification thereof. Let me now discuss what I mean by a 'suitable modification'.

Contact Terms in Gauge Theory

Previously I compared contact terms to counterterms, and indeed we saw how certain contact terms in the superstring arise from regulating singularities that arise when two operators collide on the worldsheet. One could imagine regulating the theory in some way in order to render these divergences finite. One natural regulator, which is suggested by both the dual gauge theory and the string bit model, is to discretize the worldsheet.

Discretizing the worldsheet leads to a spacetime Hamiltonian which depends on J:

$$H(J) = H_2(J) + H_3(J) + \sum_{k=1}^{\infty} J^k C_k(J), \tag{20}$$

where $C_k(J)$ are the contact terms (which we suppose have a finite $J \to \infty$ limit). When J is finite, the cubic interaction $H_3(J)$ will fail to precisely cover the moduli space, but the 'contact terms' will be finite and will cover regions of moduli space of small but nonzero measure. In the continuum limit $J \to \infty$, the contact terms become infinite but restricted to regions of measure zero.

Physical quantities (such as S-matrix elements) should of course be independent of the cutoff J. The precise way to say this is that the amplitudes obtained from $H(J_1)$ differ from those obtained from $H(J_2)$, $J_1 \neq J_2$ by something which is BRST-exact. Anything BRST-exact integrates to zero over moduli space, so this would prove that amplitudes are indeed independent of the cutoff J.

The previous two paragraphs have been well-motivated, but not precise: I do not know how to construct $H(J)$ in string theory. Certainly discretized string theories have been considered in the past, but including interactions is frequenly problematic. The bit model is intended as a step in the direction of constructing $H(J)$.

The conclusion is that I see three possibilities for the BMN correspondence:

- There is a precise way to define $H(J)$ in string theory. When this is done, one will find that matrix elements of $H(J)$, computed in string theory, will equal matrix elements of $(\Delta - J)$ in the gauge theory (after an appropriate basis transformation).
- There is no precise analogue of $H(J)$ that makes sense in the string theory. Instead, one must calculate the amplitudes directly. The amplitudes calculated from H in string theory will agree with amplitudes calculated from $\Delta - J$ in gauge theory.
- Other.

Some Problems and Puzzles

`[calculational, conceptual]` Define $H(J)$ Precisely and Provide a Construction for it.

`[calculational --- TOO HARD]` Can we say more about light-cone string field theory in the plane wave background? In particular, is it possible to determine the measure factor? Is it possible to honestly calculate a 4-particle interaction, or a 1-loop mass renormalization. What can be said about the contact terms?

`[conceptual, calculational]` **Does an S-matrix Exist in the Plane Wave?** Obviously, if the answer is no, then quite a bit of these lectures will have to be rewritten... In any case, the existence of an S-matrix has several predictions which should be possible to check in the gauge theory. For example, it implies that the one-loop mass renormalization of any k-particle state should be equal to the sum of the one-loop mass renormalizations of the individual particles. This is because the existence of an S-matrix presupposes that the particles can be well-separated from each other. For the most trivial case, where one particle has 2 impurities, and the other $k-1$ particles are all in the ground state, $\mathrm{Tr}[Z^J]$, this has indeed been shown to be true (it is essentially due to the fact that 'disconnected' diagrams contribute over 'connected' ones in the large J limit — see the gauge theory papers for details). Can this proof be generalized to k operators, each of which has more than zero (ideally, arbitrarily many) impurities?

`[calculational]` **Calculate the Gauge Theory Inner Product.** The gauge theory inner product is defined as the coefficient of the free ($g_{\mathrm{YM}} = 0$) two-point function. For example, in the simplest case of two vacuum operators it has been shown that

$$\langle \mathrm{Tr}[\overline{Z}^J]\mathrm{Tr}[Z^J]\rangle = \frac{\sinh(g_2/2)}{g_2/2}. \tag{21}$$

For BMN operators with impurities, the inner product has been calculated only to a couple of orders in g_2. Since this is a free gauge theory calculation, it reduces to a simple Gaussian matrix model, with graphs of genus g contributing at order g_2^{2g}. It should be possible to prove (or disprove!) the conjectured formula that the inner product is given in general by $e^{g_2\Sigma}$, where Σ is the splitting-joining operator of the bit model.

`[conceptual, calculational]` **'Solve' the Quantum Mechanics of BMN Operators.** Recent studies have uncovered the exciting possibility that the BMN limit may be the tip of an integrable sector in $\mathcal{N} = 4$ gauge theory. One interesting question is whether there is any more hidden structure in the quantum mechanical Hamiltonian we wrote down. For example, one can check (at least in the two-impurity sector) that the interaction W in the string Hamiltonian $\widetilde{H}$ commutes with Σ! Are there any other hidden symmetries which will allow one to make progress towards 'solving' this quantum mechanics?

`[conceptual, calculational]` **Do Higher-Point Functions Play any Role in the BMN Limit?** In my discussion I limited my interest purely to two-point functions in the gauge theory (although I did consider two-point functions of k-trace operators with l-trace operators, but that contains only a tiny remnant of the information encoded in the $k+l$-point function). These two-point functions were interpreted (essentially) as S-matrix elements. It is natural to wonder (as many papers have) what (if any) role is played by the higher-point functions.

In the previous lecture I explained the correspondence between three-point functions, matrix elements of $|V\rangle$, and matrix elements of Σ at $\lambda' = 0$. I avoided attempting to extend this correspondence to $\lambda' > 0$ for the following reason. In a conformal field theory, three-point functions of operators with well-defined scaling dimension have simple three-point functions. But BMN operators do **not** have well-defined scaling dimension! They are eigenstates of the 'free' Hamiltonian H_0, but they are not eigenstates of $H = \Delta - J$. So in general, the three-point function of BMN operators looks like

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)\rangle = \text{very complicated function of} (x_1, x_2, x_3). \tag{22}$$

I do not know how to recover the x_1, x_2, x_3 dependence of (22) from the string theory side.

Another point to keep in mind is that higher-point functions of BMN operators typically diverge in the large J limit. An interesting alternative is to consider n-point functions where 2 operators are BMN, and $n-2$ operators have finite charge (for example, $\mathrm{Tr}[\phi Z]$). Such an n-point function might be interpreted as an amplitude for propagation from some initial state to some final state, with some operators inserted on the worldsheet which perturb the spacetime away from the pure pp-wave background. The correspondence between spacetime perturbations and operator insertions can be read off from the familiar dictionary of $AdS_5 \times S^5$.

`[conceptual, calculational]` **Can Continuum Light-Cone Superstring Field Theory be Honestly Discretized?** Discretized string theories have been studied for a long time (in particular by Thorne), but it seems somewhat problematic to discretize an interacting type IIB string. In particular, there are problems with the fermions. There is the usual 'fermion doubling' problem, that Herman mentioned briefly. There is also the basic question of how to implement the hermiticity constraint, which in the continuum theory singles out the fermionic zero modes from the non-zero modes. However, a discretized string doesn't really have any fermionic zero modes, there is just one fermionic oscillator on each 'site' along the string, and the natural 'adjoint' just takes the adjoint of each fermion at each site.

`[conceptual, calculational]` **Which Quantities can be Calculated Perturbatively in the Gauge Theory?** It is important to not forget that the BMN limit still involves taking the 't Hooft coupling to infinity, so the weak coupling expansion is not really valid. However, it is empirically observed that some quantities, notably the conformal dimensions of BMN operators at $g_2 = 0$:

$$\Delta - J = \sum_{n=-\infty}^{\infty} N_n \sqrt{1 + \lambda' J^2}, \tag{23}$$

may be calculated at small λ (i.e., in gauge theory perturbation theory) and finite J, and then extrapolated to $\lambda, J \to \infty$ where magically they agree with the corresponding string calculation!

These quantities are not BPS, so we had no right to expect this miracle to occur. The basic question behind the BMN correspondence is simply this: for which (if any!) other quantities does this miracle occur? For example, I commented that there have been successful comparisons of off-shell matrix elements of the light-cone Hamiltonian, at one loop (order λ').

`[conceptual]` **How is the Order of Limits Problem Resolved?** The order of limits problem is that in the gauge theory, we want to expand around $\lambda' = 0$, which on the string theory corresponds to $\mu = \infty$ and seems quite singular. In particular, several steps in the derivation of the light-cone string vertex depended on the assertion that at small distances (on the worldsheet), the physics is essentially unchanged by the addition of the parameter μ. However this doesn't really make sense if $\mu = \infty$.

In particular, the prefactor of the superstring is a local operator insertion at the point where the string splits. If we view the gauge theory as a discretized version of the string theory (with J 'bits'), how can the prefactor possibly be recovered when all calculations are necessarily done at finite J, where there is no notion of 'locality' on the worldsheet?

We remarked above that we should not expect that non-BPS quantities calculated perturbatively in λ should agree with any corresponding predictions from the string theory, where λ is infinite. Another way to say this is that suppose we have some quantity $Q(\lambda, J)$ which we can calculate in the gauge theory

$$Q(\lambda, J) = Q_0 J^{k_0} + \lambda Q_1 J^{k_1} + \cdots = \sum_{n=0}^{\infty} \lambda^n Q_n J^{k_n}. \tag{24}$$

Naively, we would think that Q is well-defined in the BMN limit only if

$$k_n - 2n \leq 0 \qquad \forall n, \tag{25}$$

in which case it would be given by

$$Q(\lambda') = \lim_{J\to\infty} \sum_{n=0}^{\infty} (\lambda')^n Q_n J^{k_n - 2n}, \tag{26}$$

where $\lim_{J\to\infty} J^{k_n-2n}$ is always either zero or 1, thanks to (25).

However, there is no *a priori* reason why the λ perturbation expansion has to agree with the λ' expansion. The BMN limit tells us that we should calculate $Q(\lambda, J)$ at *finite* λ, and then take the $\lambda \to \infty$ limit holding $J^2 \sim \sqrt{N}$. The resulting expansion

$$\lim_{BMN} Q(\lambda, J) = \sum_{n=0}^{\infty} (\lambda')^n \widetilde{Q}_n \tag{27}$$

might have coefficients $\widetilde{Q}_n$ that look completely different from Q_n. String theory calculates $\widetilde{Q}_n$, but gauge theory calculates Q_n.

`[calculational]` **What Happens at Finite λ in the Bit Model?** But how do we take λ to be finite in a gauge theory calculation? One way that it might be possible to make sense of this is to take λ finite in the bit model, with the hope that the bit model successfully encapsulates all of the relevant degrees of freedom.

Let a_x, $a_x^\dagger$ denote the usual operators living on site x on the string and satisfying $[a_x, a_y^\dagger] = \delta_{xy}$. Now at $\lambda = 0$, these operators are just raising and lowering operators of the Hamiltonian. That is, creating an excitation at some position x on the string precisely increases the energy by one unit.

However at $\lambda \neq 0$ it is no longer the case that the Hamiltonian is diagonal in this 'local' basis. Instead, the Hamiltonian is diagonal in the Fourier basis, and the eigenmodes are not those which are located at some point x on the string, but rather are those will well-defined Fourier momentum n around the string.

One can turn on finite λ in the bit model by doing a Bogoluybov transformation; the operator a_x will now be expressed in terms of both raising and lowering operators a_n and $a_n^\dagger$. The cubic vertex, which is trivial at $\lambda = 0$ because it is just the delta-function overlap matching excitations pointwise in σ, needs to be reexpressed in terms of the true eigenmodes of the $\lambda > 0$ Hamiltonian.