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# *SPRING SCHOOL ON SUPERSTRING THEORY AND RELATED TOPICS*

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## SUPPLEMENT TO LECTURES ON

## TOPOLOGICAL STRING AND ITS APPLICATION

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Please note: These are preliminary notes intended for internal distribution only.

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1. Preliminaries

Calabi-Yau manifold

Alvarez-Gaume's lecture

Topological sigma-model — A-model, B-model

Dijkgraaf's lecture

Moore's lecture

2. Geometry of the moduli space of TCFT

Zamolodchikov metric, restricted Kähler geometry

Special geometry

3. Coupling to 2d gravity - topological string theory

higher genus amplitudes
& its geometric meaning

→ also Narain's lectures

holomorphic anomalies

How to solve anomalies
(c.f chiral anomaly → WZ action)

4 mirror symmetry at $g \geq 1$ ← Greene's lecture

The Kodaira-Spencer theory
as a closed string field theory

## Kähler manifold and $N=2$ $\sigma$-model

$M$ : Kähler manifold $\Leftrightarrow$ complex manifold $(z^1 \cdots z^n)$ : complex coordinates
Kähler metric $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$
$K$ : Kähler potential

Cohomologies

- $H^{p,q}(M) \ni \omega = \omega_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} \, dz^{i_1} \cdots dz^{i_p} d\bar{z}^{\bar{j}_1} \cdots d\bar{z}^{\bar{j}_q}$

$$d\omega = 0 \quad \text{modulo} \quad \omega \to \omega + d\nu$$

- $H^q_{\bar{\partial}}(M, \overset{p}{\wedge} TM) \ni v = v_{\bar{j}_1 \cdots \bar{j}_q}{}^{i_1 \cdots i_p} \, d\bar{z}^{\bar{j}_1} \cdots d\bar{z}^{\bar{j}_q} \, \partial_{i_1} \cdots \partial_{i_p}$

$$\bar{\partial} v = 0 \quad \text{modulo} \quad v \to v + \bar{\partial} u$$

## Calabi-Yau manifold

, canonical line bundle [illegible]

Ricci-tensor $R_{i\bar{j}} \, dz^i \wedge d\bar{z}^{\bar{j}} = -i \, \partial\bar{\partial} \log \det g = 0$

$\Rightarrow$ $\exists^1$ holomorphic $n$-form $\Omega = \Omega_{i_1 \cdots i_n} dz^{i_1} \cdots dz^{i_n}$
$(\dim H^{n,0}(M) = 1)$ $\qquad \Omega \neq 0$ everywhere

$$\Rightarrow \quad H^q_{\bar{\partial}}(M, \overset{p}{\wedge} TM) \simeq H^{n-p,q}(M)$$

$$\begin{array}{ccc} \Psi & & \Psi \\ v_{\bar{j}_1 \cdots \bar{j}_q}{}^{i_1 \cdots i_p} & \longrightarrow & \Omega_{i_1 \cdots i_p i_{p+1} \cdots i_n} v_{\bar{j}_1 \cdots \bar{j}_q}{}^{i_1 \cdots i_p} \end{array}$$

$$\left\{ \begin{array}{l} \bigoplus_{p,q} H^{p,q}(M) \Rightarrow \text{chiral ring of the A-model} \\ \bigoplus_{p,q} H^q_{\bar{\partial}}(M, \overset{p}{\wedge} TM) \Rightarrow \text{chiral ring of the B-model} \end{array} \right.$$

Kähler class and complex structure ··· moduli of Calabi-Yau manifold

$g_{i\bar{j}}$ : Ricci-flat Kähler metric

deformation $\delta g_{i\bar{j}}$, $\delta g_{ij}$, $\delta g_{\bar{i}\bar{j}}$

Ricci-flatness of the deformed metric, modulo diffeomorphism.

$$\begin{cases} \delta g_{i\bar{j}}\, dz^i d\bar{z}^{\bar{j}} & : \text{harmonic} \in H^{1,1}(M) \\ g^{i\bar{j}} \delta g_{\bar{j}\bar{k}}\, d\bar{z}^{\bar{k}} \partial_i & \in H^1_{\bar{\partial}}(M, TM) \end{cases}$$

$\mathcal{M}$ : moduli space of Calabi-Yau manifold

$$\begin{aligned} T\mathcal{M}|_M &\simeq H^{1,1}(M) \oplus H^1_{\bar{\partial}}(M, TM) \\ &\simeq H^{1,1}(M) \oplus H^{n-1,1}_{\bar{\partial}}(M) \end{aligned}$$

Kähler form $J = i g_{i\bar{j}}\, dz^i d\bar{z}^{\bar{j}} \in H^{1,1}(M)$ $\because$) $M$ : Kähler

$$= \sum_{a=1}^{h^{1,1}} r^a \omega_a$$

$r^a \in \mathbb{R}$, $\omega_a$ : basis of $H^{1,1}(M)$

define Kähler class of $M$.

$h^{1,1} = \dim H^{1,1}$

deformation of complex structure

$$z^i \to z'^i, \quad \frac{\partial}{\partial \bar{z}^{\bar{i}}} \to \frac{\partial}{\partial \bar{z}^{\bar{i}}} + v_{\bar{i}}{}^{j} \frac{\partial}{\partial z^j} = \frac{\partial}{\partial \bar{z}'^{\bar{i}}}$$

integrability $\bar{\partial} v = 0$ diffeomorphism $v \to v + \bar{\partial}\varepsilon$ $\therefore v \in H^1_{\bar{\partial}}(M, \ldots$

$$\begin{cases} H^{1,1}(M) & : \text{deformation of Kähler class} \\ H^1_{\bar{\partial}}(M, TM) & : \text{deformation of complex structure} \end{cases}$$

$N=2$ $\sigma$-model

$\Sigma$ : 2dim surface , $M$ : Calabi-Yau manifold

$(X^i, X^{\bar{i}}) : \Sigma \to M$

$(\psi_L^i, \psi_L^{\bar{i}})$ : left-moving spin-½ fermion
$(\psi_R^i, \psi_R^{\bar{i}})$ : right-moving " "

action

$$S = \int_\Sigma d^2z \left[ \frac{1}{2} g_{i\bar{j}} (\partial_z X^i \partial_{\bar{z}} X^{\bar{j}} + \partial_{\bar{z}} X^i \partial_z X^{\bar{j}}) + \right.$$
$$+ i g_{i\bar{j}} \psi_R^{\bar{j}} D_z \psi_R^i + i g_{i\bar{j}} \psi_L^{\bar{j}} D_{\bar{z}} \psi_L^i +$$
$$\left. + R_{i\bar{j}k\bar{l}} \psi_R^i \psi_R^{\bar{j}} \psi_L^k \psi_L^{\bar{l}} \right]$$
$$- \frac{1}{2} \sum_{a=1}^{h^{1,1}} \theta^a \int_\Sigma \omega_{a,i\bar{j}} (\partial_z X^i \partial_{\bar{z}} X^{\bar{j}} - \partial_{\bar{z}} X^i \partial_z X^{\bar{j}}) \quad \leftarrow \text{instanton term}$$

$$D_z \psi_R^i = \partial_z \psi_R^i + \Gamma^i_{jk} \partial_z X^j \psi_R^k \ , \quad D_{\bar{z}} \psi_L^i = \cdots$$

$U_L(1) \times U_R(1)$ symmetry

$$\psi_L^i \to e^{i\alpha_L} \psi_L^i \ , \quad \psi_L^{\bar{i}} \to e^{-i\alpha_L} \psi_L^{\bar{i}}$$

similarly for $(\psi_R^i, \psi_R^{\bar{i}})$

$\downarrow$

$$D_z \psi_R^i \to (D_z + iA_z) \psi_R^i \ , \quad D_{\bar{z}} \psi_L^i \to (D_{\bar{z}} + iA_{\bar{z}}) \psi_L^i$$

$$S \to S + \int d^2z \, (A_{\bar{z}} J_z + A_z J_{\bar{z}})$$

$$J_z = g_{i\bar{j}} \psi_L^i \psi_L^{\bar{j}} \ , \quad J_{\bar{z}} = g_{i\bar{j}} \psi_R^i \psi_R^{\bar{j}}$$

renormalization ( $g_{i\bar{j}}$ as coupling "constant" )

$$\beta_{i\bar{j}} = -\mu \frac{\partial}{\partial\mu} g_{i\bar{j}}^{\text{ren}} \propto \underbrace{R_{i\bar{j}}}_{\substack{\text{one-loop} \\ (\text{nontrivial in } H^{1,1})}} + \underbrace{\partial_i \partial_{\bar{j}} \chi}_{\text{higher loops}} , \quad (\chi : \text{polynomials in } R)$$

$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$ : Ricci-flat Kähler metric on $M$

$\Rightarrow$ renormalization of $K$ does not affect the Kähler class and complex structure.

at each order in the perturbation, we can adjust $g_{i\bar{j}}$ so that $\beta_{i\bar{j}} = 0$.

$N=2$ superconformal symmetry

left mover $\quad T_{zz} = g_{i\bar{j}} \partial_z X^i \partial_z X^{\bar{j}} + \frac{1}{2} g_{i\bar{j}} (\psi_L^i \partial_z \psi_L^{\bar{j}} + \psi_L^{\bar{j}} \partial_z \psi_L^i)$

$$G^+ = g_{i\bar{j}} \psi_L^i \partial_z X^{\bar{j}} , \quad G^- = g_{i\bar{j}} \psi_L^{\bar{j}} \partial_z X^i$$

$$J_z = g_{i\bar{j}} \psi_L^i \psi_L^{\bar{j}}$$

They are all holomorphic.

$$\partial_{\bar{z}} J_z = (\text{chiral anomaly}) \propto R_{i\bar{j}} (\partial_z X^i \partial_{\bar{z}} X^{\bar{j}} - \partial_z X^{\bar{j}} \partial_z X^i) + (\partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z)$$

$\Rightarrow \quad R_{i\bar{j}} = 0$

N.B $A$ . c-number while $R_{i\bar{j}}$ : q-number

Topological $\sigma$-model

twistings A-twist

(c.f. $E_8 \to E_6$ in String Compactification)

spin connection

$$\begin{cases} A_z = \frac{i}{2}\,\omega_z = \frac{i}{2}\,\partial_z \log(\det g) \\ A_{\bar z} = -\frac{i}{2}\,\omega_{\bar z} = -\frac{i}{2}\,\partial_{\bar z} \log(\det g) \end{cases}$$

$$\psi_R^{i} = \rho_{\bar z}^{\,i} : \text{one-form}, \quad \psi_R^{\bar i} = \chi^{\bar i} : \text{zero-form}$$

$$\psi_L^{i} = \chi^{i} : \text{zero-form}, \quad \psi_L^{\bar i} = \rho_z^{\,\bar i} : \text{one-form}$$

· left-mover

$$G^+ = g_{i\bar j}\, \chi^i \partial_z X^{\bar j} \quad : \text{one-form} \Rightarrow \text{BRST current}$$

$$G^- = g_{i\bar j}\, \rho_{\bar z}^{\,\bar j} \partial_z X^i \quad \text{two-form}$$

$$Q_L = \oint dz\, G^+$$

$$\frac{1}{2}\{ Q_L, G^- \} = T + \frac{1}{2}\partial J = \tilde T$$

In the following we will omit this

$$\left( \because \; -\frac{1}{\sqrt g}\frac{\delta}{\delta g^{zz}} \int A_{\bar z} J = \frac{1}{2}\partial J \right)$$

$G^-$ is a BRST partner of $T$ (i.e. anti-ghost)

· right-mover

$$\bar G^+ = g_{i\bar j}\, \chi^{\bar j} \partial_{\bar z} X^i \quad : \text{BRST current}$$

$$\bar G^- = g_{i\bar j}\, \rho_{\bar z}^{\,i} \partial_{\bar z} X^{\bar j} \quad : \text{anti-ghost}$$

$$\frac{1}{2}\{ Q_R, \bar G^- \} = \bar T$$

TCFT = topological conformal field theory

BRST transformation

$$\delta_L X^i = \chi^i \quad , \quad \delta_R X^{\bar{i}} = \chi^{\bar{i}}$$

$\chi^i, \chi^{\bar{i}}$ : invariant

$$\delta_L \rho_z^{\bar{i}} = \partial_z X^{\bar{i}} + \cdots \quad , \quad \delta_R \rho_{\bar{z}}^{i} = \partial_{\bar{z}} X^i + \cdots$$

fermion bi-linear

BRST invariant observables

- $\omega \in H^{p,q}(M) \rightarrow \Phi[\omega](z,\bar{z}) = \omega_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q}(X(z,\bar{z}))\chi^{i_1} \cdots \chi^{i_p}\chi^{\bar{j}_1} \cdots \chi^{\bar{j}_q}$

  dim = (0,0) $\qquad dx^i \rightarrow \chi^i , \quad dx^{\bar{i}} \rightarrow \chi^{\bar{i}}$

  $\boxed{Q_L = \partial , \quad Q_R = \bar{\partial}} \qquad [Q, \Phi] = 0 , \quad \Phi \neq Q(\cdots)$

- $\int_\Sigma G_{-1}\bar{G}_{-1}\Phi[\omega](z,\bar{z})\, d^2z$

$$\delta_L(\cdot\cdot) = 2\int_\Sigma \partial_z(\bar{G}_{-1}\Phi[\omega])\, d^2z$$

Deformation of Kähler structure $J = i g_{i\bar{j}} dz^i dz^{\bar{j}} = \sum_a r^a \omega_a$

$$r^a \longrightarrow r^a + \delta r^a , \quad \theta^a \longrightarrow \theta^a + \delta\theta^a$$

$$S \rightarrow S + \sum_a \delta t^a \int G_{-1}\bar{G}_{-1}\Phi[\omega](z,\bar{z})$$
$$+ \delta\bar{t}^a \int G_0^+ \bar{G}_0^+ \bar{\Phi}[\omega](z,\bar{z})$$

$$\bar{\Phi} = \omega_{a\, i\bar{j}}\, \rho_z^{\bar{j}} \rho_{\bar{z}}^{i} \qquad \begin{aligned} \delta t^a &= \delta(r^a + i\theta^a) \\ \delta\bar{t}^a &= \delta(r^a - i\theta^a) \end{aligned}$$

B-twist $\quad A_z = \frac{1}{2}\omega_z \quad , \quad A_{\bar z} = \frac{1}{2}\omega_{\bar z}$

( N.B. $\quad A_{\bar z} \neq (A_z)^* \Rightarrow$ make sense only for CY )

$\psi_R{}^i = \rho_{\bar z}{}^i$ : one-form , $\quad \psi_R{}^{\bar i} = \tilde\chi^{\bar i}$ : zero-form

$\psi_L{}^i = \rho_z{}^i$ : one-form , $\quad \psi_L{}^{\bar i} = \chi^{\bar i}$ : zero-form

BRST currents $\quad G'^+ = g_{i\bar j}\, \chi^{\bar j} \partial_z X^i \ , \quad \bar G^+ = g_{i\bar j}\, \tilde\chi^{\bar j} \partial_{\bar z} X^i$

$G^-$ of the A-model

anti-ghosts : $G'^-$, $\bar G^-$

BRST transformation $\qquad X^i \ , \ \chi^{\bar i} \ , \ \tilde\chi^{\bar i}$ : invariant

$\delta_L X^{\bar i} = \chi^{\bar i} \qquad \delta_R X^{\bar i} = \tilde\chi^{\bar i}$

$\delta_L \rho_{\bar z}{}^i = \partial_{\bar z} X^i + \cdots \ , \quad \delta_L \rho_z{}^i = \partial_z X^i + \cdots$

BRST invariant observables

$$\chi_{(+)}^{\bar i} = \chi^{\bar i} + \tilde\chi^{\bar i} \ , \quad \chi_{(-)i} = g_{i\bar j}\,(\chi^{\bar j} - \tilde\chi^{\bar j})$$

- $v \in H^q_{\bar\partial}(M, \overset{p}{\wedge} T_M)$

$$\rightarrow \ \varphi[v] = v_{\bar j_1 \cdots \bar j_q}{}^{i_1 \cdots i_p}\, \chi_{(+)}^{\bar j_1} \cdots \chi_{(+)}^{\bar j_q}\, \chi_{(-)i_1} \cdots \chi_{(-)i_p}$$

$$\boxed{Q_L + Q_R = \bar\partial} \qquad Q_L = \frac{1}{2}(\bar\partial + \partial^+)$$

$$Q_R = \frac{1}{2}(\bar\partial - \partial^+)$$

$F_{L,R} = \oint J_{L,R}$ are <u>not</u> equal to the degrees of $v$

$$= \frac{1}{2}(ik - ik^+ \pm (p-q))$$

Especially for $v \in H^1_{\bar\partial}(M, T_M)$,

$$\varphi[v] = v_{\bar j}{}^{i}\, g_{i\bar k}\, \chi_{(+)}^{\bar j}\, \chi_{(-)}^{\bar k}$$

Deformation of complex structure

$$\frac{\partial}{\partial z^{\bar i}} \rightarrow \frac{\partial}{\partial z^{\bar i}} + \eta_{\bar i}{}^{j} \frac{\partial}{\partial z^{j}} , \quad \eta = \sum_\alpha \delta y^\alpha v_\alpha$$

$$S \rightarrow S + \sum_\alpha \delta y^\alpha \int G'^{-}_{-1} \bar G^{-}_{-1}\, \varphi[v_\alpha] + \delta \bar y^{\alpha} \int G'^{+}_{0} \bar G^{+}_{0}\, \varphi[v_\alpha]^\dagger$$

Summary

N=2 σ-model $\left\langle \begin{array}{l} \text{A-twist : physical observables} \leftrightarrow \bigoplus_{p,q} H^{p,q}(M) \\ \text{B-twist : } \quad \text{"} \quad \leftrightarrow \bigoplus_{p,q} H^{q}_{\bar\partial}(M, \wedge^{p} T_M) \end{array} \right.$

In the A-model, deformation of complex structure is BRST trivial.

In the B-model, Kähler deformation is BRST trivial.

Geometry of moduli space of TCFT

TCFT = twisted $N=2$ conformal field theory

$T,\ G^+,\ G^-,\ J,\ \hat{c}$ = central charge
( = dim $M$ for $N=2$ $\sigma$-model )

In this lecture, we consider the case when the original $N=2$ SCFT is unitary.

notations

hermitian conjugate in the original $N=2$ SCFT

$G^+,\ \bar{G}^+$ : BRST currents $\quad Q_L = \oint G^+ dz$, $\quad Q_R = \oint \bar{G}^+ d\bar{z}$

$G^-,\ \bar{G}^-$ : anti-ghosts $\quad T = \frac{1}{2}\{Q_L, G^-\}$, $\quad \bar{T} = \frac{1}{2}\{Q_R, \bar{G}^-\}$

twisted EM tensor

$J,\ \bar{J}$ : $U(1)$ currents $\quad F_L = \oint J dz$, $\ F_R = \oint \bar{J} d\bar{z}$

chiral primary fields

$\phi_a(z,\bar{z})$, $\quad [Q, \phi_a] = 0$, $\quad \phi_a \neq \{Q, *\}$

( In the A-model, $\phi = \omega_{i_1\cdots i_p \bar{j}_1\cdots \bar{j}_q} \chi^{i_1}\cdots\chi^{i_p}\chi^{\bar{j}_1}\cdots\chi^{\bar{j}_q}$

$[F_L, \phi_a] = q_a \phi_a$, $\quad [F_R, \phi_a] = \bar{q}_a \phi_a$

chiral ring

$\phi_a(z)\,\phi_b(w) = C_{ab}{}^{c}\,\phi_c(w) + \{Q, *\}$

topological metric
$$\eta_{ab} = \langle \phi_a(z_1)\, \phi_b(z_2) \rangle \quad \text{on } S^2$$

Yukawa coupling
$$C_{abc} = \langle \phi_a(z_1)\, \phi_b(z_2)\, \phi_c(z_3) \rangle = C_{ab}{}^{c'}\, \eta_{cc'}$$

From the point of view of the original $N=2$ SCFT:

twisting $\quad \frac{1}{2}\int d^2z\,(\omega_z J_{\bar z} + \omega_{\bar z} J_z) \underset{\uparrow}{=} -\frac{i}{2}\int d^2z\, R_{z\bar z}\,\varphi$

( bosonization $J_z = i\,\partial_z \varphi$ )

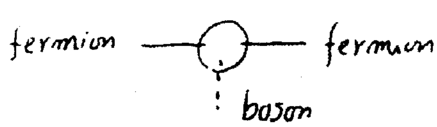
$$\int_\Sigma d^2z\, R_{z\bar z} = 2-2g \qquad (g : \text{genus of } \Sigma).$$

On $S^2$: we can choose a metric so that
$$R_{z\bar z} = \delta(z-z_1) + \delta(z-z_2)$$

We then have
$$e^{\pm\frac{1}{2}\int d^2z\,\omega_{\bar z} J_z} = e^{-\frac{i}{2}\varphi(z_1)}\, e^{-\frac{i}{2}\varphi(z_2)}.$$

$$\therefore \quad \eta_{ab} = \langle \phi_a(z_1)\, \phi_b(z_2) \rangle^{TCFT}$$
$$= \langle \phi_a(z_1)\, e^{-\frac{i}{2}\varphi(z_1)} \quad \phi_b(z_2)\, e^{-\frac{i}{2}\varphi(z_2)} \rangle^{N=2\,SCFT}$$
$$C_{abc} = \langle \underline{\phi_a(z_1)\, e^{-\frac{i}{2}\varphi(z_1)}} \quad \underline{\phi_b(z_2)\, e^{-\frac{i}{2}\varphi(z_2)}} \quad \phi_c(z_3) \rangle^{N=2\,SCFT}$$

Ramond operator $\qquad\qquad$ Neveu-Schwarz operator



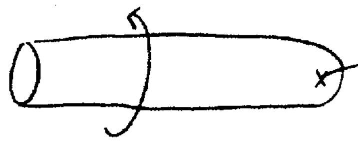
: Yukawa coupling

$$J(z)J(w) \sim \frac{\hat{c}}{(z-w)^2} \quad \Rightarrow \quad e^{-\frac{1}{2}\phi} \;:\; \text{charge } \frac{\hat{c}}{2} \text{ for both } L/R$$

$$\Rightarrow \quad \eta_{ab} \neq 0 \;:\; q_a + q_b = \bar{q}_a + \bar{q}_b = \hat{c}.$$

$$C_{abc} \neq 0 \;:\; q_a + q_b + q_c = \bar{q}_a + \bar{q}_b + \bar{q}_c = \hat{c}$$

$|a\rangle$ =

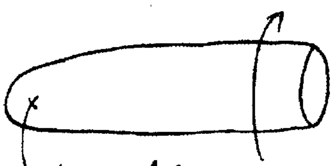


$\phi_a e^{-\frac{1}{2}\varphi}$

Ramond

in the original $N=2$ SCFT

$\langle a|$ =



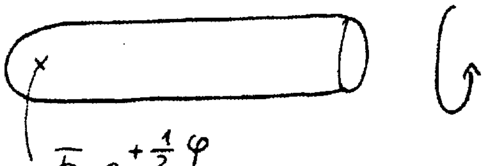
$\phi_a e^{-\frac{1}{2}\varphi}$ Ramond

$$\eta_{ab} = \langle a|b\rangle, \quad C_{abc} = \langle a|\phi_b|c\rangle$$

*$tt^*$ metric*

$\langle a|$ is not hermitian conjugate to $|a\rangle$ in the original $N=2$ SCFT.

$\langle \bar{a}|$ =



Ramond

$\bar{\phi}_a e^{+\frac{1}{2}\varphi}$

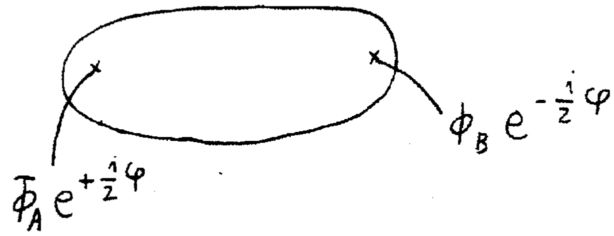
↑ hermitian conjugate to $\phi_a$ in the original $N=2$

*$tt^*$ metric*

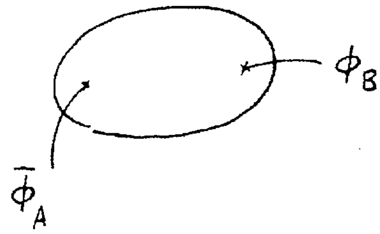
$$g_{\bar{a}b} = \langle \bar{a}|b\rangle$$

Relation to Zamolodchikov metric (for marginal operator, $\phi_A$ $q_A = \bar{q}_A = 1$)

$\langle \bar{A} | B \rangle$ =




= (const) x




For marginal operators, $\dfrac{\langle \bar{A} | B \rangle}{\langle 0 | 0 \rangle} = G_{\bar{A}B}$ : Zamolodchikov metric

marginal deformation of TCFT

$\phi_A$ : chiral primary field with $q_A = \bar{q}_A = 1$

$$\left\{ \begin{array}{l} [Q, \int d^2z \, G^-_{-1} \bar{G}^-_{-1} \phi_A] = 0 \\ [F, G^-_{-1} \bar{G}^-_{-1} \phi_A] = 0 \end{array} \right.$$

$$S \to S + \sum_A \Big( t^A \int d^2z \, G^-_{-1} \bar{G}^-_{-1} \phi_A + \bar{t}^A \int d^2z \, G^+_0 \bar{G}^+_0 \bar{\Phi}_A \Big)$$

This preserves the topological invariance.

In the A-model,

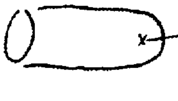
$$\phi_A = \omega_{A i \bar{j}} \chi^i \chi^{\bar{j}}, \quad \bar{\Phi}_A = \bar{\omega}_{A \bar{i} j} \rho^{\bar{i}}_{z} \rho^{j}_{\bar{z}}$$

$$t^A = r^A + i\theta^A, \quad r^A : \text{Kahler class}$$

$$\frac{\partial}{\partial \bar{t}^A} C_{abc} = 0$$

Special geometry $\quad \mathcal{M}$ : moduli space of TCFT

$(t^A, \bar{t}^A)$ : coordinates

- Normalization of $|0\rangle =$  $e^{-\frac{1}{2}\phi(0)}$ is not unique.

Since $\displaystyle \frac{\partial}{\partial \bar{t}^A} |0\rangle = \left(\int d^2z\, G_0^+ \bar{G}_0^+ \bar{\Phi}_A \right)|0\rangle = Q(\cdots)$,

the normalization ambiguity is holomorphic in $t$ ;

i.e $\quad |0\rangle \to e^{f(t)} |0\rangle$

The vacuum $|0\rangle$ defines a line-bundle $\mathcal{L}$ over $\mathcal{M}$.

$$e^{-K} = \langle \bar{0} | 0 \rangle \quad , \quad K(t,\bar{t}) \to K(t,\bar{t}) + f(t) + \bar{f}(\bar{t}).$$

- $$G_{\bar{A}B} = \frac{\langle \bar{A} | B \rangle}{\langle \bar{0} | 0 \rangle} = \partial_{\bar{A}} \partial_B K$$

$G_{\bar{A}B}$ is $\left\{ \begin{array}{l} \text{Kähler metric} \\ \text{curvature of } \mathcal{L} \end{array} \right.$ $\qquad \mathcal{M}$ : restricted Kähler

$$\left( \because \right) \left( \frac{\partial}{\partial t^B} - \frac{\langle \bar{0} | \partial_B | 0 \rangle}{\langle \bar{0} | 0 \rangle} \right) |0\rangle \sim |B\rangle \text{ ; orthogonal to } |0\rangle$$

For A-model on $M$ ($\dim M = 3$),

$$\phi_A = \omega_{A i \bar{j}} \chi^i \chi^{\bar{j}} \qquad \omega_A \in H^{1,1}(M),$$

$$C_{ABC} = \int_M \omega_A \wedge \omega_B \wedge \omega_C + \sum_n N_n^{(0)} n_A n_B n_C \frac{e^{-\langle n,t \rangle}}{1 - e^{-\langle n,t \rangle}}$$

$$\left( \because \right) \quad \frac{\partial}{\partial \bar{t}} C_{ABC} = 0 \,, \quad \bar{t} \to \infty : \text{holomorphic map } S^2 \to M \Big)$$

Greene's talk

For B-model on $M$.

$$\phi_A = v_{A \bar{j}}{}^{i} g_{i\bar{k}} \chi_{(+)}^{\bar{j}} \chi_{(-)}^{\bar{k}} \qquad v_A \in H^1_{\bar{\partial}}(T_M)$$

In the B-model, Kähler moduli are BRST trivial.

volume$(M) \to \infty$ : semi-classical limit of $\sigma$-model

$$G_{\bar{A}B} = - \frac{1}{\int_M \bar{\Omega} \wedge \Omega} \int (\bar{\eta}_A \bar{\Omega}) \wedge (\eta_B \Omega)$$

$$= \partial_{\bar{A}} \partial_B K \quad : \text{Weil-Peterson metric}$$

$$e^{-K} = \int_M \bar{\Omega} \wedge \Omega$$

$$C_{ABC} = - \int_M \bar{\Omega} \wedge \partial_A \partial_B \partial_C \Omega = \int_M \partial_A \Omega \wedge (\partial_B \Omega)^\vee \wedge \partial_C \Omega$$

$$\left( \text{N.B. } \partial_A \Omega = \underbrace{v_A \Omega}_{(2,1)} + \underbrace{\cdots}_{(3,0)} \,, \quad (\partial_B \Omega)^\vee = v_B + \cdots \right)$$

Coupling of topological $\sigma$-model to 2d gravity (topological string

$$T = \frac{1}{2}\{Q_L, G^-\}, \quad \bar{T} = \frac{1}{2}\{Q_R, \bar{G}^-\}$$

$G^-, \bar{G}^-$ : anti-ghost

topological string amplitude on genus $g \geq 2$

$$F_g = \int_{\mathcal{M}_g} \langle \prod_{k=1}^{3g-3} \int_\Sigma \mu_k G^- \int_\Sigma \bar{\mu}_k \bar{G}^- \rangle$$

$\mu_k$ $(k=1,\cdots,3g-3)$ : Beltrami differentials

$$\int_\Sigma R = 2-2g \Rightarrow \langle (\cdots) \rangle^{TCFT} = \langle (\cdots) \prod_{i=1}^{2g-2} e^{\frac{1}{2}\varphi(r_i)} \rangle^{N=2\ SC}$$

$\therefore \langle (\cdots) \rangle^{TCFT} = 0$

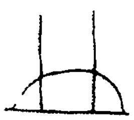
unless $(\cdots)$ has charge $-\hat{c}(g-1)$

$\therefore F_g \neq 0$ only when $\hat{c} = 3$

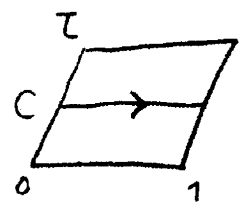
e.g. $\sigma$-model on $M$ with $\underline{\dim M = 3}$.

topological string amplitude at $g=1$, with one-puncture

$$\frac{1}{2}\int d^2\tau \langle \phi_A(0) \int_\Sigma \mu G^- \int_\Sigma \bar{\mu} \bar{G}^- \rangle =$$



$$= \frac{1}{2}\int d^2\tau \langle \phi_A(0) \oint_C G^- \oint_C \bar{G}^- \rangle$$




$q_A = \bar{q}_A = +1$

$$F_1 = \frac{1}{2} \int \frac{d^2\tau}{\mathrm{Im}\,\tau} \langle \oint_C J \oint_C \bar{J} \rangle$$

$$\frac{\partial}{\partial t^A} F_1 = \frac{1}{2} \int \frac{d^2\tau}{\mathrm{Im}\,\tau} \langle \int_\Sigma G^-_{-1} \bar{G}^-_{-1} \phi_A \oint_C J \oint_C \bar{J} \rangle$$

$$= \frac{1}{2} \int d^2\tau \langle \phi_A(0) \oint_C G^- \oint_C \bar{G}^- \rangle$$

$F_1$ : topological string amplitude at $g=1$

At $g=0$, $\frac{\partial}{\partial \bar{t}} C_{ABC} = 0$. What about $g \geq 1$?

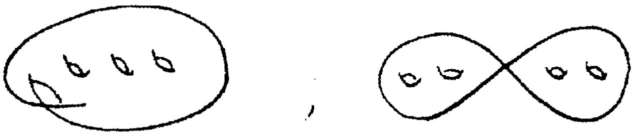
$$\frac{\partial}{\partial \bar{t}^A} F_g = \int_{\mathcal{M}_g} \langle \int_\Sigma G_0^+ \bar{G}_0^+ \bar{\phi}_A \prod_{k=1}^{3g-3} \int_\Sigma \mu_k G^- \int_\Sigma \bar{\mu}_k \bar{G}^- \rangle$$

$$= \int_{\mathcal{M}_g} \sum_{k',k''} \langle \int \bar{\phi}_A \int 2\mu_{k'} T \int 2\bar{\mu}_{k''} \bar{T} \times$$
$$\times \prod_{k \neq k'} \int \mu_k G^- \prod_{k \neq k''} \int \bar{\mu}_k \bar{G}^- \rangle$$

$$= \int_{\mathcal{M}_g} 4 \sum_{k',k''} \frac{\partial^2}{\partial m_{k'} \partial \bar{m}_{k''}} \langle \int \bar{\phi}_A \prod_{k \neq k'} \int \mu_k G^- \prod_{k \neq k''} \int \bar{\mu}_k \bar{G}^- \rangle$$

$\Downarrow$

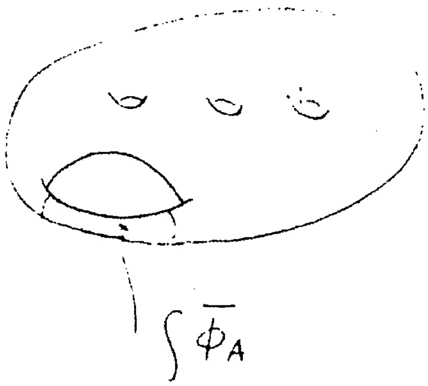
boundary of $\mathcal{M}_g$

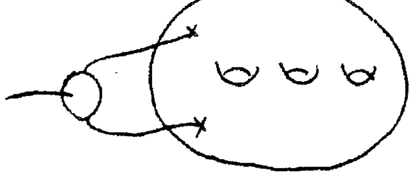
$\partial \mathcal{M}_g \ni$ 

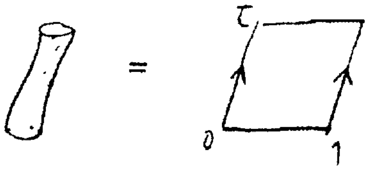
$$\langle \int_\Sigma \bar{\Phi}_A \cdots \rangle \longrightarrow ? \quad \text{near } \partial\mathcal{M}_g$$

If this tends to be finite at $\partial\mathcal{M}_g$, $\frac{\partial}{\partial \bar{t}^A} F_g = 0$.


$\longrightarrow$ finite




$\simeq \mathrm{Im}\,\tau \times \frac{1}{2} \times \bar{\Phi}_A$ 




$$\parallel$$

$$\frac{1}{2} \bar{C}_{\bar{A}\bar{B}\bar{C}}\, g^{\bar{B}B} g^{\bar{C}C} \langle \int G_{-1}^- \bar{G}_{-1}^- \phi_B \int G_{-1}^- \bar{G}_{-1}^- \cdots$$

$$\times \prod \int \mu_a G^- \int \bar{\mu}_a \bar{G}^- \cdots$$

$$\parallel$$

$$\frac{1}{2} \bar{C}_{\bar{A}\bar{B}\bar{C}}\, e^{2K} G^{B\bar{B}} G^{C\bar{C}} \underbrace{D_B D_C}_{\text{Some covariant derivative}} F_{g-1}$$

$$\frac{\partial}{\partial \bar{t}^{\bar{A}}} F_g = \frac{1}{2} \bar{C}_{\bar{A}\bar{B}\bar{C}}\, e^{2K} G^{\bar{B}B} G^{\bar{C}C} \times$$

$$\times \left( D_B D_C F_{g-1} + \sum_r D_B F_r D_C F_{g-r} \right)$$

Holomorphic anomaly

Comparing the both hand sides $\rightarrow$ $F_g$ : section of $\mathcal{L}^{2-2g}$

This also follows from the sewing axiom of TCFT.

$$D_A F_g = (\partial_A - (2g-2)\partial_A K) F_g$$

- Integrability of the holomorphic anomaly

master equation $\mathcal{F}(\lambda; t, \bar{t}) = \sum_{g=1}^{\infty} \lambda^{2g-2} F_g(t, \bar{t})$

$\mathcal{F}$ : function on the total space of $\mathcal{L}$

($\lambda$ : coordinate on the fiber)

$$(\partial_{\bar{A}} - \partial_{\bar{A}} F_1) e^{\mathcal{F}} = \frac{1}{2}\lambda^2 \bar{C}_{\bar{A}\bar{B}\bar{C}} e^{2K} G^{\bar{B}B} G^{\bar{C}C} \hat{D}_B \hat{D}_C e^{\mathcal{F}}$$

$$\hat{D}_i \mathcal{F} = (\partial_A - \partial_A K \lambda \frac{\partial}{\partial \lambda}) \mathcal{F} \quad \text{etc}$$

$$d_{\bar{A}} = \partial_{\bar{A}} - \partial_{\bar{A}} F_1 - \frac{1}{2}\lambda^2 \bar{C}_{\bar{A}\bar{B}\bar{C}} e^{2K} G^{\bar{B}B} G^{\bar{C}C} \hat{D}_B \hat{D}_C$$

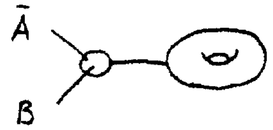
$$[d_{\bar{A}}, d_{\bar{B}}] = 0$$

For this equation to hold, it is sufficient that $F_1$ satisfies

$$\partial_{\bar{A}} \partial_B F_1 = \frac{1}{2} \operatorname{Tr} \bar{C}_{\bar{A}} C_B - \frac{\chi}{24} G_{\bar{A}B} \quad \text{for some } \chi$$

This in fact is the case.

$$\partial_B F_1 = \frac{1}{2}\int d^2\tau \, \langle \Phi_B(0) \oint_C G^- \oint_C \bar{G}^- \rangle$$

$$\partial_{\bar{A}} \partial_B F_1 = \frac{1}{2}\Big($$  $$+$$  $$\Big)$$

$$\text{first term} = \mathrm{Tr}\, \bar{C}_{\bar{A}} C_B \qquad \text{second term} = -\frac{1}{12} G_{\bar{A}B} \cdot \chi \,, \quad \chi = \mathrm{Tr}(-1)^F$$

- *Integration of the anomalies.*

$\underline{g=1}$ $(\hat{c} = 3)$

$$\partial_{\bar{A}} \partial_B F_1 = \frac{1}{2} \bar{C}_{\bar{A}\bar{C}\bar{D}} C_{BCD} \, e^{2K} G^{C\bar{C}} G^{D\bar{D}} + G_{\bar{A}B} - \frac{\chi}{24} G_{\bar{A}B}$$

$$= -\frac{1}{2} R_{\bar{A}B} + \frac{1}{2}\left(h + 3 - \frac{\chi}{12}\right) G_{\bar{A}B} \,,$$

$h = \#$ marginal operators $= \dim \mathcal{M}$

$$R_{\bar{A}B} = \partial_{\bar{A}} \partial_B \log \det G$$

$$\Rightarrow \quad F_1 = \log\left[ (\det G)^{-1/2} \, e^{\frac{1}{2}(h+3-\frac{\chi}{12})K} \cdot |f(t)|^2 \right]$$

$f(t)$ : (projective) holomorphic section of

$$\mathcal{L}^{\frac{1}{2}(h+3-\chi/12)} \otimes \left(\overset{h}{\wedge} T^*_{\mathcal{M}}\right)^{1/2}$$

To determine $f(t)$, we need to know behavior of $F_1$ near boundaries of $\mathcal{M}$.

$g \geq 2$

$$\partial_{\bar{A}} F_g = \frac{1}{2} \bar{C}_{\bar{A}\bar{B}\bar{C}}\, e^{2K} G^{B\bar{B}} G^{C\bar{C}} \left( D_B D_C F_{g-1} + \sum_{r=1}^{g-1} D_B F_r D_C F_{g-r} \right)$$

$C_{ABC}$ : Yukawa coupling

$$D_A C_{BCD} = D_B C_{ACD} \Rightarrow \quad \begin{aligned} C_{ABC} &= D_A C_{BC} \\ C_{BC} &= D_B C_C \,, \quad C_C = D_C C \end{aligned}$$

$C$ : local section of $\mathcal{L}^2$

In fact, in many cases, we can construct such $C$ globally on $\mathcal{M}$.

$$\bar{C}_{\bar{A}\bar{B}\bar{C}}\, e^{2K} G^{B\bar{B}} G^{C\bar{C}} = \partial_{\bar{A}} \left[ \bar{C}_{\bar{B}\bar{C}}\, e^{2K} G^{B\bar{B}} G^{C\bar{C}} \right]$$

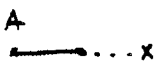
$$\therefore \partial_{\bar{A}} \left[ F_g - \frac{1}{2} \bar{C}_{\bar{B}\bar{C}}\, e^{2K} G^{B\bar{B}} G^{C\bar{C}} \left( D_B D_C F_{g-1} + \sum_{r=1}^{g-1} D_B F_r D_C F_{g-r} \right) \right]$$

$$= -\frac{1}{2} \bar{C}_{\bar{B}\bar{C}}\, e^{2K} G^{B\bar{B}} G^{C\bar{C}}\, \partial_{\bar{A}} \left( \cdots \right)$$

By repeating this procedure, we find

$\partial_{\bar{A}} [ F_g + \cdots ] = 0$ where $[\cdots]$ is a sum over

connected Feynmann graphs with propagators



$= - e^{2K} G^{A\bar{A}} G^{B\bar{B}} \bar{C}_{\bar{A}\bar{B}} \times \lambda^2$

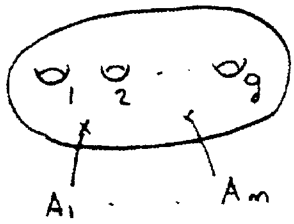


$= - e^{2K} G^{A\bar{A}} \bar{C}_{\bar{A}} \times \lambda^2$

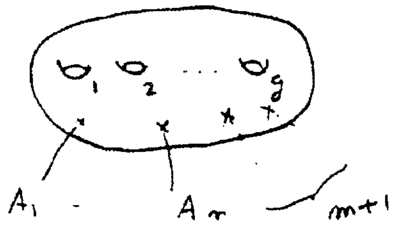


$= -2 e^{2K} \bar{C} \times \lambda^2$

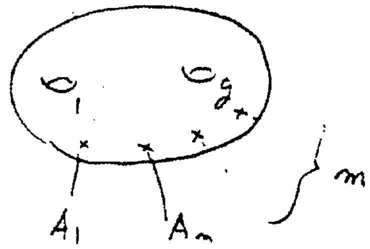
and vertices




$$= D_{A_1} \cdots D_{A_m} F_g \cdot \lambda^{2g-2}$$




$$= (2g-2+n+m)$$




$$= \frac{\chi}{24} - 1$$

We have proven this to all order in $g$

by using the Schwinger-Dyson equation for a finite-dim system.

$$\therefore \; F_g = -\,(\text{connected vacuum graphs at } g\text{-loop}) + f_g(t)$$

$f_g(t)$ : holomorphic section of $\mathcal{L}^{2-2g}$

Is it possible to choose $\bar{C}$, $\bar{C}_{\bar{A}}$, $\bar{C}_{\bar{A}\bar{B}}$ so that $f_g(t) = 0$

Back to topological $\sigma$-model

$M$ : Calabi-Yau manifold with Kähler moduli $(t^a, \bar{t}^a)$
complex moduli $(y^\alpha, \bar{y}^\alpha)$

$\Downarrow$

$N=2$ $\sigma$-model

$\Downarrow$ twistings

$$\begin{cases} \text{A-twist} & \phi_a : \text{marginal} \Leftrightarrow \omega_a \in H^{1,1}(M) \Leftrightarrow \delta t^a \\ \text{B-twist} & \varphi_\alpha : \text{marginal} \Leftrightarrow v_\alpha \in H^1_{\bar\partial}(M, T_M) \Leftrightarrow \delta y^\alpha \end{cases}$$

Under variations $t \to t + \delta t$, $y \to y + \delta y$,
the $\sigma$-model action with the <u>A-twist</u> changes as

$$S \to S + \sum_a \delta t^a \int G^-_{-1} \bar{G}^-_{-1} \phi_a + \delta \bar{t}^a \int G^+_0 \bar{G}^+_0 \bar{\Phi}_a +$$
$$+ \sum_\alpha \delta y^\alpha \int G^+_0 \bar{G}^-_{-1} \varphi_\alpha + \delta \bar{y}^\alpha \int G^-_{-1} \bar{G}^+_0 \varphi_\alpha$$

(In the B-model $G^+_0 \to G'^-_{-1}$, $G^-_{-1} \to G'^+_0$)

We saw on general ground that

$\dfrac{\partial}{\partial \bar{t}^a} F_g \neq 0$ due to the holomorphic anomaly.

On the other hand, $\dfrac{\partial}{\partial y^\alpha} F_g = 0$, $\dfrac{\partial}{\partial \bar{y}^\alpha} F_g = 0$.

$\Big[\because)$ $\displaystyle\int_\Sigma \bar{G}^-_{-1} \varphi_\alpha$ commutes with $\displaystyle\int_\Sigma \bar{\mu}_k \bar{G}^-$.

c.f. $\displaystyle\int \bar{G}^+_0 \bar{\Phi}_a$ does not commute with $\displaystyle\int_\Sigma \bar{\mu}_k \bar{G}^+$ $\Big]$

Similarly $\frac{\partial}{\partial t^a} F_g = 0$ , $\frac{\partial}{\partial \bar{t}^a} F_g = 0$ in the B-model

Geometric meaning of $F_g$

$g=1$

In the B-model , $F_1$ is independent of vol(M).

$$F_1 = \frac{1}{2}\int \frac{d^2\tau}{\mathrm{Im}\,\tau} \left\langle \oint_C J \oint_C \bar{J} \right\rangle$$

When vol(M) $\to \infty$ , for a fixed value of $\tau$,
only constant maps $\Sigma \to p \in M$ survive.

However , if we send $\tau$ to the boundary of the moduli space

 simultaneously as vol(M) $\to \infty$

we get contributions from point particles propagations along the long thin tube.

In the point particle limit , the Hilbert space of the B-model becomes

$$\bigoplus_{p,q} \Omega^{(0,q)}(M, \overset{p}{\wedge} T_M)$$
$$\ni \quad v_{\bar{j}_1 \cdots \bar{j}_q}{}^{i_1 \cdots i_p}\, d\bar{z}^{\bar{j}_1} \cdots d\bar{z}^{\bar{j}_q}\, \partial_{i_1} \cdots \partial_{i_p}$$

$\oint J = \frac{1}{2}(ik - ik^+ + p - q)$ k : wedging with the Kähler form

$\oint \bar{J} = \frac{1}{2}(ik - ik^+ - p + q)$

$$-(k-k^+)^2 = (p+q-n)^2 \quad \text{on } \Omega^{(0,q)}(\overset{p}{\wedge} T_M)$$

Therefore, in the large volume limit.

$$F_1 = \frac{1}{2} \log \prod_{p,q} \left(\det{}' \Delta^{(p,q)}\right)^{(-1)^{p+q}\, p \cdot q}$$

Ray-Singer torsion

$\Delta^{(p,q)}$ : Laplacian acting on $\Omega^{(0,q)}(\overset{p}{\wedge} T_M)$

In this case, the holomorphic anomaly

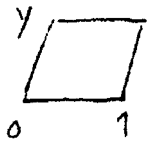
$$\frac{\partial^2}{\partial \bar{y}^\alpha \partial y^\beta} F_1 = \frac{1}{2} \mathrm{Tr}\, \bar{C}_\alpha C_\beta - \frac{\chi}{24} G_{\bar{\alpha}\beta}$$

reduces to the Quillen's formula

(first term) : zero-mode subtraction

(second term) : $\pi_*\int_M Td(T_M) \sum_{q=0}^{n} (-1)^q\, q\, Ch(\overset{n-q}{\wedge} T^*)$

example 2d torus




$$F_1 = \frac{1}{2} \log \left( \frac{1}{\mathrm{Im}\, y\, |\eta(y)|^2} \right)$$

In the A-model, if $\frac{\partial}{\partial \bar{t}^a} F_1 = 0$ were true.

$F_1$ would be given as a sum over holomorphic maps.

However the holomorphic anomaly gives an obstruction.

We can recover a sum over holomorphic maps by taking the limit $\bar{t} \to \infty$, after computing $F_1$

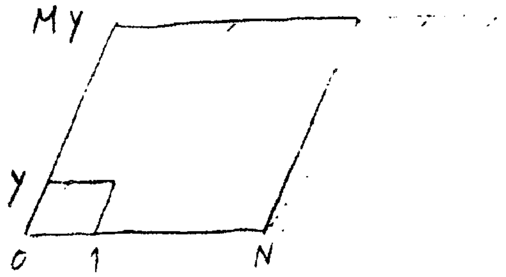
example : 2d torus $\quad t = \mathrm{vol}(M) + i\theta$

$$-\frac{\partial}{\partial t} F_1 = \frac{1}{2}\int d^2\tau \, \langle \phi(0) \oint_C G^- \oint_{\bar C} \bar G^- \rangle$$

$$\cong -\frac{1}{24} + \sum_{N,M=1}^{\infty} N e^{-NMt} \qquad (\bar t \to \infty)$$

constant map : $\displaystyle\int \frac{d^2\tau}{4\pi (\mathrm{Im}\,\tau)^2} = \frac{1}{12}$

holomorphic map of degree $NM$ :




There are $N$ maps which cover the target space $N$ times in 1-direction and $M$ times in $y$-direction.

$$= \frac{1}{2}\frac{\partial}{\partial t} \log\left(\frac{1}{\eta(it)}\right)$$

$$\therefore \; F_1 = \frac{1}{2}\log\left(\frac{1}{\mathrm{vol}(M)\,|\eta(it)|^2}\right)$$

---

In general,

$$-\frac{\partial}{\partial t^a} F_1 = \frac{(-1)^{\dim M}}{24}\int_M \omega_a \wedge C_{\dim M-1} + \sum_n n_a N_n^{(1)} \times \sum_{m=1}^{\infty} \frac{e^{-m\langle n,t\rangle}}{1-e^{-m\langle n,t\rangle}}$$

$$+ \frac{1}{12}\sum_n n_a N_n^{(0)} \frac{e^{-\langle n,t\rangle}}{1-e^{-\langle n,t\rangle}}$$

$$F_1 = \log\left[ (\det G)^{-1/2} e^{\frac{1}{2}(h+3-\frac{\chi}{12})K} |f|^2 \right]$$

If there is a mirror pair $M$. $\tilde{M}$,

we can compute $G_{a\bar{b}}$. $K$ for the A-model on $M$
by using the B-model on $\tilde{M}$.

The boundary condition on $f(t)$ can be imposed
by studying $t \to \infty$ limit of the A-model.

$g \geq 2$

- A-model , as $\bar{t} \to \infty$

$$F_g \doteq \frac{1}{2}\chi(M)\int_{\mathcal{M}_g} c_{g-1}^{3} + \sum_n N_n^{(g)} e^{-\langle n,t\rangle} + \text{bubblings}$$

$$\left( N_n^{(g)} = \#\ \Sigma_g \text{ of degree } n \text{ in } M \right)$$

For $g=2$

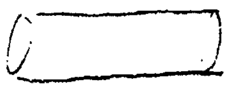
$$F_2 = \frac{\chi}{5760} + \sum_n N_n^{(2)} e^{-\langle n.t\rangle} + \frac{1}{240}\sum_n N_n^{(0)} \frac{e^{-\langle n,t\rangle}}{(1-e^{-\langle n,t\rangle})^2}$$

Euler character of $\mathcal{M}_{g=2}$ ?

- B-model

$g$-loop amplitude of QFT on $M$ ?

string propagator $\quad G_0^-, \bar{G}_0^-$ . anti-ghost

 $$= \frac{(G_0^- + \bar{G}_0^-)(G_0^- - \bar{G}_0^-)}{L_0 + \bar{L}_0} \to \frac{\bar{\partial}^{+}\partial}{\Delta}$$

in the limit $\mathrm{vol}(M) \to \infty$

interaction $\quad \int_M v_1 \wedge v_2 \wedge v_3 \quad$ ?

Kodaira - Spencer theory

deformation of complex structure

$$\partial_{\bar{i}} \rightarrow \partial_{\bar{i}} + \mathcal{A}_{\bar{i}}{}^{j} \partial_j \quad , \quad \mathcal{A} \in \Omega^{(0,1)}(M, T_M)$$

integrability : $\bar{\partial}\mathcal{A} + \frac{1}{2} \underbrace{[\mathcal{A}, \mathcal{A}]}_{\mathcal{A}_{\bar{i}}{}^{l} \partial_l \mathcal{A}_{\bar{j}}{}^{k} - \mathcal{A}_{\bar{j}}{}^{l} \partial_l \mathcal{A}_{\bar{i}}{}^{k}} = 0$

linearization $\bar{\partial}\mathcal{A} = 0$ , diffeo $\mathcal{A} \rightarrow \mathcal{A} + \bar{\partial}\varepsilon$

$\therefore$ at the linearized level , $\mathcal{A} \in H^1_{\bar{\partial}}(M, T_M)$

Therefore we set

$$\mathcal{A} = x + A \qquad \text{where}$$

- $x \in H^1_{\bar{\partial}}(T_M)$
- $A$ : orthogonal to $H^1_{\bar{\partial}}(T_M)$

i.e. $\int_M A' \wedge \omega = 0$ for $\forall \omega \in H^{1,2}(M)$

$$A'_{ij\bar{k}} = A_{\bar{k}}{}^{l} \Omega_{ijl} \in \Omega^{(2,1)}(M)$$

Since $x \in H^1_{\bar{\partial}}(T_M)$ , $x' \in H^{2,1}(M)$ and $\partial x' = 0$ , $\bar{\partial} x' = 0$ .

The Kodaira - Spencer equation has a unique solution of the form

$$\mathcal{A} = x + A(x)$$

which satisfies

(1) $\partial \mathcal{A}' = 0$ $\Leftarrow$ holomorphic 3-form exists after deformation

(2) $\bar{\partial}^{\dagger} \mathcal{A} = 0$ : gauge-fixing for diffeo.

[proof]

lemma $\partial A' = 0$, $\partial B' = 0$ $\Rightarrow$ $[A, B]' = \partial (A \wedge B)'$

Therefore the K-S equation can be written as

$$\bar{\partial} \mathcal{A}' + \frac{1}{2} \partial (\mathcal{A} \wedge \mathcal{A})' = 0$$

Let us expand $\mathcal{A} = \sum_{n=1}^{\infty} \varepsilon^n A_n$,

$$A_1 = x, \quad A_{n \geq 2} \perp H^1_{\bar{\partial}}(T)$$

$O(\varepsilon)$ : $\bar{\partial} x' = 0$

$O(\varepsilon^2)$ : $\bar{\partial} A_2' + \frac{1}{2} \partial (x \wedge x)' = 0$

Is there an obstruction to solve this equation?

$$\boxed{\begin{array}{l} \partial\bar{\partial}\text{-lemma} : \quad \omega = \partial \rho \ \ \& \ \ \bar{\partial}\omega = 0 \\ \qquad\qquad\qquad\qquad \Downarrow \\ \qquad\qquad\qquad \exists \phi, \ \omega = \partial\bar{\partial}\phi \end{array}}$$

$\rho = (x \wedge x)'$

$\bar{\partial} \rho = 0$

$\because)$ $\bar{\partial} \Omega = 0$

Because of the $\partial\bar\partial$-lemma, we can solve the equation as

$$A_2' = -\bar\partial^{+}\frac{1}{\Delta}\partial(\chi\wedge\chi)' \qquad \Delta = 2(\bar\partial\bar\partial^{+} + \bar\partial^{+}\bar\partial)$$

In fact $\quad \bar\partial^{+}A_2' = 0, \quad \partial A_2' = 0$

and $\quad \bar\partial A_2' = -\frac{1}{2}(A_1\wedge A_1)'$ .

$O(\varepsilon^3)$ : $\quad \bar\partial A_3' + \partial(A_2\wedge\chi)' = 0$

The $\partial\bar\partial$-lemma works :

$$\partial\bar\partial(A_2\wedge\chi)' = \partial([A_1, A_1]\wedge A_1)'$$
$$\sim [[A_1, A_1], A_1] = 0 \quad : \text{Jacobi id}$$

Therefore $\quad A_3' = 2\bar\partial^{+}\frac{1}{\Delta}\partial\left(A_1\wedge[\bar\partial^{+}\frac{1}{\Delta}\partial(A_1\wedge A_1)']^{\vee}\right)'$

( $\vee$ : inverse of $'$ )

$O(\varepsilon^n)$ : $\quad \bar\partial A_n' + \frac{1}{2}\sum_{i=1}^{n}\partial(A_{n-i}\wedge A_i)' = 0$

$\partial\bar\partial$-lemma : $\quad \sum_i \partial\bar\partial(A_{n-i}\wedge A_i) \sim \sum_{i,j}[[A_i, A_j]A_{n-i-j}]$
$$= 0$$

Tian has proven that the series $\sum_{n=1}^{\infty}\varepsilon^n A_n$ has a finite radius of convergence.

We have a map $x \in H^1_{\bar\partial}(T) \to \mathcal{A} = x + A_0(x)$

$$A_0(x) \perp H^1_{\bar\partial}(T)$$

such that

$\mathcal{A}$ solves the Kodaira-Spencer equation.

- How the holomorphic 3-form changes under the deformation?

$$\bar\partial \Omega_0 = 0$$

$$\downarrow \quad \bar\partial \to \bar\partial + \mathcal{A}\partial$$

$$\bar\partial\Omega + \frac{1}{2}\partial(\Omega^\vee \wedge \mathcal{A})' = 0 \qquad (\,',\vee : \text{refer to } \Omega_0\,)$$

$$\downarrow$$

$$\Omega(x) = \Omega_0 + \mathcal{A}' + (\mathcal{A}\wedge\mathcal{A})' + (\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A})'$$

$x$ gives local coordinates on the moduli space of complex structure on $M$.

- Kodaira-Spencer gravity

$$S = \frac{1}{2}\int_M A'\frac{1}{\partial}\bar\partial A' + \frac{1}{6}\int_M [(x+A)\wedge(x+A)]'\wedge(x+A)$$

Since $A' \perp H^1_{\bar\partial}(T_M)$, the kinetic term is well-defined by the $\partial\bar\partial$-lemma.

$$\frac{\delta S}{\delta A} = 0 \Rightarrow \bar\partial A' + \frac{1}{2}\partial[(x+A)\wedge(x+A)]' = 0$$

The K-S equation for $\mathcal{A} = x + A$.

Classical theory

$$e^{\frac{1}{\hbar}W_0(x)} = \int \mathcal{D}A \; e^{\frac{1}{\hbar}S} \Big|_{\hbar \to 0}$$

$\therefore$ The classical effective action $W_0(x) = S(x, A = A_0(x))$

$$A_0(x) = O(x^2) \Rightarrow W_0(x) = O(x^3)$$

To the leading order $W_0(x) \simeq \frac{1}{6} \int_M (x \wedge x)' \wedge x'$

$$= \frac{1}{6} C_{\alpha\beta\gamma} x^\alpha x^\beta x^\gamma$$

where $x = \sum_\alpha x^\alpha v_\alpha$ $\quad v_\alpha$ : basis of $H^1_{\bar\partial}(T_M)$

To all order in $x$,

$$\frac{\partial^3 S(x, A_0(x))}{\partial x^\alpha \partial x^\beta \partial x^\gamma} = \int_M \left[ \frac{\partial A}{\partial x^\alpha} \wedge \frac{\partial A}{\partial x^\beta} \right]' \wedge \left( \frac{\partial A}{\partial x^\gamma} \right)'$$

$$A = x + A_0(x)$$

$$= \int_M \left[ (\partial_\alpha \Omega)^\vee \wedge (\partial_\beta \Omega)^\vee \right]' \wedge \partial_\gamma \Omega$$

$$= - \int_M \Omega \wedge \frac{\partial^3 \Omega}{\partial x^\alpha \partial x^\beta \partial x^\gamma}$$

$$= C_{\alpha\beta\gamma}(x)$$

This proves that tree $n$-point amplitudes of the topological string of the B-model

is equivalent to the Feynmann amplitudes of the Kodaria - Spencer gravity.

Especially, no higher-order contact term is needed.

- Kodaria - Spencer gravity as closed string field theory

  $\Phi$ : string field $\in$ Hilbert space of the 1st quantized string theory

$$\begin{cases} \text{constraints} & b_0^- \Phi = 0 \\ \text{Siegel gauge} & b_0^+ \Phi = 0 \end{cases} \qquad b_0^\pm = b_0 \pm \bar{b}_0 : \text{anti-ghost}$$

Action

$$S = \frac{1}{2} (\Phi , c_0^- (Q + \bar{Q}) \Phi ) + \text{interactions} .$$

In this case, $b_0^- = \partial$, $b_0^+ = \bar{\partial}^\dagger$, $Q + \bar{Q} = \bar{\partial}$

$c_0^-$ is define as $\{ b_0^- , c_0^- \} = 1$

Therefore $c_0^- ( Q + \bar{Q} ) = \dfrac{1}{\partial} \bar{\partial}$

EXAMPLE $\quad \sum_{i=1}^{5} X_i^5 = 0$ IN $\mathbb{C}P^4$ $\quad (\dim H^{1,1} = 1)$

$N_1^{(1)} = 0$ $\qquad$ DEGREE 1 → LINE → RATIONAL

$N_2^{(1)} = 0$ $\qquad$ DEGREE 2 → PLANE CONIC → RATIONAL

$N_3^{(1)} = 609250 \qquad = N_2^{(0)}$

$N_4^{(1)} = 3721431625 \qquad = 3718024750 + 1185 \times N_1^{(0)}$

$N_5^{(1)} = 12129909700200$

$N_6^{(1)} = 31147299732677250$

$N_7^{(1)} = 7157840602280761750$

$N_8^{(1)} = 154990541752957846986500$

$N_9^{(1)} = 324064464310279585656399500$

$N_{10}^{(1)} = 662863774391414084612496876100$

$\vdots$

$N_n^{(1)} \simeq a\, n^{-1} e^{\beta n} \quad , \quad n \gg 1$

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$$2X_1^3 + X_2^6 + X_3^6 + X_4^6 + X_5^6 = 0$$

$$N_1^{(1)} = 0$$

$$N_2^{(1)} = 7884$$

$$N_3^{(1)} = 145114704$$

$$\vdots$$

$$4X_1^2 + X_2^8 + X_3^8 + X_4^8 + X_5^8 = 0$$

$$N_1^{(1)} = 0$$

$$N_2^{(1)} = 41312$$

$$N_3^{(1)} = 21464350592$$

$$\vdots$$

$$5X_1^2 + 2X_2^5 + X_3^{10} + X_4^{10} + X_5^{10} = 0$$

$$N_1^{(1)} = 280$$

$$N_2^{(1)} = 207680680$$

$$N_3^{(1)} = 161279120326560$$

$$\vdots$$

$g = 2$

$$F_2 = \underbrace{\frac{\chi(K)}{5760}}_{\frac{1}{2}\int_{\mathcal{M}_2} (c_1)^3} + \sum_n N_n^{(2)} e^{-\langle t, n\rangle} + \underbrace{\frac{1}{240}}_{(-1)\times \text{EULER CHARACTERISTIC OF } \mathcal{M}_2} \sum_n \frac{N_n^{(0)} e^{-\langle t, n\rangle}}{(1 - e^{-\langle t, n\rangle})^2}$$

EXAMPLE

$$\sum_{i=1}^{5} X_i^{5} = 0$$

$$N_1^{(2)} = 0$$

$$N_2^{(2)} = 0$$

$$N_3^{(2)} = 0$$

$$N_4^{(2)} = 534750$$

$$N_5^{(2)} = 75478987900$$

$$N_6^{(2)} = 8717081396382250$$

$$N_7^{(2)} = 5185462556617269625$$

$$N_8^{(2)} = 90067364252423675345000$$

$$N_9^{(2)} = 325859687147358266010240500$$

$$\vdots$$

$$N_n^{(2)} \simeq a' n \cdot (\log n)^2 e^{\beta n}, \quad n \gg 1$$

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