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Large-Scale Rossby Waves & their Instability

Joe Pedlosky
Woods Hole Institution
MA, USA

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Large Scale Rossby Waves and Their Instabilities

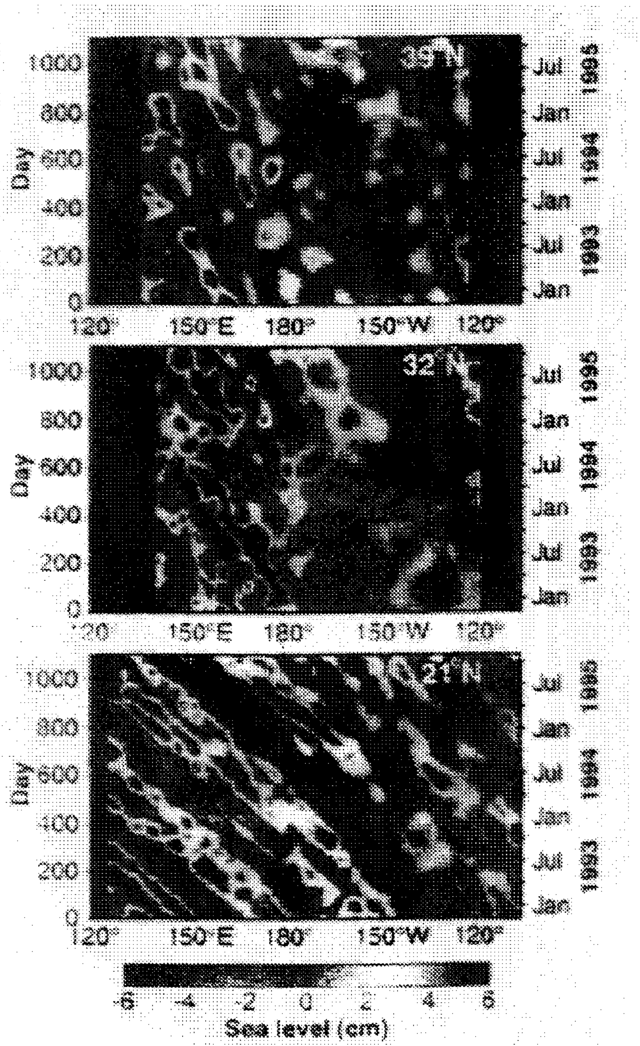
(with Joe LaCasce)

A proper description of the coupling between the atmosphere and the oceans requires an understanding of the natural modes of response of the ocean basins to variable forcing.

This leads naturally to the question of the low frequency modes of oscillation of ocean basins.

Observations e.g. Chelton and Schlax show large scale westward propagating waves which have been identified as long (slow) Rossby waves although there is considerable discussion about the detailed nature of their dynamics.

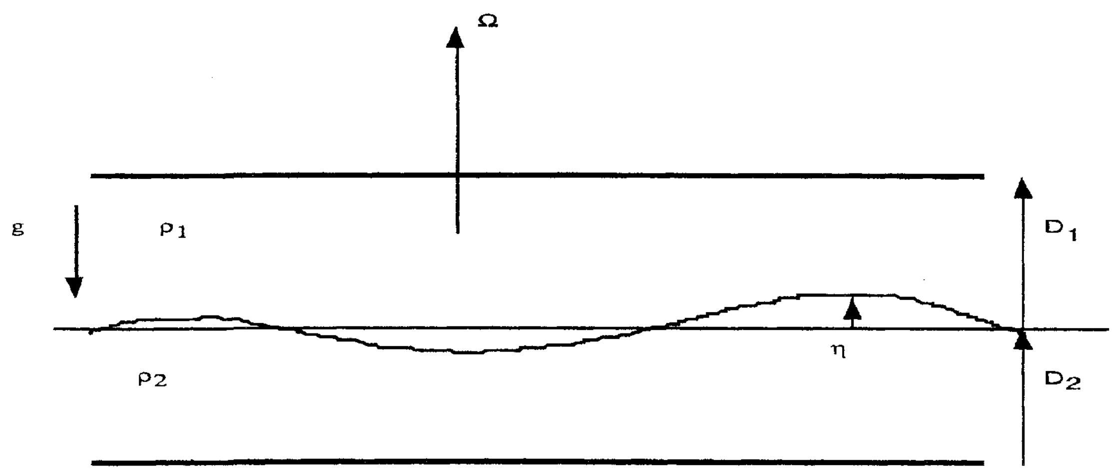
We examine the question of Rossby wave modes of oscillation and, as a natural extension of these ideas, their instability properties.



Chelton & Schlax
Science '96
272 234-238

www.coas.oregonstate.edu/po/research/rossby_waves/chelton.html

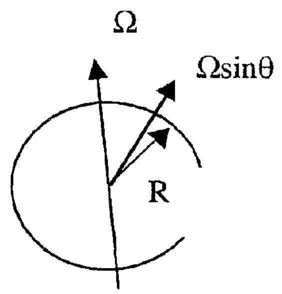
The model:



The model is a two layer model with slightly different densities across an interface. The motion is in geostrophic balance.
Consider a linear model which we will apply to the basic wave field whose stability we will investigate later.

The linearized equations can be written in terms of a barotropic mode, in which the motion in the two layers is identical, and a baroclinic mode in which the vertically averaged velocity is zero. ($\vec{u}_1 D_1 = -\vec{u}_2 D_2$)
Let ψ **be the geostrophic streamfunction from which the horizontal velocities can for either mode, be obtained,** $u = -\psi_y$, $v = \psi_x$

It is not hard to show that the linear governing equation for the modes is of the form,



$$\frac{\partial}{\partial t}\left[\nabla^2\psi - F\psi\right] + \beta\psi_x = T - Diss(\psi)$$

β is the planetary vorticity gradient $2\Omega\cos(lat)/R$, and for the baroclinic mode, $F = \frac{f^2(D_1+D_2)}{g' D_1 D_2}$, while for the barotropic mode it is zero. We define the characteristic scale $L_d = 1/F^{1/2}$ $(deformation\,radius)$

T is a vorticity source, say due to the wind, and $Diss(\psi)$ is a representation of some form of dissipation, e.g. $\delta\nabla^2\psi$. On the boundaries of the basin the geostrophic streamfunction must be a constant.

Free Rossby waves in the infinite region and with no dissipation can be found in the form:

$$\psi = A e^{i(kx+ly-\omega t)},$$

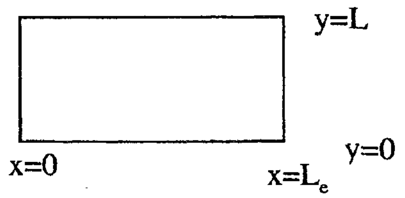
$$\omega = \frac{-\beta k}{k^2+l^2},\ \textit{barotropic},$$

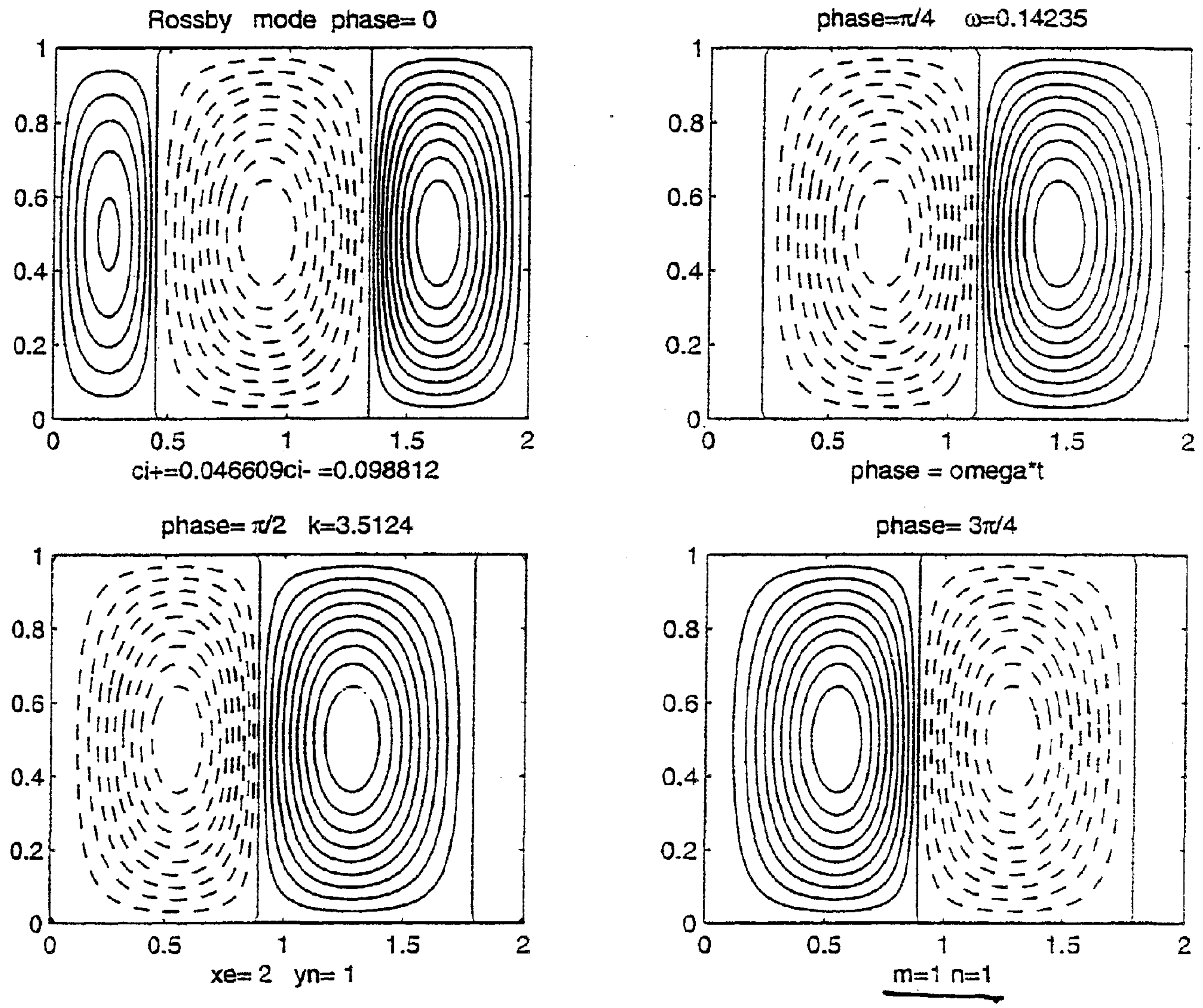
$$\omega = \frac{-\beta k}{k^2+l^2+F},\ \textit{baroclinic}$$

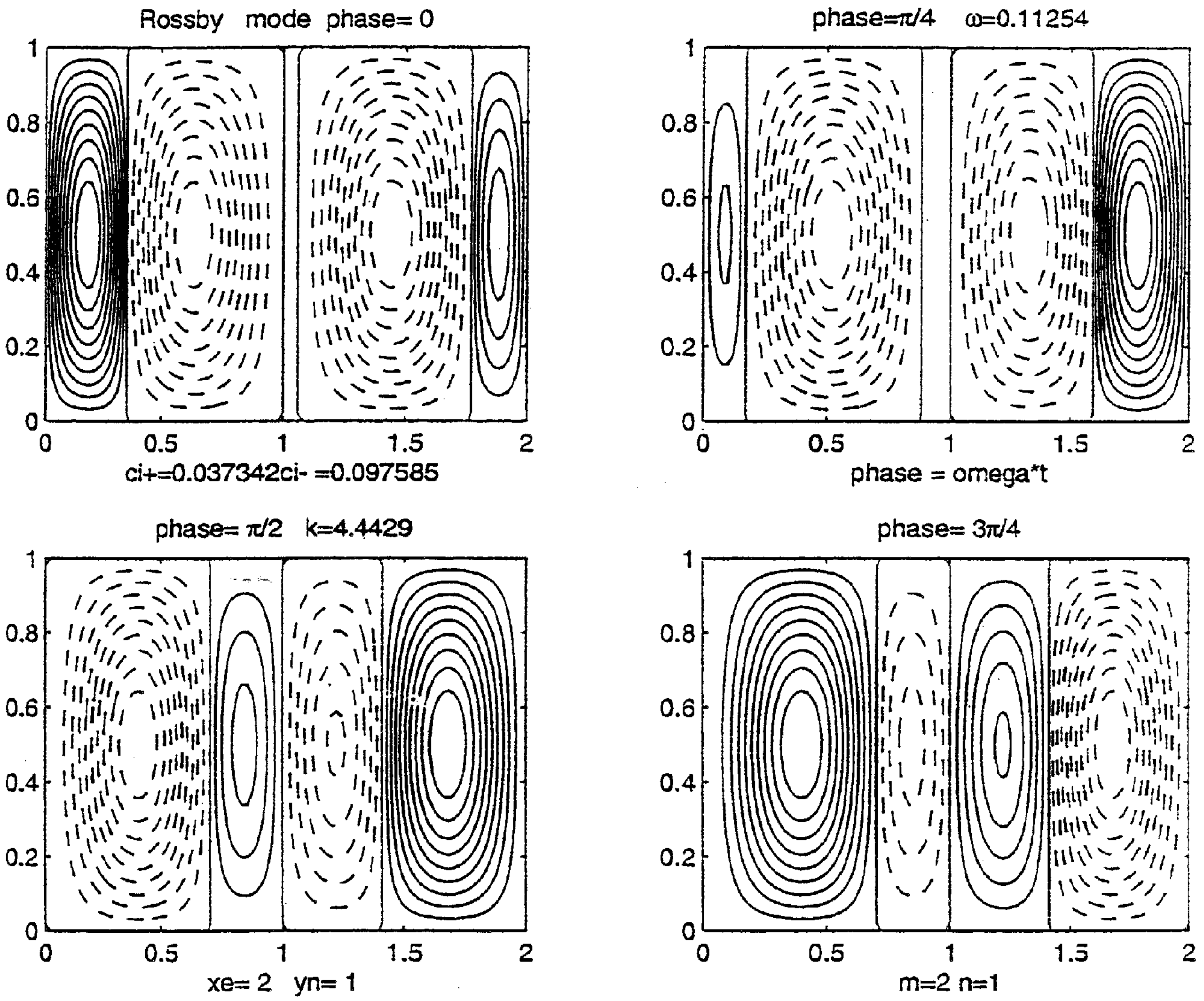
For wavelengths long compared to the deformation radius the frequency and phase speed of the baroclinic waves are $\omega = -\beta k / F, \quad c_r = -\beta / F = -\beta L_d^{\,2}$ **Note, nondispersive.**

<u>Normal modes in a basin.</u>

Consider a rectangular basin




Rossby mode phase= 0
ci+=0.046609ci- =0.098812
phase=π/4 ω=0.14235
phase = omega*t
phase= π/2 k=3.5124
xe= 2 yn= 1
phase= 3π/4
m=1 n=1


Rossby mode phase= 0
ci+=0.037342ci- =0.097585
phase=π/4 ω=0.11254
phase = omega*t
phase= π/2 k=4.4429
xe= 2 yn= 1
phase= 3π/4
m=2 n=1

The plane wave solutions in the infinite domain can be synthesized to find solutions that satisfy the boundary conditions.

For the barotropic mode (F=0)the free solutions are simply

$$\psi = Ae^{i(\omega t + \beta x / 2\omega)} \sin(m\pi x / L_e) \sin(n\pi y / L)$$

with, $\omega = \dfrac{\beta}{2\left[m^2\pi^2 / L_e^{\,2} + n^2\pi^2 / L^2\right]^{1/2}}$.

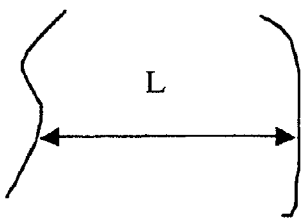
The solutions propagate <u>phase</u> (but no energy) to the west and consist of a set of vortical motions which expand and contract regularly.

Note that the stream function = 0 on the basin boundaries and this constant value is used with no loss of generality <u>for the barotropic mode.</u>

Some review;

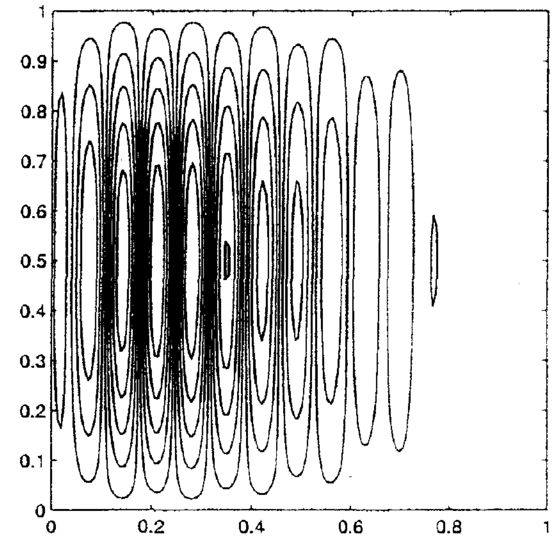
We are going to use idealized models on the β -plane and we are interested, for the coupling question, as to whether there are modes that are long lived in the presence of dissipation.

Consider an ocean basin :

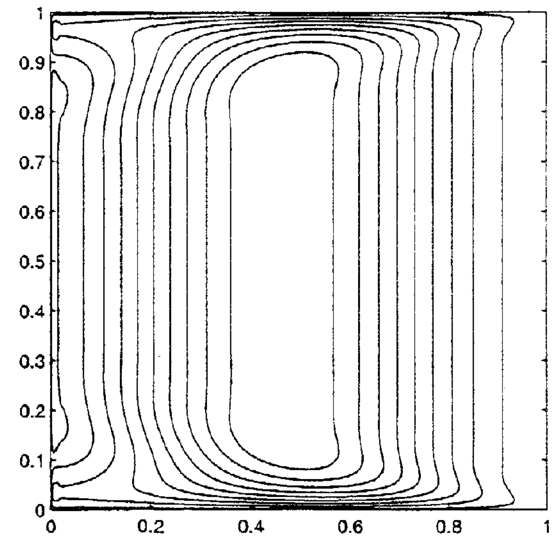


Let us suppose that the travel time for a Rossby wave to cross the basin is $T_r = L/c_r$. {for a Rossby wave, $c_r = 0(\beta N^2 D^2 / f^2)$} We will be particularly interested in the parameter σT_r where σ is either the dissipation rate of the wave, or the <u>growth</u> rate of its instability. If σT_r >1 this would imply that the wave will either dissipate or share its energy with eddies before it crosses the basin. Effective coupling modes will require this parameter to be small.

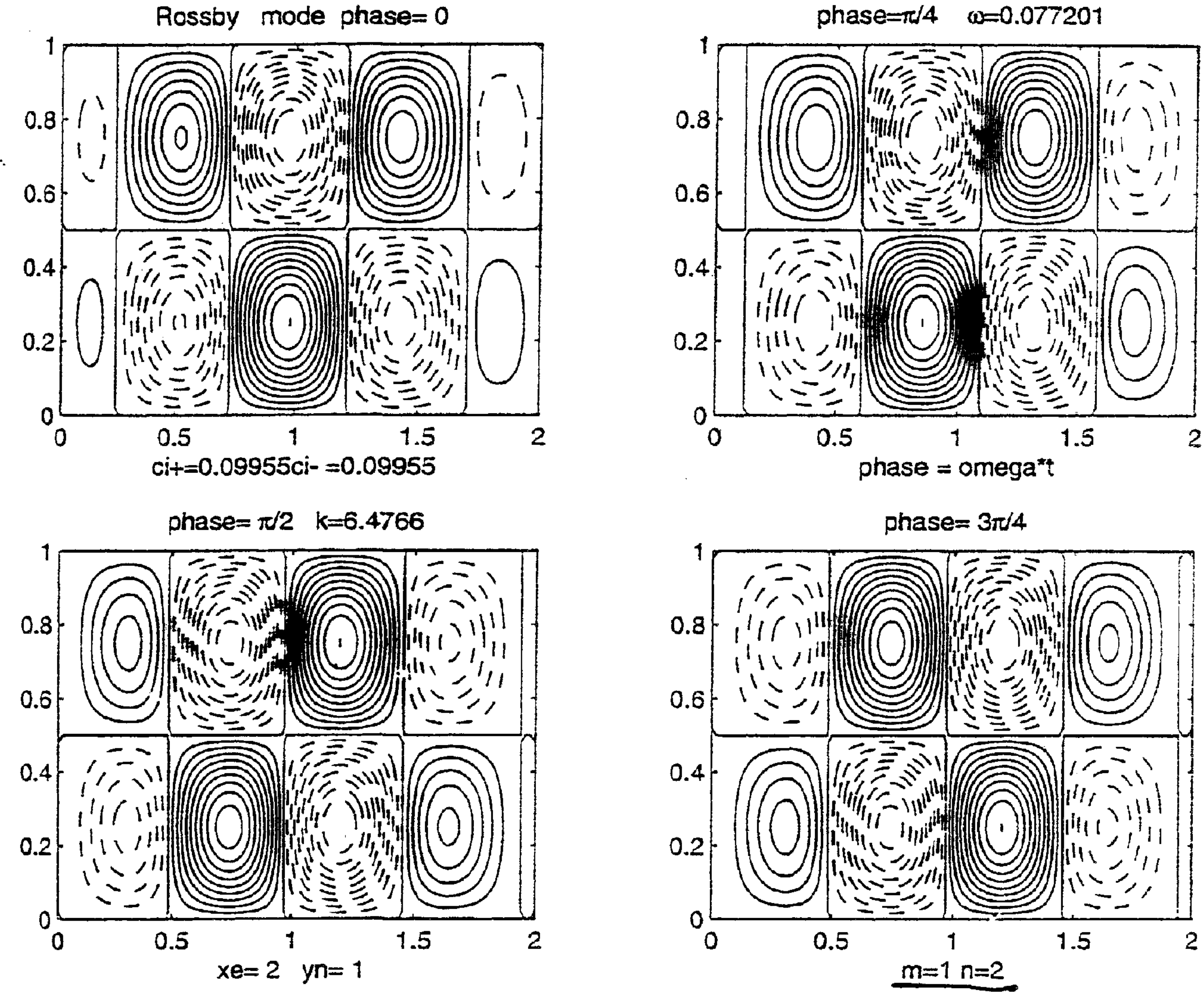
Gravest mode with F=2000, δ=.002; ψ=0; ω=(0.0111,−0.001)



with $\psi=\Gamma(t)$; ω=(0.0029,−1.32e−4)



note short waves
with $\psi = 0$ b.c.


Rossby mode phase= 0
ci+=0.09955ci- =0.09955
phase=π/4 ω=0.077201
phase = omega*t
phase= π/2 k=6.4766
xe= 2 yn= 1
phase= 3π/4
m=1 n=2

Note that with damping the frequency is only shifted by a constant imaginary part $\omega = \omega_{inviscid} - i\delta$.

The situation is quite different for the baroclinic mode. For the baroclinic mode the geostrophic stream function is directly related to the interface displacement,

$$\psi_{baroclinic} = \psi_1 - \psi_2 = -\frac{g'}{f}\eta.$$

The boundary condition requires that the stream function be spatially constant on the boundary but it may be a <u>nontrivial</u> function of time (the interface height) and is unknown. Thus for the baroclinic mode we have the extra unknown

$\psi = \Gamma(t)$ on the boundary. How do we determine it?

To determine the constant we use the condition of mass conservation within each layer (if there are cross isopycnal fluxes the condition must be modified by the heating function responsible for the fluxes).

The condition is equivalent to the statement:

$$\frac{\partial}{\partial t} F \iint_A \psi dA = 0$$

and for purely oscillating solutions this yields the constraint

$$\omega F \iint_A \psi dA = 0$$

There have been a number of calculations of such modes in simple basin shapes. (e.g. Cessi and Primeau, 2001, JPO 31, LaCasce, 2000, JPO 30, LaCasce and Pedlosky, 2002, JPO 32) and we will describe the results and a bit of the analysis.

First, let's do the baroclinic problem wrong (it's almost always easier that way) by applying the b.c. $\psi = 0$. Then, as in the barotropic case the solutions can be found directly in the form, including dissipation,

$$\psi = e^{-i(kx+\omega t)} \sin(m\pi x / L_e)\sin(n\pi y / L)$$

$$k = \frac{\beta}{2(\omega + i\delta)}, \quad K^2 \equiv \pi^2\left[m^2 / L_e^{\,2} + n^2 / L^2\right]$$

$$\Rightarrow$$

$$\omega = -i\delta\left[\frac{2K^2+F}{2K^2+2F}\right] \pm \frac{\beta}{2}\left\{\frac{1}{K^2+F} - \frac{\delta^2 F}{(K^2+F)^2}\right\}^{1/2}$$

(note: if n is <u>even</u> this satisfies integral constraint, if n is <u>odd</u> it does not). If δ is zero this yields the barotropic formula with $K^2 \rightarrow K^2 + F$.

As *K* varies the decay rate of the mode varies between the limits

$$-\delta/2 > \omega_i > -\delta$$

Now let's consider the correct solution:

The solution can be written as

$$\psi = \Gamma e^{-i\omega t} + e^{-i\omega t} e^{ikx} \phi(x,y)$$

$$k = \frac{1}{2(\omega + i\delta)}$$

$$\Rightarrow \nabla^2 \phi + (k^2 - \tilde{F})\phi = \tilde{F}\Gamma e^{ikx}$$

$$\tilde{F} = F \frac{\omega}{\omega + i\delta}$$

If F=0, the solution is independent of Γ.

The solution for ϕ is quite complicated, requiring a complete Fourier expansion in sin modes in y and the details are not give here (see LaCasce and Pedlosky, 2002). The basic idea however is evident from a simpler model.

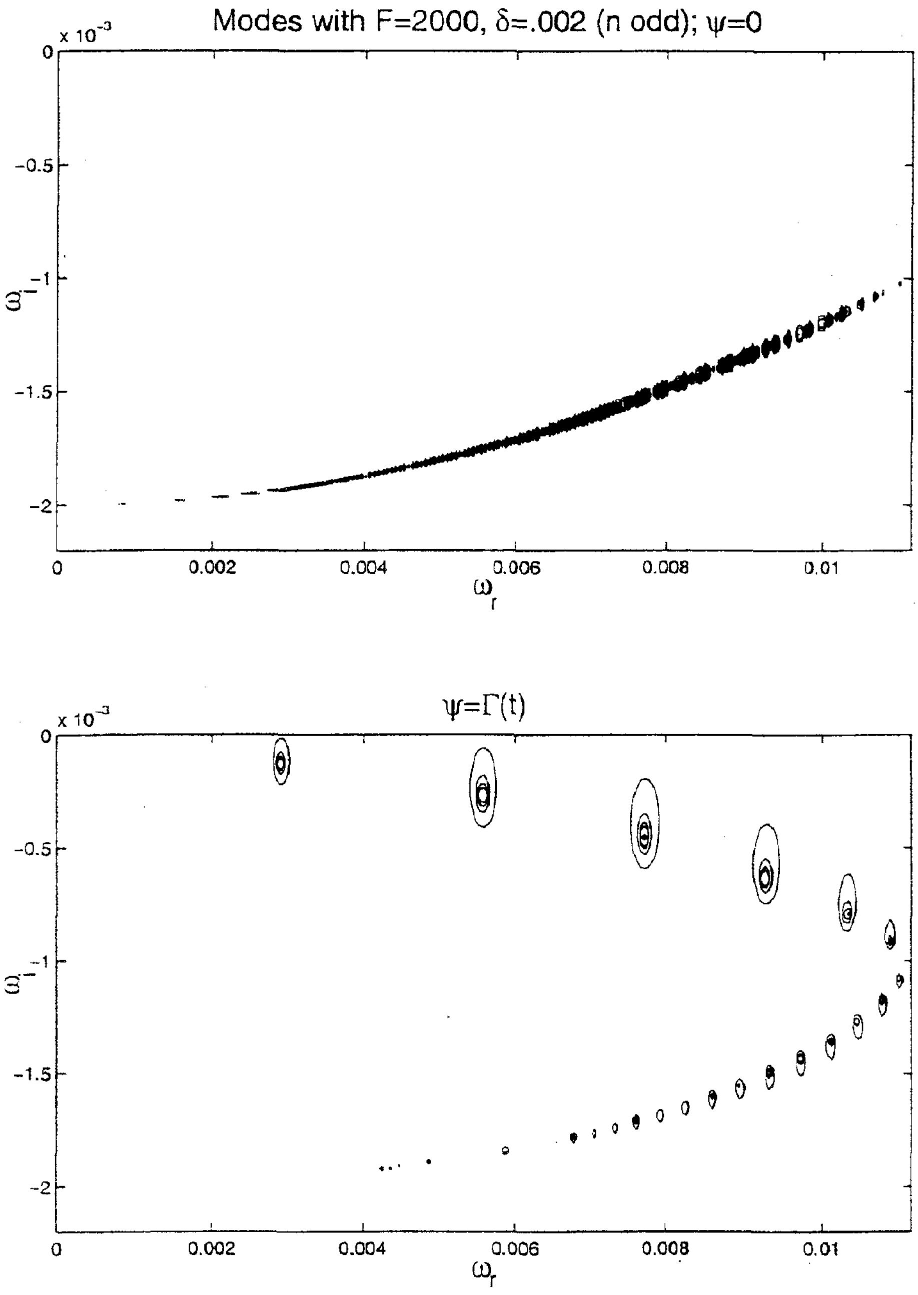

Modes with F=2000, δ=.002 (n odd); ψ=0
x 10^-3
0
-0.5
-1
-1.5
-2
ω_i
0
0.002
0.004
0.006
0.008
0.01
ω_r
ψ=Γ(t)
x 10^-3
0
-0.5
-1
-1.5
-2
ω_i
0
0.002
0.004
0.006
0.008
0.01
ω_r

Figure 2:

Suppose we have a basin which is infinitely long in the y direction and let's look for solutions independent of y. At the same time, let's suppose that the wavelength of the wave is very much longer than the deformation radius. Now, with the <u>incorrect</u> boundary conditions the zero condition on the western wall is satisfied by radiating small scale, rapidly damped waves. We shall see that this is not necessary for the baroclinic mode. In the long wave limit (if we are successful in finding a solution)

$$-F\frac{\partial \psi}{\partial t} + \beta \frac{\partial \psi}{\partial x} = 0$$

Solutions satisfying the b.c. $\psi = \Gamma e^{-i\omega t}$, $x = 0, L_e$ are

$$\psi = \Gamma e^{-i\omega(t + xF/\beta)}, \quad \mathit{iff} \ \ \omega L_e / c_r = 2j\pi,$$

$$c_r = \beta / F = \beta L_d{}^2$$

An even number of wavelengths "fit" across the basin.

This satisfies the integral condition exactly and

$$\omega = \beta k_j / F, \qquad k_j = 2\pi j / L_e, j = 1,2,3...$$

Most importantly, if friction is added in this long wave limit the frequency is shifted to

$$\omega = \beta k_j / F - i\delta k_j^2 / F$$

i) **The frequency when the wave length is large compared to L_D is much lower than the incorrect solution.** ($\omega = O(\beta L_d^2 / L_e) << \beta L_d$

ii) **The decay rate is very much smaller**

$$\omega_i = O\left(\delta L_d^2 / L_e^2\right) << \delta / 2$$

These basin modes are very weakly damped and the expectation is that they may survive long enough to couple efficiently with long term atmospheric variability. The reason the mode is not damped as much is the absence of small scale structure.

Physically, the way the mode works is that the long Rossby wave propagates westward, hits the western boundary and instead of reflecting as short Rossby waves, produces a Kelvin wave which races around the basin to the eastern boundary where the Γ on the eastern boundary forces another long Rossby wave.

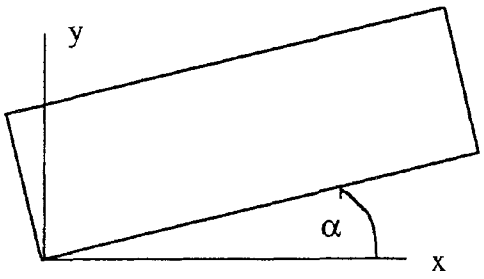
Quasi-geostrophic theory does not explicitly describe the Kelvin wave but the role of the wave is taken instead by the integral condition previously described.

The correct solution in a basin with finite scale in y behaves similarly.

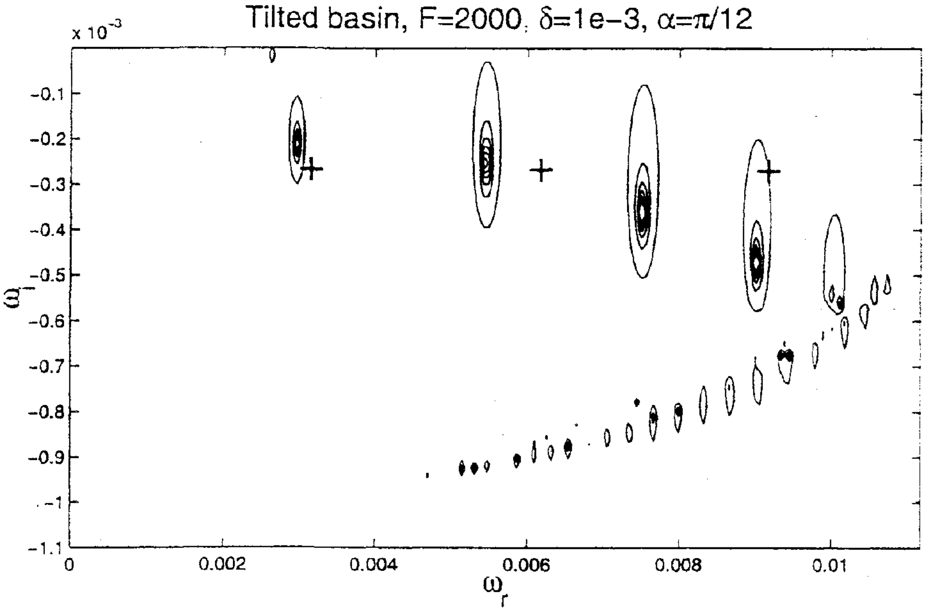
A key feature of the solution is that an integral number of complete wavelengths fit across the basin at <u>all</u> latitudes.

If the basin does not have uniform width (or if c_r) is a function of latitude (as it is). The story is more complex.

LaCasce and Pedlosky looked at a simple model of a tilted rectangle



As expected, it is no longer possible to "fit" an even number of wavelengths across the basin at all latitudes. Small scale motions are produced, more dissipation occurs but still <u>less than the incorrect solution would predict.</u> These slightly damped modes resonate easily with wind forcing (also in a circular basin)


Tilted basin, F=2000, δ=1e−3, α=π/12
x 10^-3
ω_i
ω_r
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
-0.9
-1
-1.1
0
0.002
0.004
0.006
0.008
0.01

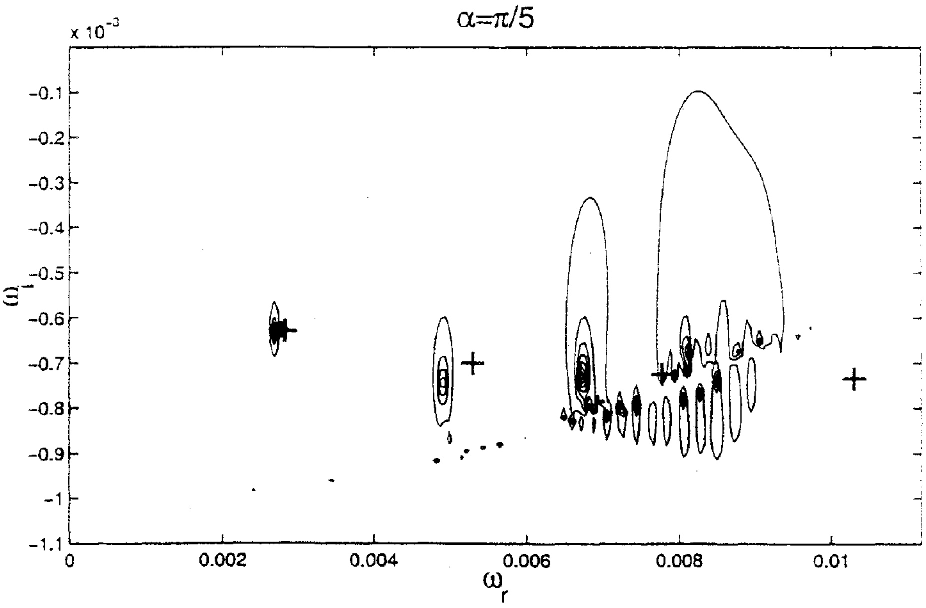

α=π/5
x 10^-3
ω_i
ω_r
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
-0.9
-1
-1.1
0
0.002
0.004
0.006
0.008
0.01

Figure 4:

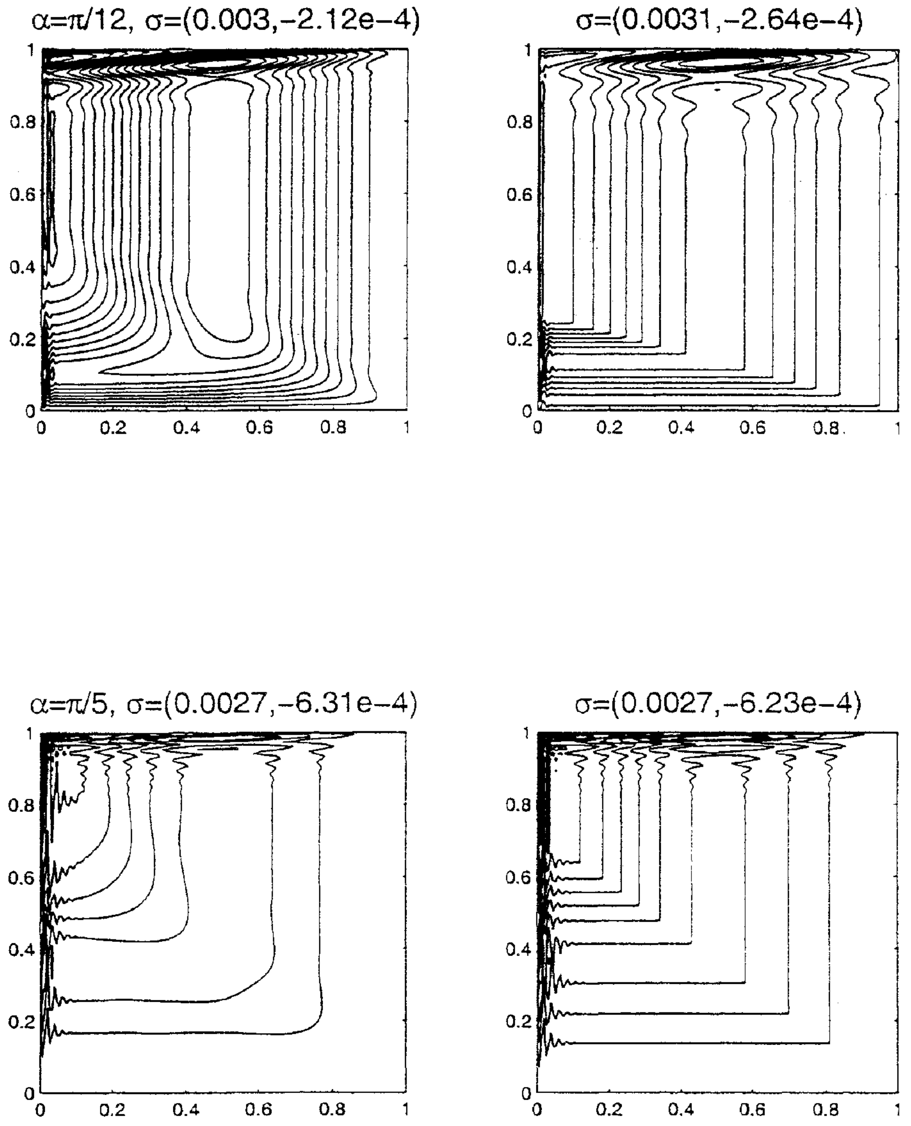

α=π/12, σ=(0.003,−2.12e−4)
σ=(0.0031,−2.64e−4)
α=π/5, σ=(0.0027,−6.31e−4)
σ=(0.0027,−6.23e−4)
0
0.2
0.4
0.6
0.8
1

Figure 5:

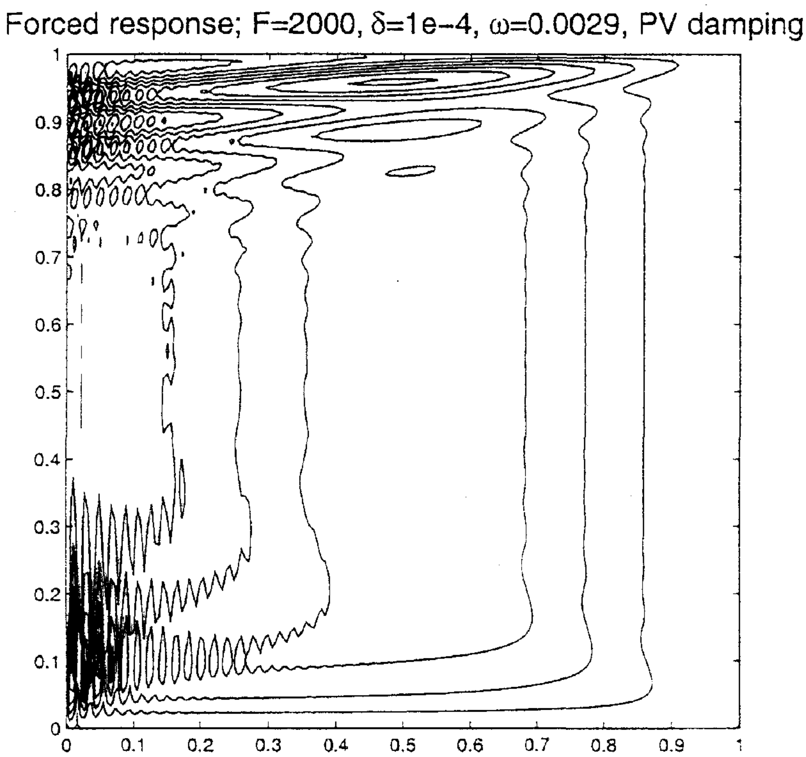

Forced response; F=2000, δ=1e-4, ω=0.0029, PV damping
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Figure 14:

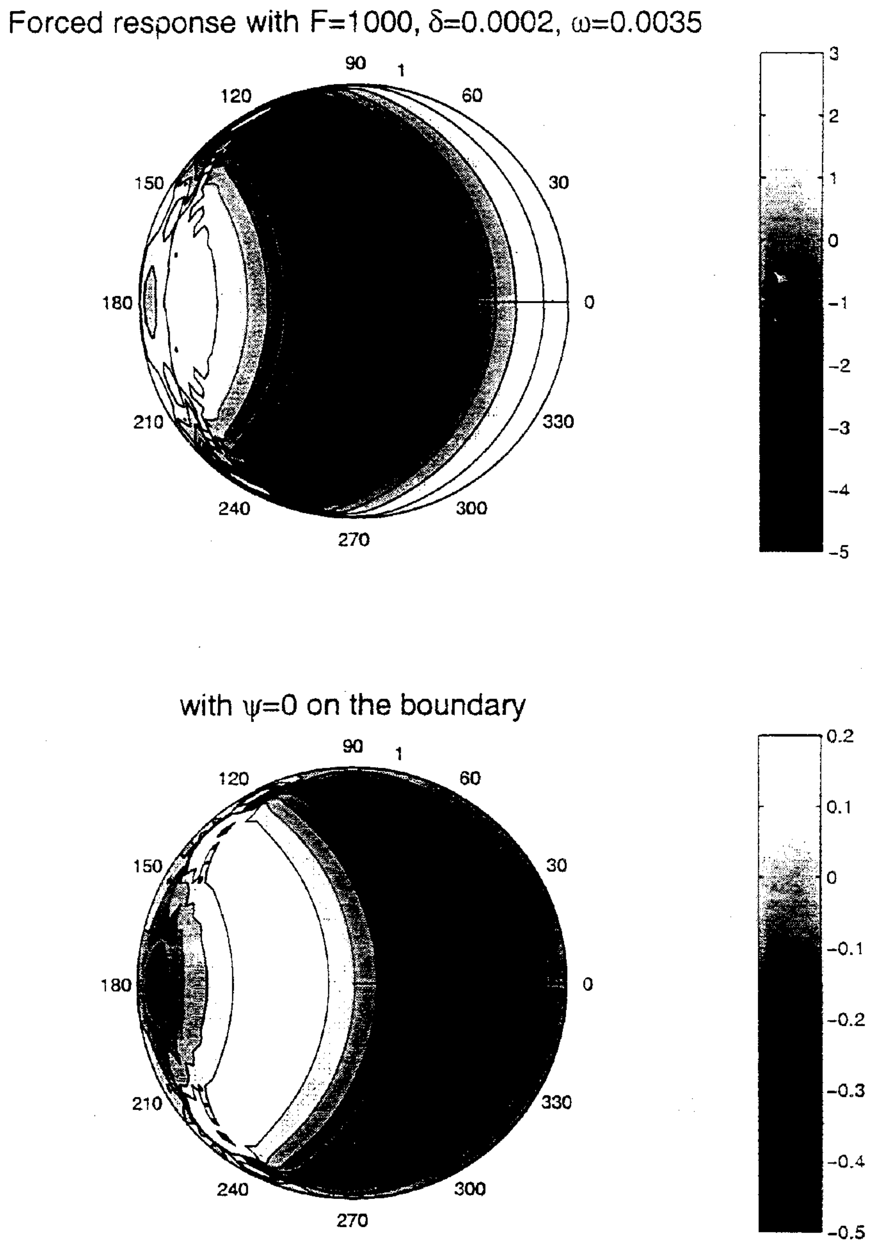

Forced response with F=1000, δ=0.0002, ω=0.0035
90
1
120
60
150
30
180
0
210
330
240
300
270
3
2
1
0
-1
-2
-3
-4
-5
with ψ=0 on the boundary
90
1
120
60
150
30
180
0
210
330
240
300
270
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5

Figure 11:

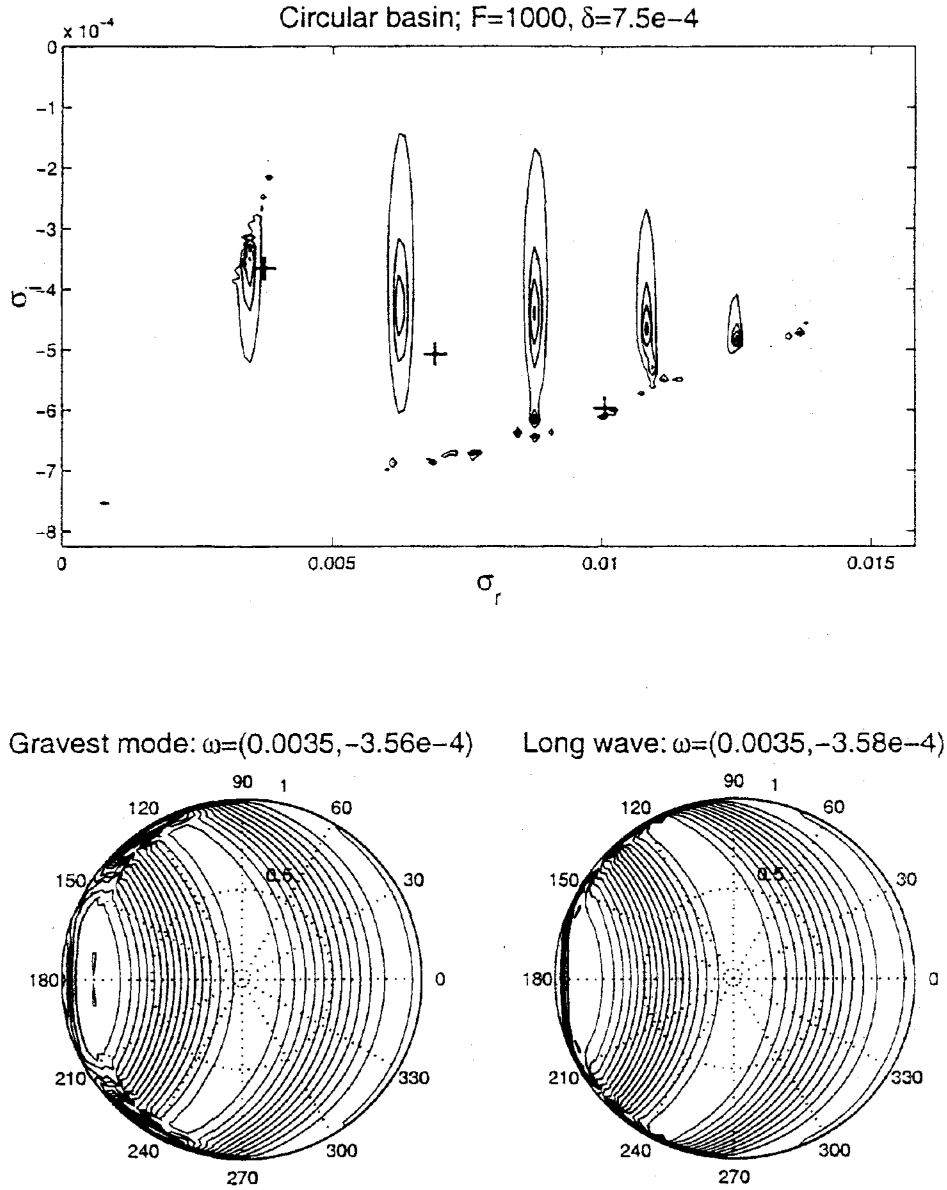

Circular basin; F=1000, δ=7.5e−4
x 10^-4
σ_i
σ_r
Gravest mode: ω=(0.0035,−3.56e−4)
Long wave: ω=(0.0035,−3.58e−4)

Figure 7:

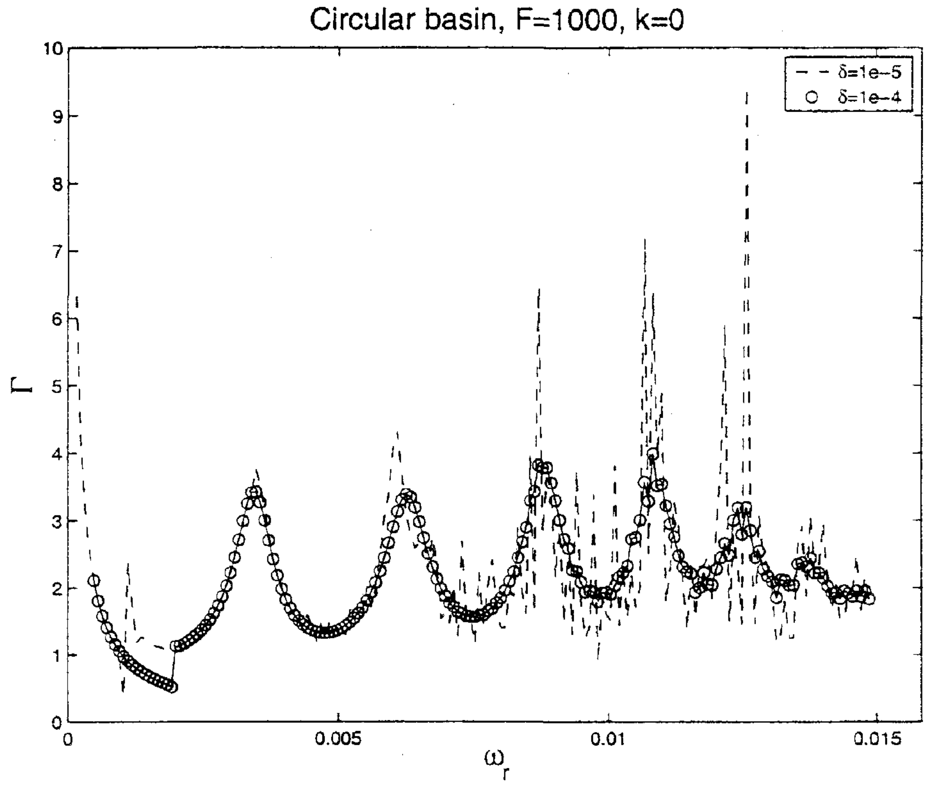

Circular basin, F=1000, k=0
δ=1e−5
δ=1e−4
Γ
ω_r
0
1
2
3
4
5
6
7
8
9
10
0.005
0.01
0.015

Figure 10:

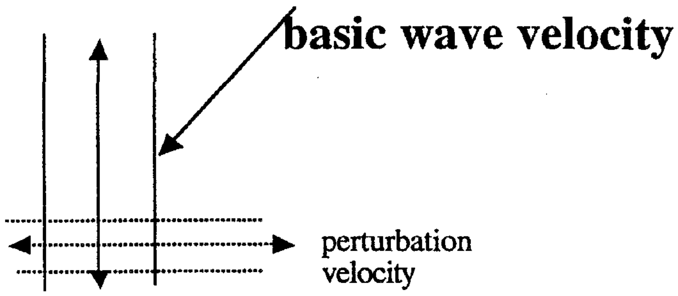
Summary

- **The baroclinic response of an ocean basin is very sensitive to the correct boundary condition.**
- **With the correct condition ($\psi = \Gamma(t)$) on bdy. A class of long waves with small dissipation arise when the Rossby wave travel times do not vary by a large amount as a function of latitude.**
- **The long waves resist dissipation by "fitting" into the basin and satisfy the mass conservation condition.**
- **Strongly tilted basins will tend to damp the long wave response to forcing although the modes are still identifiable.**
- **Basins, like the circle, which have a latitude interval for which $dL_e/dy = 0$, resemble the weakly tilted rectangle.**
- **The long wave modes persist even when the dissipation is not scale selective although they are privileged when it is.**

THE STABILITY OF THE MODES

We see that the linear, slow, large scale modes are weakly damped by small scale friction.
The modes have a structure in which, over most of the basin the velocity is largely meridional (i.e. ψ contours run north south-- ψ after all, is constant on the eastern boundary).

The velocity is baroclinic. There is a vertical shear in the wave. If the motion in the wave were zonal the β effect would stabilize the wave if the motion were weak (linear). However, the meridional motion can't be stabilized by β since perturbations which have east-west trajectories release potential energy but don't "feel" the β effect. Are these modes unstable then? If so, what is the growth rate?



Model: We will use the two layer quasi-geostrophic model to describe the instability.

We will introduce dimensionless time and space variables. We scale x and y with L the characteristic wavelength of the basic wave (close to the basin scale) L/U
and we scale time with [redacted]**, velocity with U and ψ with UL , [these are the classical scales for this system] then**

$$\frac{\partial}{\partial t} q_n + J(\psi_n, q_n) + \beta \psi_{n_x} = 0, n = 1,2$$

$$q_n = \nabla^2 \psi_n + F_n (-1)^n [\psi_1 - \psi_2]$$

$$\beta = \beta_{\text{dim}} L^2 / U, \quad F_n = \frac{f^2 L^2}{g' D_n}$$

It is useful to rewrite the system in terms of the barotropic and baroclinic fields,

$$\psi_b = h_1\psi_1 + h_2\psi_2, \quad h_n = D_n/(D_1 + D_2)$$

$$\psi_t = \psi_1 - \psi_2$$

$$q_b = \nabla^2\psi_b, \; q_t = \nabla^2\psi_t - F\psi_t,$$

$$F = F_1 + F_2 = \frac{f^2 L^2 (D_1 + D_2)}{g' D_1 D_2}$$

We can anticipate certain results (to be checked) The time scale for the basic wave to transit the basin or to travel the distance L, its wavelength, is

$$T_r = L / c_r = \frac{L}{\beta_{\text{dim}} L_d^2}$$

We have scaled time with the advective time L/U. The ratio is $b^{-1} = \frac{U}{\beta_{\text{dim}}^2 L_d^2}$. In terms of our dimensionless variables this implies a <u>new</u> time variable $t_* = b\, t$.

On the other hand, if the instability feeds on the shear and the scale of the instability is given by the classical two layer model with no β (remember the modes will not feel it strongly) then the growth rate should be of the order $\sigma = O(U / L_d)$. When this is compared to the advective time scale it suggests another time variable to measure the growth, $t_g = t\, F^{1/2}$. It is natural to expect that the solution will depend on both time scales. Note that the ratio of these time scale is,

$$\sigma T_r = \frac{F^{1/2}}{b} \equiv Z$$

(small Z= ~~large~~ [small] shear, large Z small β [(large shear)] β)

Again from standard stability theory we will expect that the length scale of the perturbation is O(L_d) and is the north south scale, this suggests a new y scale,

$$y = s\bar{F}^{1/2}$$

If these transformations are made, the problem contains a single parameter , Z

We have done analytic and numerical work to examine the instability as a function of Z and I will present only the principal (and preliminary results).

For small Z the problem is nearly linear. The wave amplitude of the basic wave is small and to the first approximation the linear wave theory we have described is apt. The basic solution looks like a plane wave with purely meridional, baroclinic, velocity (and meridional shear). In n.d. units,

$$\Psi = V_o k^{-1} \cos(kx + \omega t_*), \; \omega = -k$$

This gives the solution only on the time scale t_* which for small Z is the fast time scale. On the longer growth rate time scale we find that the basic wave is unstable to a triad perturbation consisting of the basic baroclinic wave, a barotropic wave with a y-scale of the order of the deformation radius, and the triad is completed by an baroclinic wave whose y scale is also O(L_d). The x-scale of all three waves is O(L).

The triad must satisfy the triad resonance conditions:

The x and s wave numbers must add up and the frequencies must also. Under the approximation $L_d << L$,

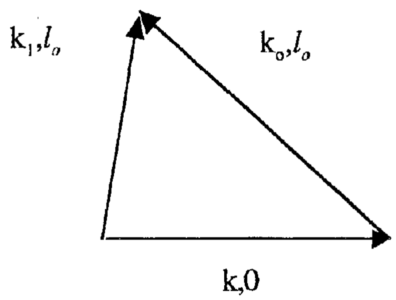
these become,

$$\overbrace{k}^{basic\,wave} + \underbrace{k_o}_{baroclinic\ pert.} = \overbrace{k_1}^{barotropic\,pert.}$$

$$\omega + \omega_o = \omega_1,$$

where,

$$\omega(k) = -k,\ \omega_1 = -k_1 / l_o^{\,2},\ \omega_o = -k_o / (l_o^{\,2} + 1)$$

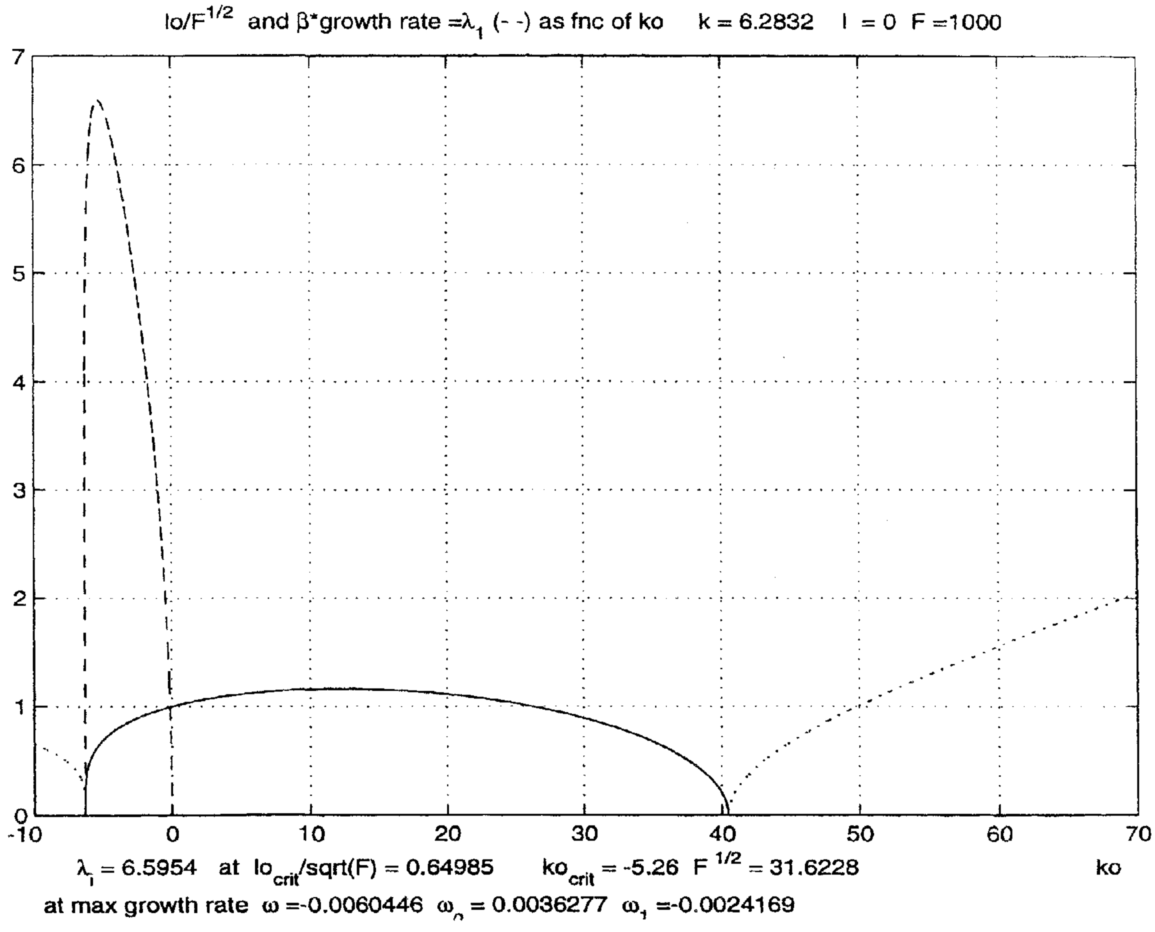


(triad)

If the resulting triad equations are linearized about the original baroclinic wave, the growth rate, on the time scale t_g is given by

$$\lambda = V_o l_o \left[\frac{h_1 h_2}{2} \frac{1 - l_o^{\,2}}{1 + l_o^{\,2}} \right]^{1/2}$$ **which is qualitatively the same as the Eady growth rate.**

The wave number l_o is related to the x-wavenumber from the resonance condition, $l_o^2 = \left(1 + \frac{k_o}{k}\right)^{1/2}$, where k is the given wave number of the basic wave.

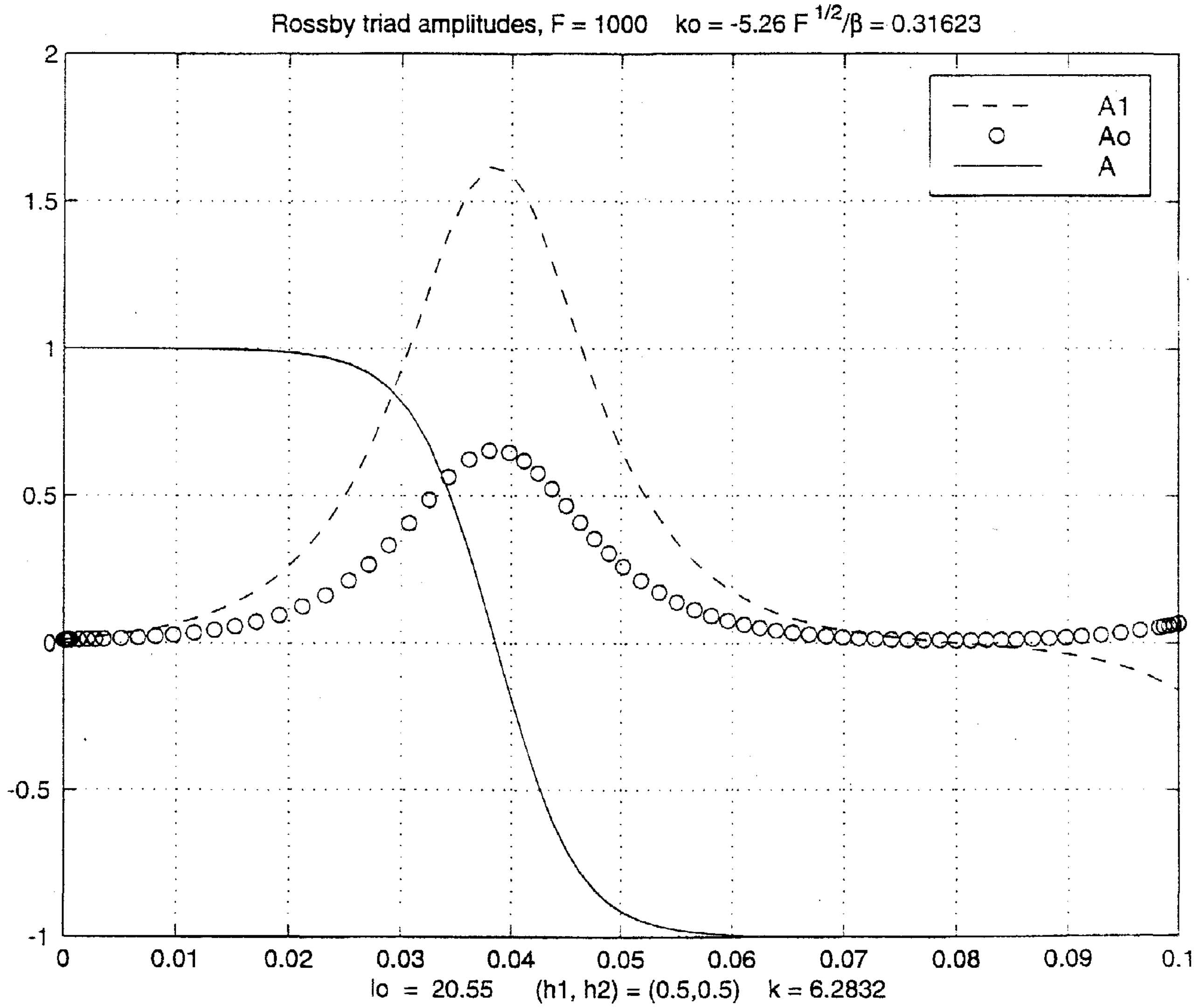


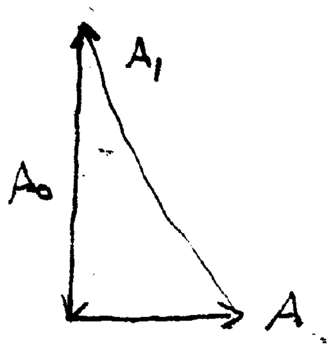
the figure shows l_o as a fnc. of k_o by the resonance condition and the growth rate. It peaks at the scale predicted by Eady's theory.

The nonlinear triad equations show the amplitude development. Realistically, this evolution is valid only initially. At larger amplitudes of the perturbation the triad will interact with other waves.

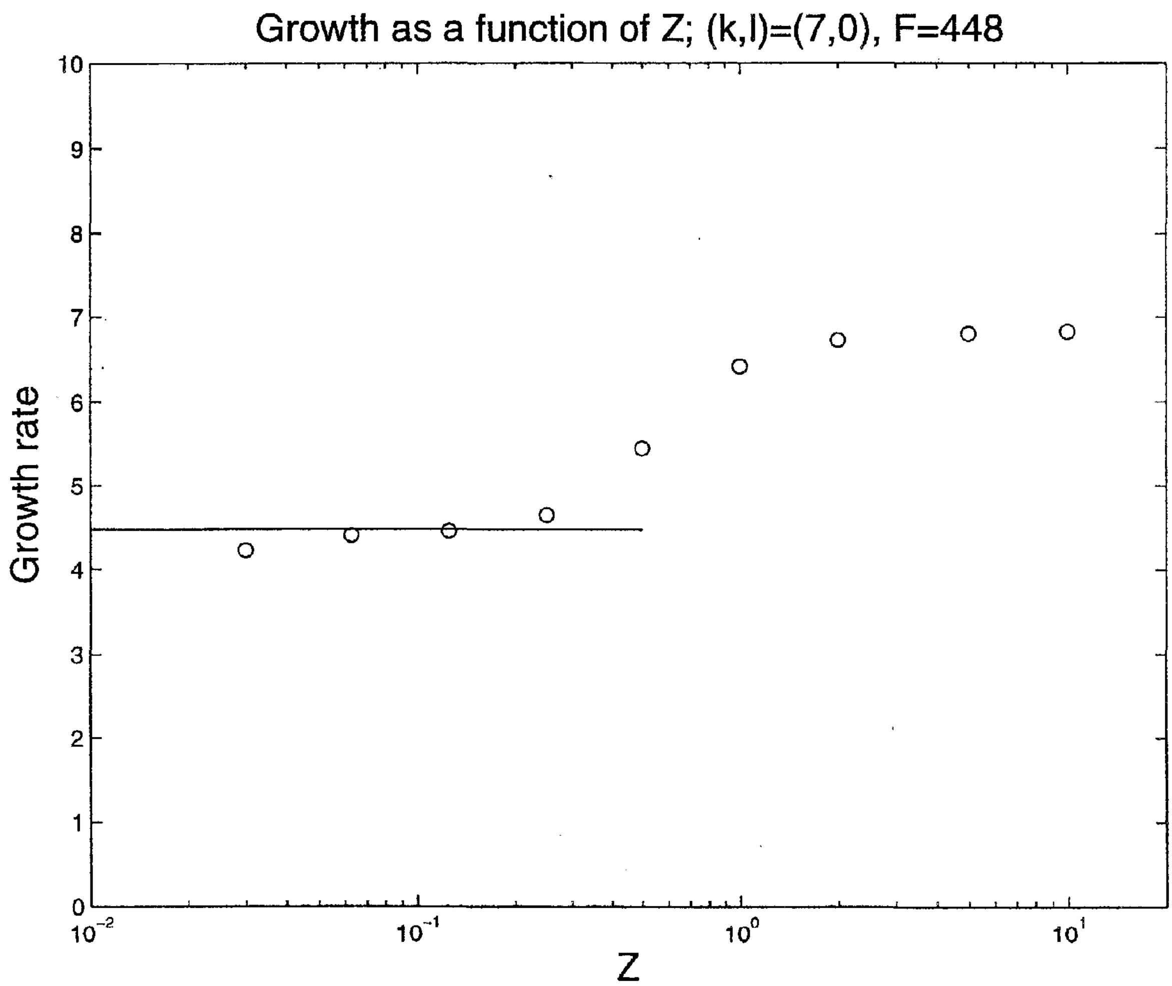
For very large Z the basic wave does not move very far in a characteristic growth time and the problem "reduces" to the instability of a steady flow $V(x)=V_0 cos(kx)$. We have not worked out the problem analytically completely (a complicated WKB problem) but local theory would yield the <u>same</u> growth rate as the triad theory with only an O(1) difference in the numerical factor.

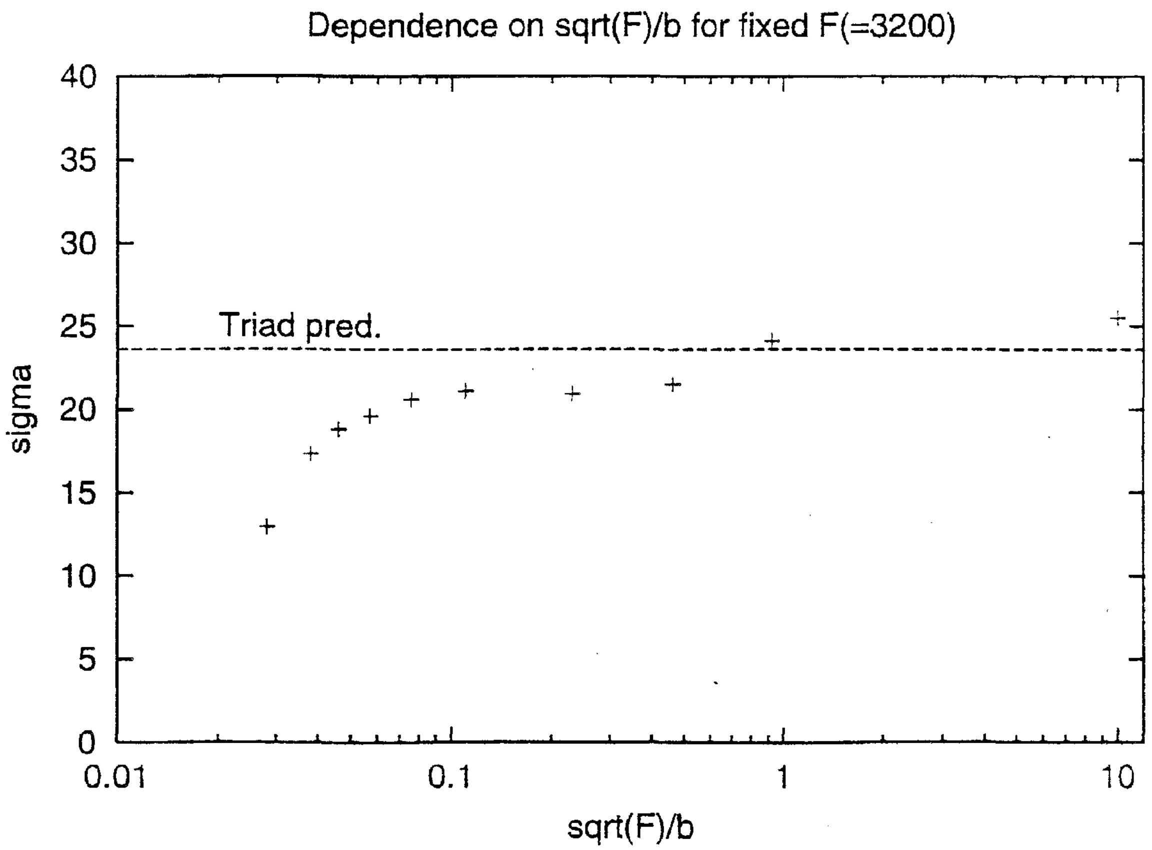
A detailed numerical calculation verifies these ideas.

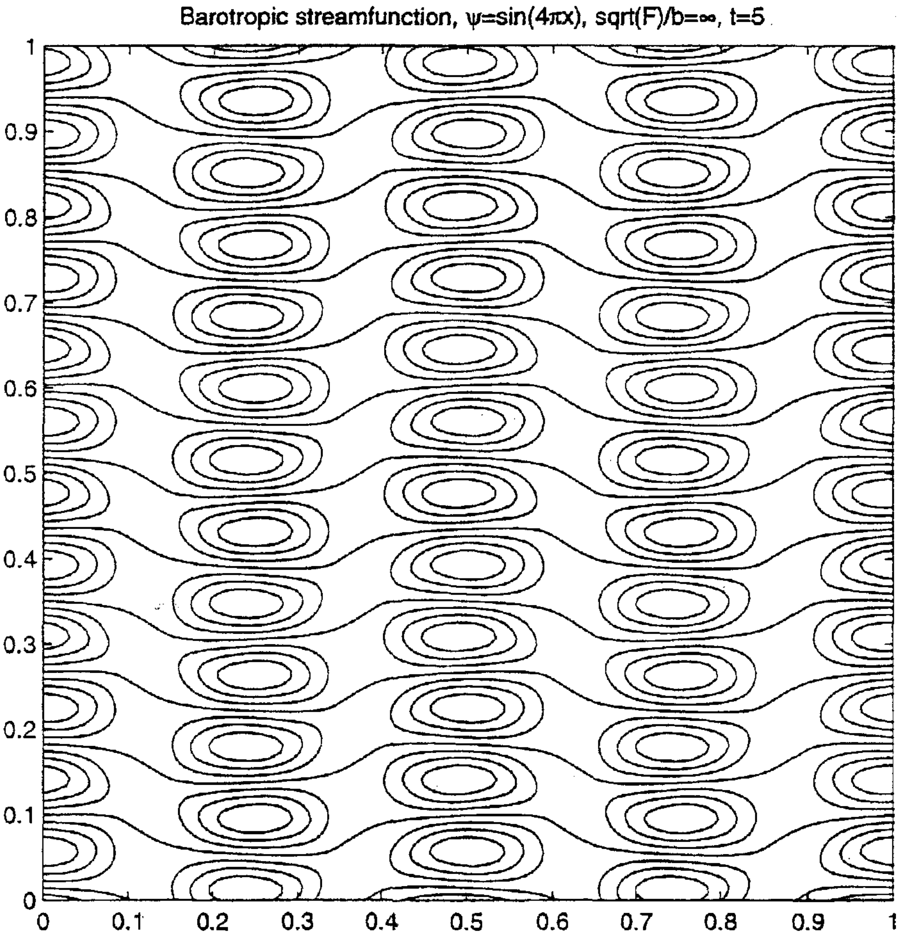

Rossby triad amplitudes, F = 1000 ko = -5.26 F 1/2/β = 0.31623
A1
Ao
A
2
1.5
1
0.5
0
-0.5
-1
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1
lo = 20.55 (h1, h2) = (0.5,0.5) k = 6.2832


A1
Ao
A

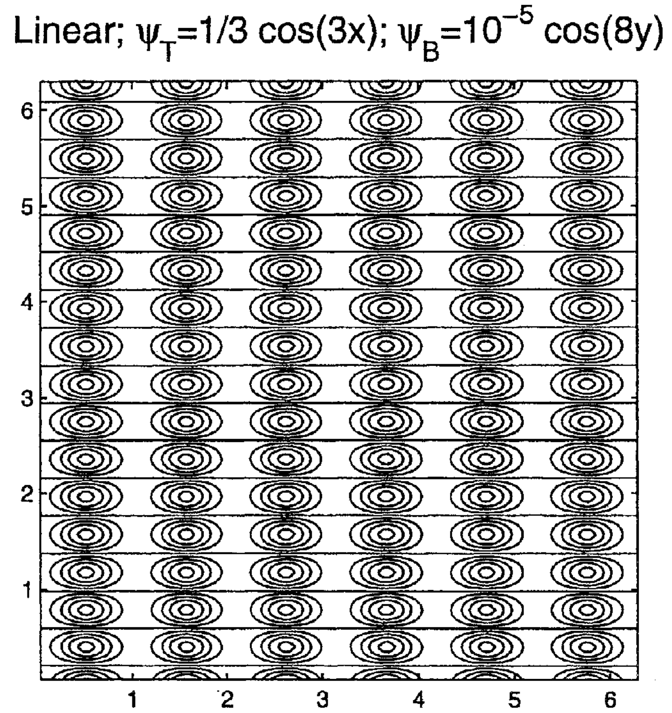
1/4

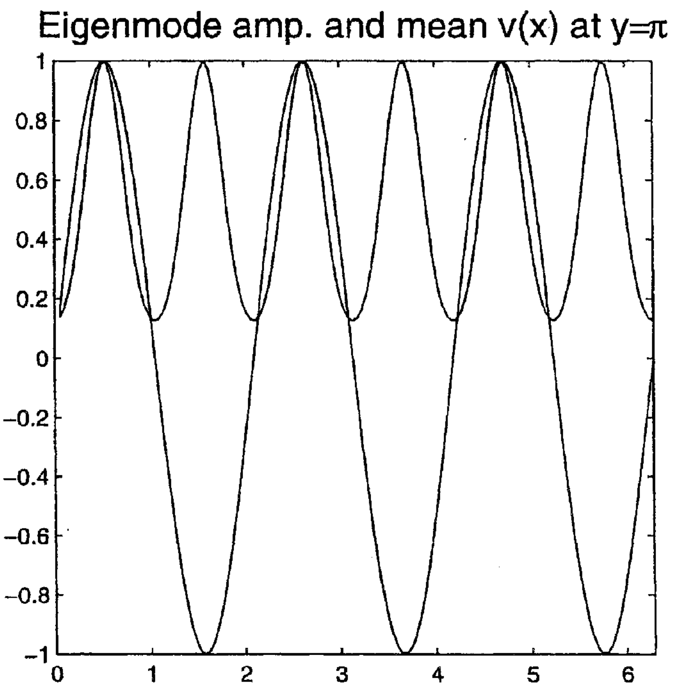

Growth as a function of Z; (k,l)=(7,0), F=448
Growth rate
Z
10
9
8
7
6
5
4
3
2
1
0
10^-2
10^-1
10^0
10^1

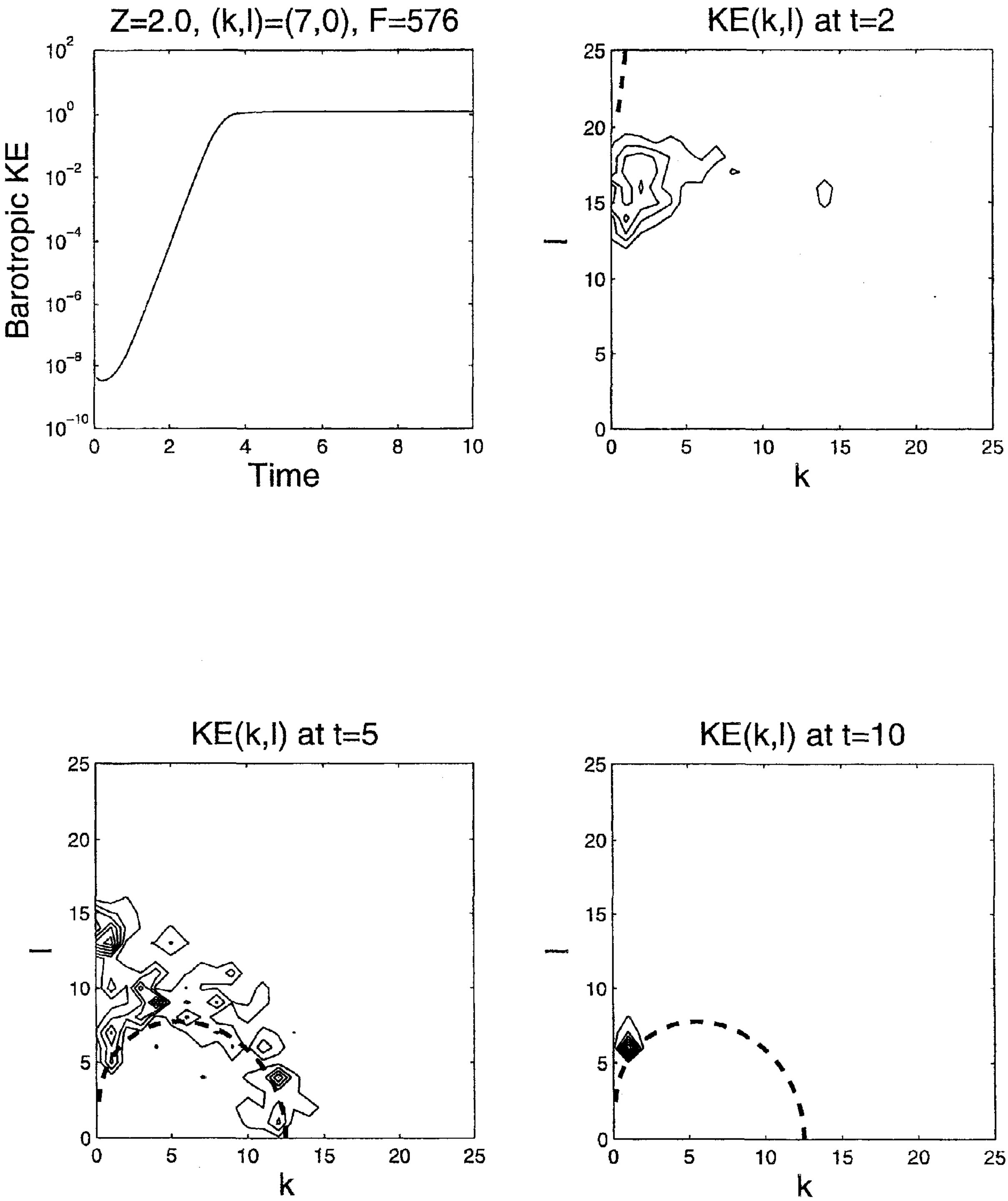

Dependence on sqrt(F)/b for fixed F(=3200)
40
35
30
25
20
15
10
5
0
sigma
Triad pred.
0.01
0.1
1
10
sqrt(F)/b


Barotropic streamfunction, ψ=sin(4πx), sqrt(F)/b=∞, t=5
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

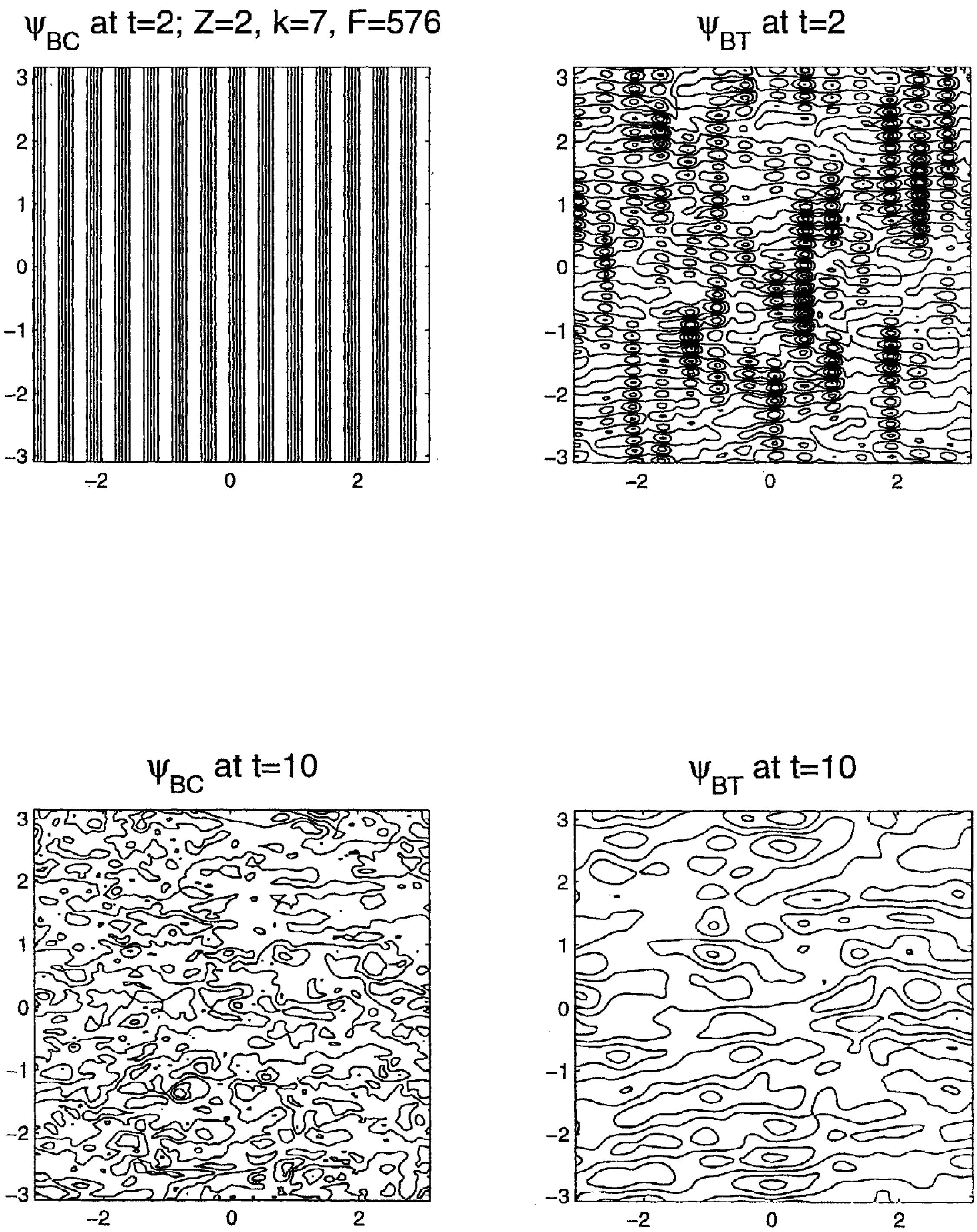
from Joe L.


Linear; ψT=1/3 cos(3x); ψB=10−5 cos(8y)
6
5
4
3
2
1
1
2
3
4
5
6

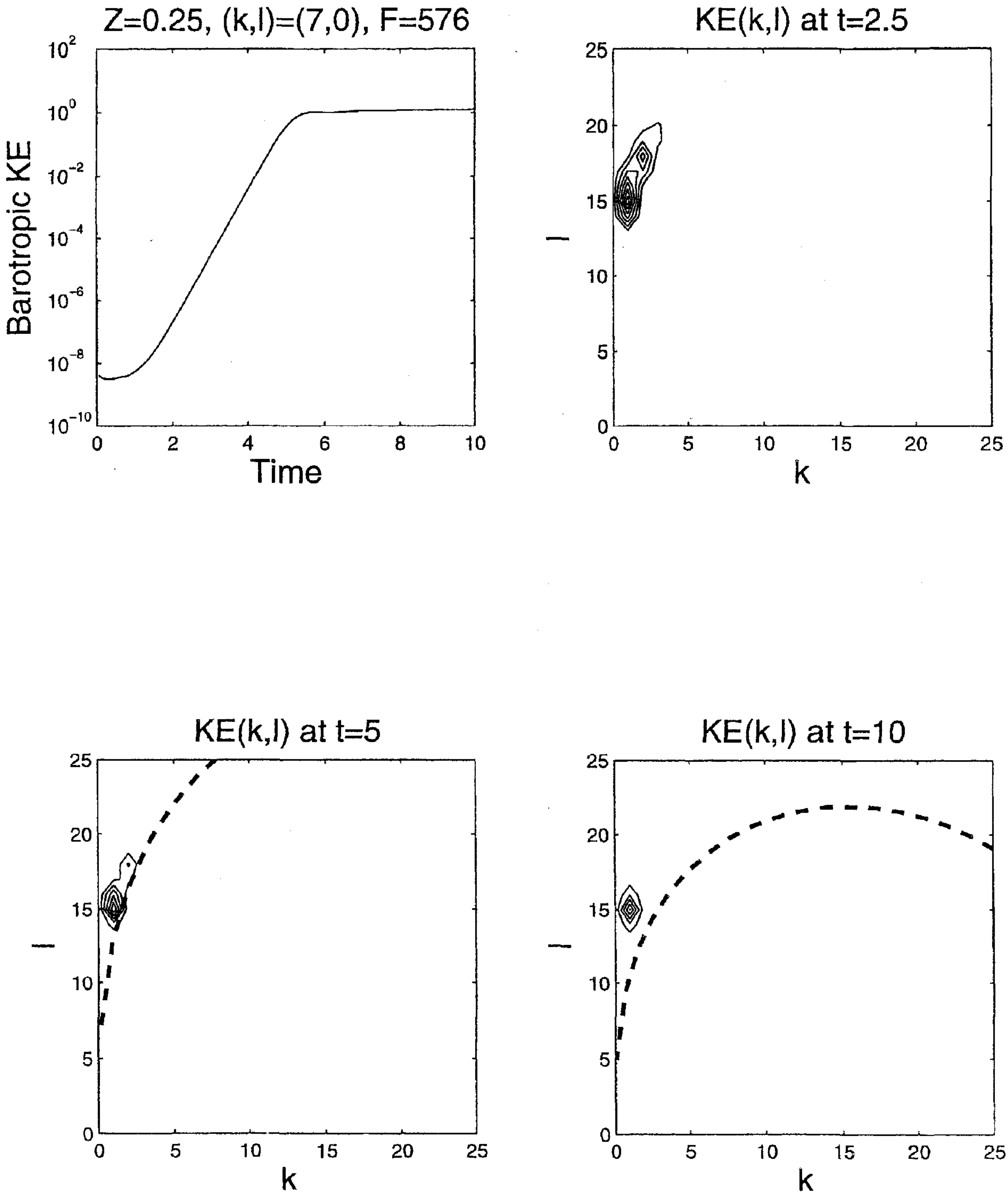

Eigenmode amp. and mean v(x) at y=π
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
0
1
2
3
4
5
6



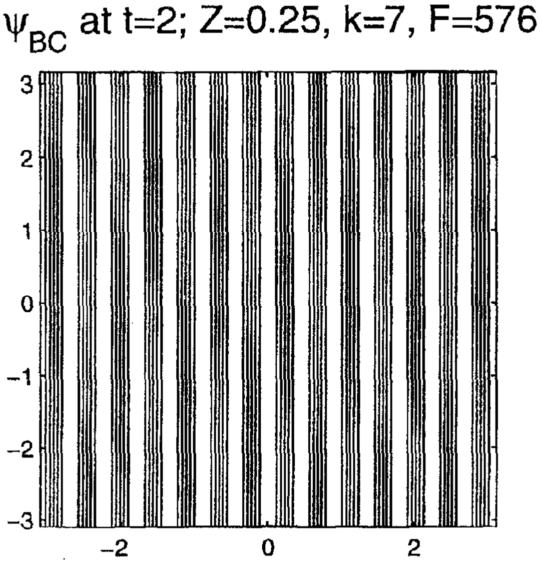
Z=2.0, (k,l)=(7,0), F=576
Barotropic KE
Time
KE(k,l) at t=2
KE(k,l) at t=5
KE(k,l) at t=10
k
l

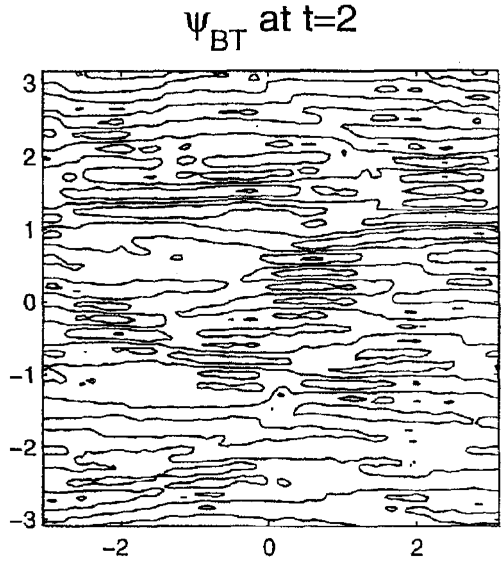


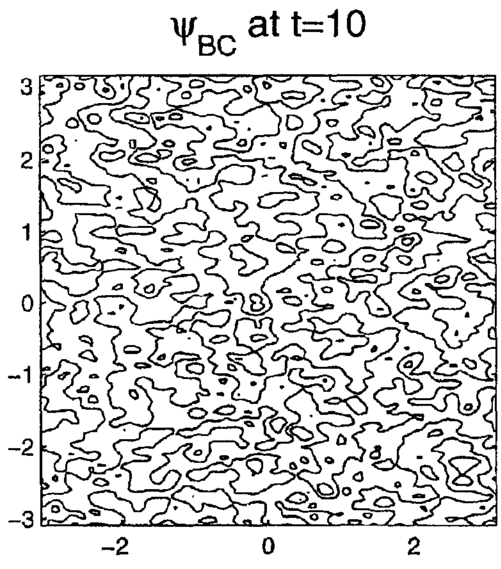
ψBC at t=2; Z=2, k=7, F=576
ψBT at t=2
ψBC at t=10
ψBT at t=10
3
2
1
0
−1
−2
−3
−2
0
2

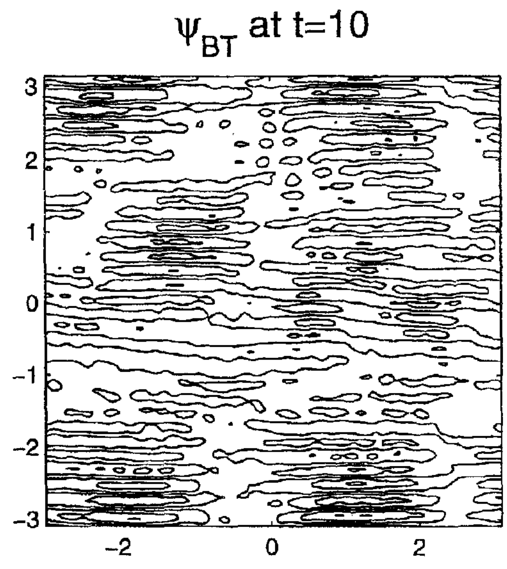


Z=0.25, (k,l)=(7,0), F=576
Barotropic KE
Time
KE(k,l) at t=2.5
KE(k,l) at t=5
KE(k,l) at t=10
k
l


ψ_{BC} at t=2; Z=0.25, k=7, F=576
3
2
1
0
−1
−2
−3
−2
0
2


ψ_{BT} at t=2
3
2
1
0
−1
−2
−3
−2
0
2


ψ_{BC} at t=10
3
2
1
0
−1
−2
−3
−2
0
2


ψ_{BT} at t=10
3
2
1
0
−1
−2
−3
−2
0
2

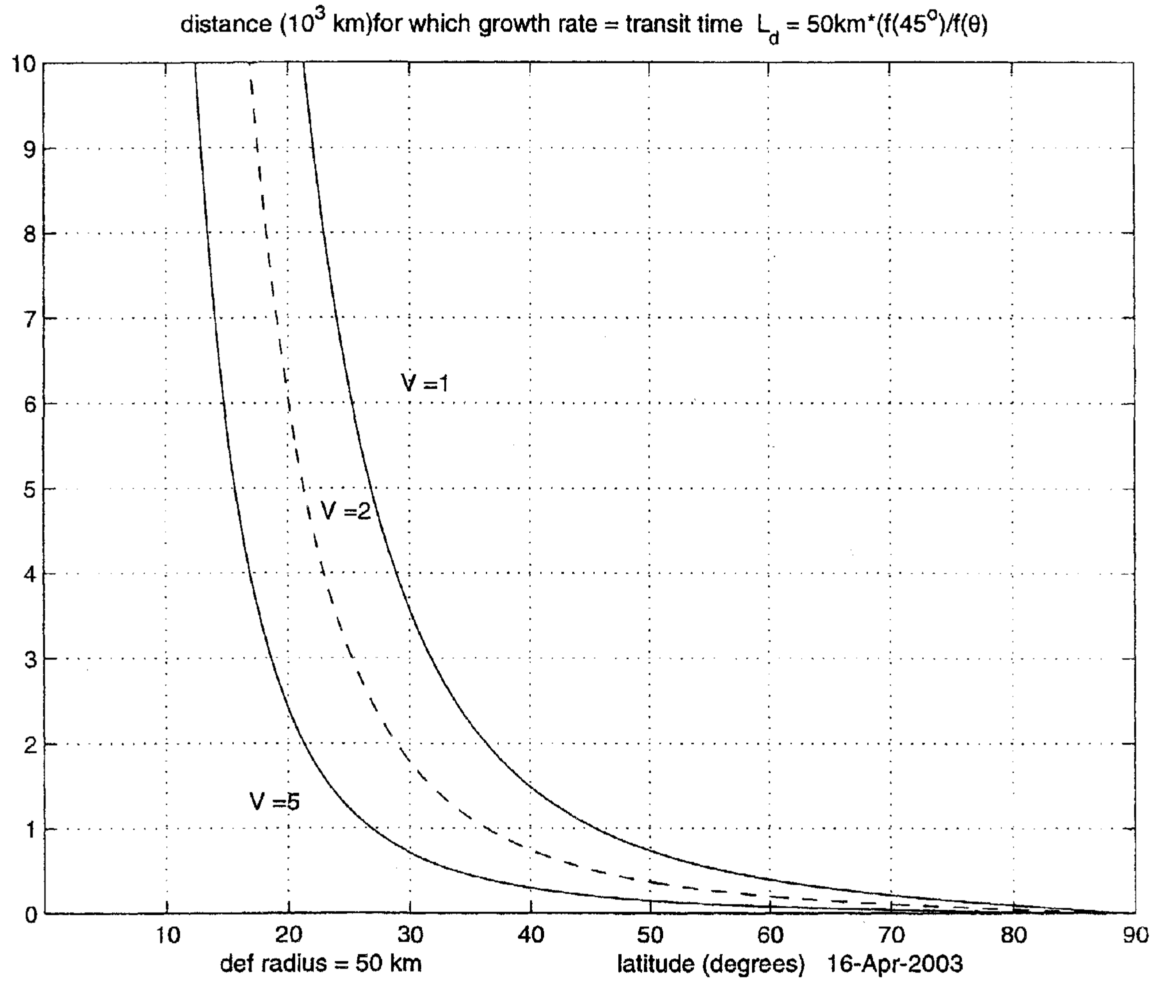
We have also run fully nonlinear calculations (doubly periodic domain) to see how the instability develops beyond linear instability. For large *Z* the instability changes structure as an inverse cascade in the *(k,l)* plane takes place, consistent with the ideas of Vallis. For small *Z* there is no turbulent cascade. In all cases the final scale is at the deformation radius.

Since the growth rate, σ, is of the order of V_o / L_d for all *Z* it follows that *Z* is a measure of the growth/per transit time.
Recall,

$$Z = F^{1/2} / b = \frac{LV}{\beta_{\dim} L_d{}^3}$$

If, for a given *L* , *Z*>1 this implies that the wave will suffer considerable growth of parasitic perturbations before it can transit the basin. On the other hand if *Z* <1 we can expect the wave to succeed in traversing the basin.

In quasi-geostrophic theory β, the deformation radius, and *f* are constants. If instead we take the bold step of including the latitude dependence of each of those parameters, we can ask, as a function of latitude what is the scale *L* for which *Z*=1. If we choose a deformation radius of 50 km at 45° N and use the known variation of the Coriolis parameter and β with latitude we obtain the following curves.



Each curve is for a different amplitude of the basic wave. What they all share in common is that for low latitudes the basin must be wide before there is substantial growth in a transit time while at mid and high latitudes the distance traversed before *Z*=1 is much less than a characteristic basin width. The stronger the wave the more true this is.

Our suggestion is that planetary scale Rossby waves can succeed in crossing a basin of the scale of the Pacific ocean only in low latitudes. In higher latitudes the wave will give much of its energy to parasitic instabilities of smaller scale.

This is consistent with the satellite observations mentioned at the beginning of the lecture of Chelton and Schlax in which the Rossby waves were seen most clearly at low latitudes and higher latitudes are dominated by eddies.

We further suggest that much of the eddy field at higher latitudes may actually <u>be due to the Rossby wave</u>. The remark has often been made that Rossby waves are artificial and that the ocean is instead dominated by eddies. We are suggesting that, in fact, it may be the Rossby waves that are the generators of the eddy field.