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Sensitivity Theory Methods for Nuclear Reactor Analysis

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SENSITIVITY THEORY METHODS FOR NUCLEAR REACTOR ANALYSIS

A. Gandini

1. Introduction

Since the beginning of nuclear reactor physics studies, perturbation theory has played an important role. As well known, it was first proposed in 1945 by Wigner *[1]* to study fundamental quantities such as the reactivity worths of different materials in the reactor core. It is also well known that this first formulation, today widely used by reactor analysts, makes a consistent use of the adjoint flux concept. The advantage of using perturbation theory lies in the fact that instead of making a new, often lengthy direct calculation of the eigenvalue (and then of the real flux) for every perturbed system configuration, a simple integration operation is required in terms of unperturbed quantities.

It is interesting that as early as 1948 Soodak *[2]* associated to the adjoint flux the concept of importance, viewing it as proportional to the contribution of a neutron, inserted in a given point of a critical system, to the asymptotic power. Along with the introduction of the concept of importance and, parallel to it, along with the development of calculation methods and machines, from the early 60' a flourishing of perturbation methods, at first in the linear domain and then in the nonlinear one, have been proposed for analysis of reactor core physics, shielding, thermohydraulics, as well as other fields. The perturbation formulations proposed by various authors may be subdivided into three main categories, according to the approach followed in their derivation:

1. The heuristic approach, making exclusive use of importance conservation concepts *[3-5]*. It will be referred to, in the following, as heuristic generalized perturbation theory (HGPT) method.
2. The variational approach *[6-10]*.
3. The differential method *[11-12]* based on a formal differentiation of the response considered.

Each of the above methods has its own merit, although all of them can be shown equivalent to each other *[13]*.

Here we shall discuss the potential applications of the HGPT methodology to the analysis of subcritical systems. A first indication of its potential use with respect to neutron kinetic analysis of critical and noncritical systems (with an external source) and to the possibility of analyzing integral experiments in reactor facilities at subcritical conditions was suggested in 1969 *[14]*. In particular, the neutron and precursor importances associated with a given response was considered.

Considering the increasing attention being given to the subcritical, accelerator driven systems (ADS), the application of the HGPT methodology for their cycle life analysis was proposed in 1997 *[15]* basing on a former procedure *[16,17]* developed for critical reactors. Here, we shall shortly review these works. In particular, the role will be discussed of the importance function associated with the power control, and the definition of the concept of "generalized reactivity", merging into the standard concept of reactivity with the system approaching criticality.

2. Theory

In the HGPT method the importance function is uniquely defined in relation to a given system response, for example, a neutron dose, the quantity of plutonium in the core at end of cycle, the temperature of the outlet coolant. The HGPT method was first derived in relation to the linear neutron density field. Then it was extended to other linear ones. For all these fields the equation governing the importance function was obtained directly by imposing that on average the contribution to the chosen response from a particle [a neutron, or a nuclide, or an energy carrier] introduced at a given time in a given phase space point of the system is conserved through time ("importance conservation principle"). Obviously such importance will result generally dependent on the time, position, and, when the case, energy and direction, of the inserted particle.

Consider a linear particle field density represented by vector **f** (e.g., the multigroup neutron density field) and a response Q of the type[+]

$$Q = \int_{t_o}^{t_F} < \mathbf{s}^+, \mathbf{f} > dt \equiv << \mathbf{s}^+, \mathbf{f} >> , \qquad (2.1)$$

where $\mathbf{s}^+$ is an assigned vector function and where < > indicate integration over the phase space. Weighting all the particles inserted into the system, let's assume a source **s**, with the corresponding importance ($\mathbf{f}^*$) will obviously give the response itself, i.e.,

$$<<\mathbf{f}^*, \mathbf{s}>> = Q = <<\mathbf{s}^+, \mathbf{f}>> , \qquad (2.2)$$

which represents an important reciprocity relationship.

From the first derivations mentioned above the rules for determining the equation governing the importance function $\mathbf{f}^*$ were learned (see in Appendix the derivation of this importance for the case relevant to the neutron field). They imply, in relation to the equation governing $\mathbf{f}^*$:

- change of sign of the odd derivatives,
- transposing matrix elements,
- reversing the order of operators,
- substitution of the source **s** with $\mathbf{s}^+$.

The first three rules will be generally called "operator reversal" rules.

The HGPT method was then extended to any field governed by linear operators for which the rules for their reversal were known. In particular, it was extended to the derivative fields, obtained from expanding to first order, around a given starting solution, a number of important nonlinear equations, as those governing:

- the coupled neutron/nuclide field, relevant to core evolution and control problems,
- the temperature field, relevant to thermohydraulics.

[+] Expression 2(5.1) is also representative of more general responses, of the type $Q = << L(\mathbf{f}) >>$, L being a given function of **f**. In fact, if we extend **f** to the field $\hat{\mathbf{f}} = \begin{vmatrix} \mathbf{f} \\ y \end{vmatrix}$, where y=L(**f**), Q reduces to the form of Eq.(1), i.e., $Q = <<\mathbf{s}^+, \hat{\mathbf{f}}) >>$, having set $\mathbf{s}^+ = \begin{vmatrix} \mathbf{0} \\ 1 \end{vmatrix}$.

2.1. General Formulation

Consider a generic physical model defined by a number of parameters p_j (j=1,2,...,J) and described by an N-component vector field **f** obeying a generally non-linear equation

$$\mathbf{m}(\mathbf{f}|\mathbf{p}) = \mathbf{0} \ . \qquad (2.3)$$

Vector **f**(**q**,t) generally depends on the phase space coordinates **q** and time t. Vector **p** represents the set of independent parameters p_j (j=1,2,...) fully describing the system and entering into Eq.(2.3). Their value generally determines physical constants, initial conditions, source terms, etc. Eq. (2.3) can be viewed as an equation comprising linear, as well as nonlinear, operators and is assumed to be derivable with respect to parameters p_j and (in the Frechet sense[+]) component functions f_n (n=1,2,...,N).

Consider now a response of interest, or functional Q given by Eq.(2.1). In the following, we shall look for an expression giving perturbatively the change δQ of the response Q in terms of perturbations δp_j of the system parameters. In particular, expressions giving the sensitivity coefficients relevant to each parameter p_j will be obtained.

Expanding equation (2.3) around a reference solution gives, setting $\mathbf{f}_{/j}=\frac{d\mathbf{f}}{dp_j}$, we obtain

$$\sum_{j=1}^{J} \delta p_j (H\,\mathbf{f}_{/j} + \mathbf{m}_{/j}) + \mathbf{O}_2 = \mathbf{0} \qquad (2.4)$$

where $\mathbf{O}_2$ is a second, or higher order term, and where $\mathbf{m}_{/j} = \frac{\partial \mathbf{m}}{\partial p_j}$.

Operator *H* is given by the expression

[+] A Frechet derivative corresponds to a formal derivation which, when applied to an expression m, function of a variable f, gives as result a linear operator (defined as $\bar{\partial} m / \partial f$) *[18]*. It coincides with a normal derivative if it is applied to an algebraic expression. If we call α a linear operator acting on a function f, the Frechet derivative of αf with respect to f is α. For example, if $\alpha \equiv$ grad, we shall have

$$\frac{\bar{\partial}(\alpha f)}{\partial f} \equiv \frac{\bar{\partial}(\mathrm{grad}\, f)}{\partial f} = \mathrm{grad}$$

In case $\alpha = \int dx\, K(x) f(x) = \int dx\, K(x)(\cdot)$, it is

$$\frac{\bar{\partial}(\alpha f)}{\partial f} \equiv \frac{\bar{\partial}}{\partial f} \int dx\, K\, f(x) = \int dx\, K\,(\cdot)$$

$$H = \begin{vmatrix} \frac{\bar{\partial} m_1}{\partial f_1} & \frac{\bar{\partial} m_1}{\partial f_2} & \cdots\cdots & \frac{\bar{\partial} m_1}{\partial f_N} \\ \frac{\bar{\partial} m_2}{\partial f_1} & \frac{\bar{\partial} m_2}{\partial f_2} & \cdots\cdots & \frac{\bar{\partial} m_2}{\partial f_N} \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ \frac{\bar{\partial} m_N}{\partial f_1} & \frac{\bar{\partial} m_N}{\partial f_2} & \cdots\cdots & \frac{\bar{\partial} m_N}{\partial f_N} \end{vmatrix} \tag{2.5}$$

Since parameters p_j, and then their changes δp_j, have been assumed independent from each other, it must follow

$$H\mathbf{f}_{/j} + \mathbf{m}_{/j} = \mathbf{0} \ , \tag{2.6}$$

which represents the (linear) equation governing the derivative functions $\mathbf{f}_{/j}$. The source term $\mathbf{m}_{/j}$ is here intended to account also, via appropriate delta functions, for the initial and, if the case, boundary conditions.

Consider now functional

$$Q_j = <<\mathbf{h}^+, \mathbf{f}_{/j}>> \ . \tag{2.7}$$

Introducing the importance ($\mathbf{f}^*$) associated with field $\mathbf{f}_{/j}$, if we use it as weight of the source term $\mathbf{m}_{/j}$, and integrate space- and time-wise, according to the source reciprocity relationship, Eq.(2.2), the resulting quantity will be equivalent to functional Q_j, i.e.,

$$Q_j = <<\mathbf{f}^*, \mathbf{m}_{/j}>> \ , \tag{2.8}$$

where the importance $\mathbf{f}^*$ obeys the (index-independent) equation

$$H^*\mathbf{f}^* + \mathbf{h}^+ = \mathbf{0} \ , \tag{2.9}$$

H^* being obtained by reversing operator H. As said above, this implies transposing matrix elements, changing sign of the odd derivatives, inverting the order of operators.

We can easily see that the sensitivities s_j (j=1,2,...,J) of system parameters can be written

$$s_j = \frac{dQ}{dp_j} = << \frac{\partial \mathbf{h}^+}{\partial p_j}, \mathbf{f}>> + <<\mathbf{f}^*, \frac{\partial \mathbf{m}}{\partial p_j} >> \ , \tag{2.10}$$

where the first term at the right-hand side represents the so called, easy to calculate, direct term.
The overall change δQ due to perturbations δp_j (j=1,2,...,J) of system parameters can be written, at first order,

$$\delta Q = \sum_{j=1}^{J} \delta p_j \ [<< \frac{\partial \mathbf{h}^+}{\partial p_j}, \mathbf{f}>> + <<\mathbf{f}^*, \frac{\partial \mathbf{m}}{\partial p_j} >>]. \tag{2.11}$$

It may occur, in certain circumstances, that one or more components (e.g., f_2) of the vector field **f** do not depend on a given space-time coordinate (e.g., x). Consistently with viewing components of **f** as (pseudo)-density functions, and without alteration of the problem specifications and results, this, or these variables may be interpreted as averaged, or integral quantities and then replaced by the proper averaging, or integral operator [e.g., $\frac{<\cdot>_{(x)}}{V_x}$, or $<\cdot>_{(x)}$] applied to the corresponding extended variable [so replacing, to exemplify, f_2 with $\frac{<\tilde{f}_2(x)>_{(x)}}{V_x}$, or, simply, $<\tilde{f}_2(x)>_{(x)}$]. These extended variables will then be assumed to depend also on this coordinate, although only their average, or integrated values with respect to it are of interest and no further specification for them is required. This rule is referred to as "coordinate dependence complementation". Its use is required in order that a correct operation reversal is made to obtain the operator governing the importance function. In particular, the above rule may be applied to those cases in which the response Q, rather than by Eq.(2.1), is given by an expression

$$Q = \int_{t_o}^{t_F} L(\hat{\mathbf{f}} \mid \mathbf{p}) dt \,, \tag{2.12}$$

$L(\hat{\mathbf{f}} \mid \mathbf{p})$ being given in terms of integral quantities [for instance, a ratio of the type $\frac{<\mathbf{w}_1^+, \hat{\mathbf{f}}>}{<\mathbf{w}_2^+, \hat{\mathbf{f}}>}$]. Consistently with the above complementation rule, we shall generally consider field **f** defined as $\begin{vmatrix} \hat{\mathbf{f}} \\ \tilde{y} \end{vmatrix}$, with variable $\tilde{y}$ such that $<\tilde{y}>= L(\hat{\mathbf{f}} \mid \mathbf{p})$. The standard expression of the response given by Eq.(2.1), will then apply. The governing Eq.(2.3) will correspondingly become

$$\mathbf{m}(\mathbf{f} \mid \mathbf{p}) \equiv \begin{vmatrix} \hat{\mathbf{m}}(\hat{\mathbf{f}} \mid \mathbf{p}) \\ <\tilde{y}> - L(\hat{\mathbf{f}} \mid \mathbf{p}) \end{vmatrix} = 0 \quad . \tag{2.13}$$

3. Neutron/nuclide field

To the neutron and fuel nuclide densities, represented by vectors **n**(**r**,t) and **c**(**r**,t), respectively, defined in the reactor cycle interval (t_o,t_F), a specified intensive control variable, ρ(t), is associated so that the assigned, overall power history W(t) is maintained. Vector **n** represents the space- and time-dependent neutron density in a multigroup energy form, whereas vector **c** the space- and time-dependent density of the various fuel nuclide species. The intensive, time-dependent, control variable ρ(t) may represent, for instance, the overall control rod bank penetration into the core (not their relative movement, which is generally described by parameters p_j), or, in a subcritical system, the extraneous source strength. The general, nonlinear governing equations can then be written formally as

$$\mathbf{m}_{(n)}(\mathbf{n},\mathbf{c},\rho|\mathbf{p}) \equiv - \frac{\partial \mathbf{n}}{\partial t} + B\,\mathbf{n} + \mathbf{s}_n = 0 \tag{3.1}$$

$$\mathbf{m}_{(c)}(\mathbf{n},\mathbf{c}|\mathbf{p}) \equiv -\frac{\partial \mathbf{c}}{\partial t} + E\mathbf{c} + \mathbf{s}_c = 0 \tag{3.2}$$

$$m_{(\rho)}(\mathbf{n},\mathbf{c}|\mathbf{p}) \equiv \langle \mathbf{c}, S\,\mathbf{n} \rangle - W = 0 \ , \tag{3.3}$$

where B is the neutron diffusion, or transport, matrix operator (depending on $\mathbf{c}$ and ρ), E the nuclide evolution matrix (depending on $\mathbf{n}$), $\mathbf{s}_n$ (generally, also depending on ρ) and $\mathbf{s}_c$ are given source terms[+] , while

$$S = \gamma \begin{vmatrix} \sigma^1_{f,1} & \dots & \sigma^1_{f,G} \\ \dots & \dots & \dots \\ \sigma^J_{f,1} & \dots & \sigma^J_{f,G} \end{vmatrix} V \tag{3.4}$$

γ being the amount of energy per fission, and $\sigma^j_{f,g}$ the microscopic g'th group fission cross-section of the j'th heavy isotope. V is the diagonal neutron velocity matrix. Quantities γ, V, W and $\sigma^j_{f,g}$ may be considered generally represented by (or function of) system parameters p_j. Source terms $\mathbf{s}_n$ and $\mathbf{s}_c$ are also parameter dependent.

In quasi-static problems, as those of interest here, the derivative $\partial \mathbf{n} / \partial t$ is negligible.

If we introduce the field

$$\mathbf{f}(\mathbf{r},t) = \begin{vmatrix} \mathbf{n}(\mathbf{r},t) \\ \mathbf{c}(\mathbf{r},t) \\ \rho(t) \end{vmatrix} \tag{3.5}$$

the system of Eqs..(3.1), (3.2) and (3.3) may be represented in the compact symbolic form, Eq.(2.3), and the HGPT methodology described above applied.

Consider a functional

$$Q = \int_{t_o}^{t_F} \left\langle \begin{vmatrix} \mathbf{s}_n^+ & \mathbf{s}_c^+ & s_\rho^+ \end{vmatrix} \begin{vmatrix} \mathbf{n}(\mathbf{r},t) \\ \mathbf{c}(\mathbf{r},t) \\ \rho(t) \end{vmatrix} \right\rangle dt \tag{3.6}$$

Q may represent, for instance, the amount of a given nuclide built up at time t_F [in this case $\mathbf{s}_n^+ = 0$, $s_\rho^+ = 0$ and $\mathbf{s}_c^+$ includes a delta function $\delta(t-t_F)$], or the control variable ρ at a final time t_F [in this case $\mathbf{s}_c^+ = 0$, $\mathbf{s}_n^+ = 0$ and $s_\rho^+ = \delta(t-t_F)$. The importance function

[+] For a critical system $\mathbf{s}_n$ is generally assumed zero during burnup, except a delta-like source at t_o for representing initial conditions (usually considered at steady state), whereas $\mathbf{s}_c$ is generally given by a sum of delta functions defined at t_o and at given times to account for fuel feed and shuffling operations.

$$\mathbf{f}^*(\mathbf{r},t)=\begin{vmatrix}\mathbf{n}^*(\mathbf{r},t)\\ \mathbf{c}^*(\mathbf{r},t)\\ \rho^*(t)\end{vmatrix} \tag{3.7}$$

can then be defined, and results governed by Eq.(2.9), with H^* and $\mathbf{h}^+$ given by the general expression:

$$H^* = \begin{vmatrix} \left(\frac{\partial}{\partial t}+B^*\right) & \Omega_c^* & S^T\mathbf{c} \\ \Omega_n^* & \left(\frac{\partial}{\partial t}+E^T\right) & S\mathbf{n} \\ <\left[\mathbf{n}^T\left(\frac{\partial B}{\partial \rho}\right)^*+\frac{\partial s_n^T}{\partial \rho}\right](\cdot)> & 0 & 0 \end{vmatrix} \tag{3.8}$$

$$\mathbf{h}^+=\begin{vmatrix}\mathbf{s}_n^+(\mathbf{r},t)\\ \mathbf{s}_c^+(\mathbf{r},t)\\ s_\rho^+(t)\end{vmatrix} \tag{3.9}$$

Ω_c^* and Ω_n^* being operators adjoint$_c$ of the coupling terms Ω_c [$\equiv \frac{\overline{\partial}(E\,\mathbf{c})}{\partial \mathbf{n}}$] and Ω_n [$\equiv \frac{\overline{\partial}(B\,\mathbf{n})}{\partial \mathbf{c}}$], respectively.

The equation relevant to function ρ^* corresponds to a relationship between $\mathbf{n}^*$ and $\mathbf{n}$, i.e., .

$$<\mathbf{n}^*\left(\frac{\partial B}{\partial \rho}\right),\mathbf{n}>+<\mathbf{n}^*,\frac{\partial s_n}{\partial \rho}>=s_\rho^+ \tag{3.10}$$

To solve the equations relevant to $\mathbf{n}^*$ and $\mathbf{c}^*$ different resolution recurrent schemes may be considered, starting from the 'final' time t_F and proceeding backward, along with the same time discretization adopted in the forward reference calculation.

The sensitivity coefficient $\frac{dQ}{dp_j}$ with respect to a given parameter p_j may then be obtained from Eq.(2.10), with vector $\mathbf{m}$ made of components $\mathbf{m}_{(n)}$, $\mathbf{m}_{(c)}$ and $m_{(\rho)}$ defined in Eqs.(3.1), (3.2) and (3.3), respectively.

A general problem we are faced with is the following: how does the control reset (ρ) strategy affect the sensitivity analysis results? To answer this question, for simplicity limiting consideration to critical systems, let us consider Eq. (3.11) governing $\mathbf{n}^*$. We note that, given a particular solution $\mathbf{n}^*_{part}$, the general one may be written as

$$\mathbf{n}^* = \mathbf{n}^*_{part}+\alpha\boldsymbol{\varphi}^* \tag{3.11}$$

where α is an arbitrary coefficient and $\boldsymbol{\varphi}^*$ the standard adjoint function obeying the homogeneous equation

$$B^* \boldsymbol{\varphi}^* = 0 \tag{3.12}$$

Once a solution $\mathbf{n}^*_{\text{part}}$ has been obtained, the solution desired can then be derived by proper filtering from the fundamental mode by imposing condition (3.10), in this case reduced to $< \mathbf{n}^T (\partial B/\partial \rho)^* \mathbf{n}^* >= s^+_\rho$ and assuming $s^+_\rho = 0$ (usually the term s^+_ρ corresponds to a delta function). It results

$$\mathbf{n}^* = \mathbf{n}^*_{\text{part}} - \frac{< \mathbf{n}^*_{\text{part}}, \frac{\partial B}{\partial \rho} \mathbf{n} >}{< \boldsymbol{\phi}^*, \frac{\partial B}{\partial \rho} \mathbf{n} >} \boldsymbol{\varphi}^* . \tag{3.13}$$

The dependence of the importance function $\mathbf{n}^*$ on the control mode adopted is evident.
When calculating the sensitivity coefficient of a response Q with respect to a given parameter p_j (or its change δQ with respect to parameter alterations δp_j), the filtering of the importance function as shown in Eq.(3.13) corresponds to *implicitly* accounting for the ρ-mode control reset of the criticality.

The above result may have important implications, in the sense that in many circumstances, prior to a sensitivity study, it may be necessary to consider the proper reactivity control mode to be adopted. On the other hand, within many existing codes used with the HGPT methodology, the fictitious "λ-mode" reset control is implicitly assumed, i.e., that related to the coefficient (eigenvalue) λ multiplying the fission source term ($F\mathbf{n}$) in the transport (or diffusion) equation. In this circumstance expression (3.13) for the importance $\mathbf{n}^*$ will result, recalling that in this case $\frac{\partial B}{\partial \lambda} = F$,

$$\mathbf{n}^* = \mathbf{n}^*_{\text{part}} - \frac{< \mathbf{n}^*_{\text{part}}, F \mathbf{n} >}{< \boldsymbol{\phi}^*, F \mathbf{n} >} \boldsymbol{\varphi}^* \tag{3.14}$$

Using this λ-mode filtering, rather than the correct ρ-mode one, may lead to erroneous results.

Consider, for instance, the case of a sensitivity analysis with respect to core breeding, or conversion ratio, a quantity clearly dependent on the neutron energy spectrum. Assuming that the reactivity compensation, corresponding to a system parameter (for instance, the initial fuel enrichment) change, is effected, as it may very well be the case for a thermal reactor, by an alteration of the average (boron) poison concentration in the coolant, the correct choice of the control mode reset would clearly have the effect of hardening (if boron is added), or softening (if boron is subtracted) the neutron spectrum. Instead, if a λ-mode reset would have been implicitly adopted (as is often done with existing codes), no significant neutron energy shift would have been taken into account, and, consequently, an erroneous sensitivity coefficient would result.

It is also true that in principle one could calculate separately the amount of control poison (referring to the above example) to reset the criticality and consider the overall parameter plus control change

along with the λ-mode methodology. But this would imply a reactivity reset calculation to be performed for each parameter considered. On the other hand, the correct fundamental ρ-mode filtering may be a quite straightforward procedure. In fact, it can be effected "a posteriori" adopting expression (3.13) in which $\mathbf{n}^{*}_{\text{part}}$ would correspond to a preliminary λ-mode calculation with an existing code.

3.1. Source Driven Systems

One of the advantages often claimed for the subcritical source driven power systems is associated to the fact that the power level may be basically controlled by the source strength (via the regulation of the accelerator current). So, no regulating elements would be necessary, if a sufficient breeding is available (and/or an appropriate core burnable poison distribution is provided at the beginning of cycle) in the core for compensating the reactivity loss during burnup. To the neutron and fuel nuclide densities, a specified intensive source control variable, ρ(t), is then associated so that the assigned, overall power history W(t) is maintained. Eqs. (3.1), (3.2) and (3.3) may be properly written now as

$$\mathbf{m}_{(n)}(\mathbf{n},\mathbf{c},\rho \mid \mathbf{p}) = -\frac{\partial \mathbf{n}}{\partial t} + B\mathbf{n} + \rho \mathbf{s}_n = 0 \tag{3.15}$$

$$\mathbf{m}_{(c)}(\mathbf{n},\mathbf{c} \mid \mathbf{p}) = -\frac{\partial \mathbf{c}}{\partial t} + E\mathbf{c} + \mathbf{s}_c = 0 \tag{3.16}$$

$$\mathbf{m}_{(\rho)}(\mathbf{n},\mathbf{c} \mid \mathbf{p}) = <\mathbf{c},S\mathbf{n}> - W = 0 \tag{3.17}$$

Since we generally consider systems at quasi-static, i.e., stationary conditions, the time derivative at second member of Eq.(3.15) may be neglected in the course of the integration process.
Any response, functional of variables **n**, **c**, and ρ, could be considered for analysis. We think instructive to limit here consideration to the response defined by the expression

$$Q = \rho(t_F) \equiv \int_{t_o}^{t_F} \delta(t - t_F)\rho(t)dt \tag{3.18}$$

which corresponds to the relative source strength required at t_F to assure the power level imposed. We may assume that, at unperturbed conditions, ρ (t)=1 in the interval (t_o,t_F). If some system parameter (for instance, the initial enrichment, or some other material density) is altered, as in an optimization search analysis, it may be of interest to evaluate the corresponding change of ρ at the end of cycle, to make sure that given upper limit specifications of the source strength are non exceeded.

Along with the HGPT methodology, the equations for the corresponding importance functions result

$$-\frac{\partial \mathbf{n}^*}{\partial t} = B^*\mathbf{n}^* + \Omega_c^*\mathbf{c}^* + S^T\mathbf{c}\rho^* \tag{3.19}$$

$$-\frac{\partial \mathbf{c}^*}{\partial t} = E^*\mathbf{n}^* + \Omega_n^*\mathbf{n}^* + S\mathbf{n}\rho^* \tag{3.20}$$

$$<\mathbf{n}^*, \mathbf{s}_n> + \delta(t-t_F) = 0 \tag{3.21}$$

Eq. (3.21) corresponds to an orthonormal condition for $\mathbf{n}^*$.

In order to determine the 'final' value $\mathbf{n}^*(t_F)$ required for starting the integration of Eq. (3.19), in consideration of the nature of the above governing equations, we shall first write $\mathbf{n}^*$ and ρ^* in the general form[+]

$$\mathbf{n}^*(\mathbf{r},t) = \mathbf{n}_F^*\ \delta(t - t_F) + \tilde{\mathbf{n}}^*(\mathbf{r},t) \tag{3.22}$$

$$\rho^*(t) = \rho_F^*\ \delta(t - t_F) + \tilde{\rho}^*(t) \tag{3.23}$$

with $\tilde{\mathbf{n}}^*(\mathbf{r},t)$ and $\tilde{\rho}^*(t)$ being finite functions, vanishing at t_F.

Replacing into Eq. (3.19), integrating in the interval $(t_F - \varepsilon,\ t_F + \varepsilon)$, and then making $\varepsilon \to 0$, we obtain the equation

$$B^*\mathbf{n}_F^* + S^T\mathbf{c}(t_F)\rho_F^* = 0 \tag{3.24}$$

Let us now define $\bar{\mathbf{n}}_F^*$ as obeying equation

$$B^*\bar{\mathbf{n}}_F^* + S^T\mathbf{c}(t_F) = 0 \tag{3.25}$$

We note that $\bar{\mathbf{n}}_F^*$ corresponds to the importance relevant to functional $<\mathbf{c}(t_F), S\mathbf{n}(t_F)>$, i.e., to the system power W. From the source reciprocity relationship (Section 2), we may write

$$<\mathbf{c}(t_F), S\mathbf{n}(t_F)>, = <\bar{\mathbf{n}}_F^*, \mathbf{s}_n> = W\ . \tag{3.26}$$

From constraint, Eq. (3.21), we easily obtain

[+] The diverging of $\mathbf{n}^*(\mathbf{r},t)$ at t_F may be explained on physical grounds recalling the meaning of importance (in this case, the contribution to the given response by a neutron with the same space/time coordinates) and considering that the response here is $\rho(t_F)$, i.e., the control assumed to maintain the power at a prefixed level. A neutron introduced at t_F into the system would in fact produce a (delta-like) impulse of control ρ to balance its effect on the power level. Then, the importance associated to such neutron would be characterized by a similar delta-like behavior. A quite similar reasoning applies in relation to the diverging of importance $\rho^*(t)$ at t_F, considering that its physical meaning corresponds to the contribution to the response [defined as $\rho(t_F)$] due to a unit energy insertion at t_F or, which is the same, to an overall power pulse $\delta(t-t_F)$.

$$\rho_F^* = -\frac{1}{< \bar{\mathbf{n}}_F^*, \mathbf{s}_n >} = -\frac{1}{W} \tag{3.27}$$

and then

$$\mathbf{n}_F^* = \bar{\mathbf{n}}_F^* \rho_F^* = -\frac{\bar{\mathbf{n}}_F^*}{W} . \tag{3.28}$$

From this 'final' value, a recurrent calculation scheme may be defined starting from t_F and proceeding backward.

Along the HGPT methodology, the sensitivity coefficient relevant to the k'th parameter p_k is found as

$$\begin{aligned} \frac{\partial \rho(t_F)}{\partial p_k} = \rho_F^* [& < \bar{\mathbf{n}}_F^* \frac{\partial}{\partial p_k} (B\mathbf{n} + \mathbf{s}_n) > + \frac{\partial}{\partial p_k} (< \mathbf{c}, S\mathbf{n} > - W)]_{t_F} \\ & + \int_{t_o}^{t_F} [< \tilde{\mathbf{n}}^*, \frac{\partial}{\partial p_k} (B\mathbf{n} + \mathbf{s}_n) > + < \mathbf{c}^*, \frac{\partial E}{\partial p_k} \mathbf{c} > + \tilde{\rho}^* \frac{\partial}{\partial p_k} (< \mathbf{c}, S\mathbf{n} > - W)] dt \end{aligned} \tag{3.29}$$

with ρ_F^* given by Eq. (3.27).

Rather than on the source term, a control on the neutron absorption in the multiplying region could be of interest. In this case, the (intensive) control variable ρ would represent the average penetration of the control elements, or the average density of the soluble boron in the coolant, and then would enter into the (transport, or diffusion) operator B. The orthonormal condition for the neutron importance $\mathbf{n}^*$ would now be, rather than Eq. (3.21),

$$< \mathbf{n}^*, \frac{\partial B}{\partial \rho} \mathbf{n} > + \delta(t - t_F) = 0 \ . \tag{3.30}$$

In this case, the sensitivity coefficient with respect to a given parameter p_k would always be given by Eq. (3.29), with $\bar{\mathbf{n}}_F^*$ obeying Eq. (3.25), but with

$$\rho_F^* = -\frac{1}{< \bar{\mathbf{n}}_F^*, \frac{\partial B}{\partial \rho} \mathbf{n} >} . \tag{3.31}$$

In general, a control strategy, by which an automatic resetting of the imposed overall power is actuated, might imply a control intervention on both the neutron source strength and the absorbing elements within the multiplying region. In this case, ρ (which remains a unique, intensive control variable) would affect both operator *B* and the neutron source [in this latter case, via an appropriate ρ- and parameter dependent coefficient $\alpha(\rho \mid \mathbf{p})$, assumed unity at unperturbed conditions]. The distribution between these two control mechanisms could be described by appropriate parameters (subject to perturbation analysis). The sensitivity coefficient, in this case, with respect to a given parameter p_k would always be given by Eq. (3.29), with $\bar{\mathbf{n}}_F^*$ obeying Eq. (3.25), but with

$$\rho_F^* = -\frac{1}{< \bar{\mathbf{n}}_F^*, (\frac{\partial B}{\partial \rho}\mathbf{n} + \frac{\partial \alpha}{\partial \rho}\mathbf{s}_n) >} . \tag{3.32}$$

3.2. Stationary Case

To study a given subcritical system at stationary conditions (which may be interpreted at the beginning of its cycle life), we may consider the same system above in which the neutron source and the nuclide density are assumed time-independent during an arbitrary time interval (t_o,t_B). We assume that at t_o the neutron density ($\mathbf{n}_o$), as well as the control (ρ_o) have already reached stationary conditions. So, also these two quantities are time-independent in the same time interval. Their governing equations can then be written, in case the power level is controlled by the source strength,

$$B\mathbf{n}_o + \rho_o \mathbf{s}_{n,o} = 0 \tag{3.33}$$

$$<\mathbf{c}_o, S\mathbf{n}_o> - W_o = 0 . \tag{3.34}$$

Also here we shall assume that at unperturbed conditions ρ_o =1.

The same equations derived previously are applicable to this case, with the advertence of replacing t_F with t_B and setting the coupling operators Ω_c^* and Ω_n^* appearing in Eqs. (3.19) and (3.20) equal to zero. The sensitivity coefficient of the response ρ_o [$\equiv\rho(t_B)$] relevant to the k'th parameter p_k can then be obtained. Since in this case $\mathbf{c}^*$, as well as $\tilde{\mathbf{n}}^*(\mathbf{r},t)$ and $\tilde{\rho}^*(\mathbf{r},t)$ vanish, recalling Eq. (3.29), we obtain

$$\frac{\partial \rho_o}{\partial p_k} = \rho_o^* [< \mathbf{n}_o^*, \frac{\partial}{\partial p_k}(B\mathbf{n}_o + \mathbf{s}_{n,o}) > + \frac{\partial}{\partial p_k}(< \mathbf{c}_o, S\mathbf{n}_o > - W_o)] \tag{3.35}$$

where

$$\rho_o^* = -\frac{1}{W_o} \tag{3.36}$$

and $\mathbf{n}_o^*$ obeys equation

$$B^*\mathbf{n}_o^* + S^T\mathbf{c}_o = 0 \quad . \tag{3.37}$$

If, rather than via the source strength, the power level reset control is assumed to be regulated via neutron absorption, so that the control ρ_o would enter into operator B, the sensitivity coefficient would be given always by Eq. (3.35), but with

$$\rho_o^* = -\frac{1}{< \mathbf{n}_o^*, \frac{\partial B}{\partial \rho}\mathbf{n} >} . \qquad (3.38)$$

We might as well consider a (fictitious) control mechanism affecting the fission source, rather than the neutron absorption, i.e., we might choose as control a coefficient multiplying the fission matrix (F) and, therefore, entering into the Boltzmann, or diffusion, operator B (=A+ρ_oF). The sensitivity coefficient would be given again by Eq. (3.35), but with

$$\rho_o^* = -\frac{1}{< \mathbf{n}_o^*, F\mathbf{n}_o >} . \qquad (3.39)$$

3.3. Reactivity of Subcritical Systems

For resetting the power level, we have considered above different control mechanisms to which the following types of equations governing the neutron density may be associated:

$$B(\mathbf{p})\mathbf{n}_o + \rho_o \mathbf{s}_{n,o}(\mathbf{p}) = 0 \qquad \text{(source control)} \qquad (3.40)$$

$$B(\rho_o|\mathbf{p})\mathbf{n}_o + \mathbf{s}_{n,o}(\mathbf{p}) = 0 \qquad \text{(neutron absorption, or fission control)} \qquad (3.41)$$

where the control and parameter dependence is indicated.

The sensitivity expression (3.35) may be generalized so that

$$\frac{d\rho_o}{dp_j} = -\frac{< \mathbf{n}_o^*, \frac{\partial \mathbf{m}_{(n,o)}}{\partial p_j} > + \frac{\partial}{\partial p_j}(< \mathbf{c}_o, S\mathbf{n}_o > - W_o)}{< \mathbf{n}_o^*, \frac{\partial \mathbf{m}_{(n,o)}}{\partial \rho_o} >} , \qquad (3.44)$$

with $\mathbf{n}_o^*$ obeying Eq. (3.37).

A corresponding perturbation expression may now be obtained. Assuming that the power W_o appearing in Eq. (3.44) is not subject to perturbation, we may write:

$$\delta\rho_o = -\frac{< \mathbf{n}_o^*, \delta\mathbf{m}_{(n,o)} > + < \mathbf{n}_o, \delta(S^T\mathbf{c}_o) >}{< \mathbf{n}_o^*, \frac{\partial \mathbf{m}_{(n,o)}}{\partial \rho_o} >} , \qquad (3.45)$$

where $\delta\mathbf{m}_{(n,o)} = \sum_j \delta p_j \frac{\partial \mathbf{m}_{(n,o)}}{\partial p_j}$ and $\delta(S^T\mathbf{c}_o) = \sum_j \delta p_j \frac{\partial (S^T\mathbf{c}_o)}{\partial p_j}$.

As said previously, $\delta\rho_o$ corresponds to the control change necessary to reestablish the power level existing before the perturbation $\delta\mathbf{m}_{(n,o)}$. We may as well say that the perturbation $\delta\mathbf{m}_{(n,o)}$ [and

$\delta(S^T\mathbf{c}_o)]$ would produce a power level change equivalent to that produced by a control change δK_ρ given by the equation

$$\delta K_\rho = \frac{< \mathbf{n}_o^*, \delta\mathbf{m}_{(n,o)} > + < \mathbf{n}_o, \delta(S^T\mathbf{c}_o) >}{< \mathbf{n}_o^*, \dfrac{\partial\mathbf{m}_{(n,o)}}{\partial\rho_o} >}. \tag{3.46}$$

In the case of the (fictitious) control on the neutron fission, setting λ in place of ρ to distinguish this peculiar case, we may explicitly write

$$\delta K_\lambda = \frac{< \mathbf{n}_o^*, \delta B\mathbf{n}_o >}{< \mathbf{n}_o^*, F\mathbf{n}_o >} + \frac{< \mathbf{n}_o^*, \delta\mathbf{s}_{n,o} >}{< \mathbf{n}_o^*, F\mathbf{n}_o >} + \frac{< \mathbf{n}_o, \delta(S^T\mathbf{c}_o) >}{< \mathbf{n}_o^*, F\mathbf{n}_o >}. \tag{3.47}$$

The first term at the right side closely resembles the reactivity expression normally used for critical systems[+]. So, we shall denote quantity δK_λ given by expression (3.47) as 'generalized reactivity'. The second term may be defined the "source reactivity", whereas the last one a "direct effect". To account for a generic ρ-mode control mechanism, we shall extend this definition to δK_ρ, similarly defined by Eq. (3.46), i.e., explicitly:

$$\delta K_\rho = \frac{< \mathbf{n}_o^*, \delta B\mathbf{n}_o >}{< \mathbf{n}_o^*, \dfrac{\partial\mathbf{m}_{(n,o)}}{\partial\rho_o} >} + \frac{< \mathbf{n}_o^*, \delta\mathbf{s}_{n,o} >}{< \mathbf{n}_o^*, \dfrac{\partial\mathbf{m}_{(n,o)}}{\partial\rho_o} >} + \frac{< \mathbf{n}_o, \delta(S^T\mathbf{c}_o) >}{< \mathbf{n}_o^*, \dfrac{\partial\mathbf{m}_{(n,o)}}{\partial\rho_o} >}. \tag{3.48}$$

and call it ‘generalized ρ-mode reactivity’.

3.4. The multiplication factor of the subcritical core

Let us first write the homogeneous “critical” equation corresponding to the inhomogeneous one, Eq. (3.3), relevant to a subcritical system, in the form:

$$A\mathbf{n}_{hom} + \frac{1}{k_{eff}} F\mathbf{n}_{hom} = \mathbf{0}, \tag{3.49}$$

where coefficient $1/k_{eff}$, k_{eff} being the multiplication factor, is introduced to "restore" the neutron balance.

The standard adjoint function $\boldsymbol{\phi}^*$ is then given by equation :

$$A^*\boldsymbol{\phi}^* + \frac{1}{k_{eff}} F^*\boldsymbol{\phi}^* = 0. \tag{3.50}$$

[+] The first term at right hand side of Eq. (3.47) can be demonstrated to formally approach the standard reactivity expression as the (reference) system considered gets close to criticality conditions (Gandini, 1997).

For systems not too far from criticality conditions $\mathbf{n}_{hom}$ may be assumed as an approximation of $\mathbf{n}_o$. Multiplying equation (3.49) by $\boldsymbol{\phi}^*$ and space integrating, recalling equation (3.33), we may then write:

$$k_{eff} = - \frac{< \phi^*, F\, \mathbf{n}_{hom} >}{< \phi^*, A\, \mathbf{n}_{hom} >} \cong \frac{< \boldsymbol{\phi}^*, F\, \mathbf{n}_{hom} >}{< \boldsymbol{\phi}^*, \mathbf{s} > + < \boldsymbol{\phi}^*, F\, \mathbf{n}_{hom} >} . \tag{3.51}$$

To improve (3.51), by taking into account the change in shape of the flux, a different definition of the subcriticality has been proposed, introducing a "k-source" coefficient (k_s). The procedure consists in considering an integral balance condition obtained by integrating equation (3.5), with $\mathbf{n}_{hom}$ replaced by $\mathbf{n}$ and k_{eff} by coefficient k_s, given by the ratio

$$k = - \frac{< \mathbf{u}, F\, \mathbf{n} >}{< \mathbf{u}, A\, \mathbf{n} >}, \tag{3.52}$$

$\mathbf{u}$ being a unit vector. Recalling that $A\mathbf{n} = -(F\mathbf{n} + \mathbf{s})$, we obtain:

$$k_s = \frac{< \mathbf{u}, F\, \mathbf{n} >}{< \mathbf{u}, F\, \mathbf{n} > + < \mathbf{u}, \mathbf{s} >} . \tag{3.53}$$

This new definition does not account for the difference of "importance" between the neutrons generated by fission and the "importance" of the source neutrons.

A more satisfactory definition of the subcriticality may be proposed, introducing the importance function $\mathbf{n}_o^*$ defined previously, i.e.,:

$$k_{sub} = \frac{< \mathbf{n}_o^*, F\, n >}{< \mathbf{n}_o^*, \mathbf{s} > + < \mathbf{n}_o^*, F\, n >} \tag{3.54}$$

which takes into account both the inhomogeneous flux distribution and the importance of neutrons with respect to the relevant "observable" of the system (i.e. its power level).

It may be shown that when approaching criticality, equation (3.54) becomes equal to equation (3.51), as required on physical grounds.

Appendix

Let us consider the generic transport equation, with obvious notation,

$$\frac{dn}{dt} = - \boldsymbol{\Omega} \cdot \text{grad}\, n - \Sigma_t\, vn(\mathbf{r}, \boldsymbol{\Omega}, E|t) + \int_{4\pi} d\boldsymbol{\Omega}' \int_0^\infty \Sigma_s(E' \to E, \boldsymbol{\Omega}' \to \boldsymbol{\Omega}) vn(\mathbf{r}, E', \boldsymbol{\Omega}'| t) dE' + \frac{\chi(E)}{4\pi} \int_{4\pi} d\boldsymbol{\Omega}' \int_0^\infty \nu\Sigma_f(E') vn(\mathbf{r}, E', \boldsymbol{\Omega}'| t) dE' + s_n \tag{A.1}$$

The boundary conditions are obtained from physical considerations. Assuming that the system is isolated, i.e., comprehending all its neutron sources, and that external boundary surfaces are convex, it will be:

Flux $\phi(\mathbf{r}, \mathbf{\Omega}, E|t)=0$ for directions of $\mathbf{\Omega}$ entering in the medium.

Let us consider now in a the interval (t_o,t_F) a generic functional

$$Q = \int_{t_o}^{t_F} dt \int_{4\pi} d\mathbf{\Omega} \int_0^{\infty} dE \int_{\text{ist}} d\mathbf{r} h^+ n(\mathbf{r}, E, \mathbf{\Omega}) = << h^+ n >> \ . \qquad (A.2)$$

with vector function h^+ given. The notation << >> here means integration over space and time.

For times $t < t_F$, we may write the balance equation governing the importance function. Let us see closer the mechanisms by which a neutron of coordinates $(\mathbf{r}, E, \mathbf{\Omega})$ gives and gains importance

At the beginning it will have an amount of importance which we shall denote as $n^*(\mathbf{r}, E, \mathbf{\Omega})$. After a time Δt the following events will occur:

a) The neutron has reached point $\mathbf{r}' = \mathbf{r} + \mathbf{\Omega}\Delta s$, where

$$\Delta s = v\Delta t$$

keeping the same velocity. The probability for the neutron of arriving at $\mathbf{r}'$ is given by the quantity.

$$\left(1 - \frac{\Delta s}{l_t(E)}\right)$$

where

$$l_t(E) = \frac{1}{\Sigma_t(E)}$$

which corresponds to the mean free path of the neutron without undergoing any type of collision.

b) The neutron undergoes a scattering collision with change of energy and angle, respectively, from E into the interval dE' around E' and from $\mathbf{\Omega}$ into the interval $d\mathbf{\Omega}'$ around $\mathbf{\Omega}'$. This occurs with probability:

$$\frac{\Delta s}{l_t} \frac{\Sigma_s(E \to E', \mathbf{\Omega} \to \mathbf{\Omega}')dE'd\mathbf{\Omega}'}{\Sigma_t} = \Delta s \Sigma_s(E \to E', \mathbf{\Omega} \to \mathbf{\Omega}')dE'd\mathbf{\Omega}'$$

which corresponds to the product of the probability that during the interval Δs the neutron undergoes a collision with and that the collision is a scattering one.

e) The neutron undergoes a fission collision. In analogy with the scattering, the probability that a fission neutron emerges in the interval energy dE' around E' and within d$\boldsymbol{\Omega}$' around $\boldsymbol{\Omega}$' is givn by the ratio:

$$\Delta s \nu \Sigma_f(E) \frac{\chi(E')}{4\pi} dE' d\boldsymbol{\Omega}' \ .$$

e) During the interval Δs, the neutron contributes to the response Q equal to

$$h^+(\tilde{\mathbf{r}}, E, \boldsymbol{\Omega}, \tilde{t})\Delta t \ .$$

f) The neutron undergoes a parassitic capture. In such case it simply disappears from the system.

To the neutrons so emerged after a time Δt we may associate the values of the importance function associated with the coordinates which characterize such neutrons. The events to be accounted for are the first four ones. For the importance conservation principle, the sum of the importances relevant to each possible event must be equal to that of the starting neutron. It will then be, recalling that $\Delta s/\Delta t = v$,

$$\begin{aligned} n^*(\mathbf{r}, E, \boldsymbol{\Omega}, t) &= (1 - \Sigma_t \Delta s) n^*(\mathbf{r} + \Delta s \boldsymbol{\Omega}, E, \boldsymbol{\Omega}, t + \Delta t) \\ &+ \Delta s \int_{4\pi} d\boldsymbol{\Omega}' \int_0^\infty dE' \Sigma_s(E \to E', \boldsymbol{\Omega} \to \boldsymbol{\Omega}') n^*(\tilde{\mathbf{r}}, E', \boldsymbol{\Omega}', \tilde{t}) \\ &+ \nu \Sigma_f(E) \frac{\Delta s}{4\pi} \int_{4\pi} d\boldsymbol{\Omega}' \int_0^\infty dE' \chi(E') n^*(\tilde{\mathbf{r}}, E', \boldsymbol{\Omega}', \tilde{t}) + h^+(\tilde{\mathbf{r}}, E, \boldsymbol{\Omega}, \tilde{t}) \frac{\Delta s}{v} \end{aligned} \qquad (A.3)$$

where $\tilde{\mathbf{r}}$ represents a point in the ($\mathbf{r}$, $\mathbf{r}$+$\boldsymbol{\Omega}\Delta s$).

Adding and subtracting at the first member of equations (14.33) $n^*(\mathbf{r}, E, \boldsymbol{\Omega}, t + \Delta t)$ and dividing by Δs, at the first member there will be the incremental ratios

$$\frac{n^*(\mathbf{r} + \Delta s \boldsymbol{\Omega}, E, \boldsymbol{\Omega}, t + \Delta t) - n^*(\mathbf{r}, E, \boldsymbol{\Omega}, t + \Delta t)}{\Delta s} \underset{\Delta t \to 0}{\longrightarrow} \boldsymbol{\Omega} \operatorname{grad} n^*(\mathbf{r}, E, \boldsymbol{\Omega}, t)$$

and

$$-\frac{1}{v} \frac{n^*(\mathbf{r}, E, \boldsymbol{\Omega}, t + \Delta t) - n^*(\mathbf{r}, E, \boldsymbol{\Omega}, t)}{\Delta t} \underset{\Delta t \to 0}{\longrightarrow} -\frac{1}{v} \frac{\partial n^*}{\partial t} \ .$$

Making $\Delta t \to 0$, we shall then obtain the equation governing the importance function:

$$\begin{aligned} -\frac{dn^*}{dt} &= \boldsymbol{\Omega} v \operatorname{grad} n^* - \Sigma_t v n^* + v \int_{4\pi} d\boldsymbol{\Omega}' \int_0^\infty dE' \Sigma_s(E \to E', \boldsymbol{\Omega} \to \boldsymbol{\Omega}') n^*(\mathbf{r}, E', \boldsymbol{\Omega}') \\ &+ \frac{\nu \Sigma_f(E) v}{4\pi} \int_{4\pi} d\boldsymbol{\Omega}' \int_0^\infty dE' \chi(E') n^*(\mathbf{r}, E', \boldsymbol{\Omega}') + h^+ \end{aligned} \qquad (A.4)$$

As may be easily verified, this equation may be obtained from that relevant to neutron density by changing the sign of the first derivatives, exchanging the arguments ($E' \to E, \boldsymbol{\Omega}' \to \boldsymbol{\Omega}$) with

$(E \to E', \boldsymbol{\Omega} \to \boldsymbol{\Omega}')$, respectively, and, similarly, for what concerns the fission source, substituting to $\chi(E)\nu\Sigma_f(E')$ the product $\chi(E')\nu\Sigma_f(E)$. In other terms, we may say that the importance function is symmetrical to the real density, this implying a reversion of the operators. This symmetry is reflected also in relation to the boundary conditions. As well known, the boundary conditions associated with the real density, in case of an isolated system, are:

$n(\mathbf{r}, E, \boldsymbol{\Omega}) = 0$ for $\mathbf{r}$ on the external boundary and $\boldsymbol{\Omega}$ directed inside the system (assumed having a convex external surface).

On the countrary, the boundary conditions relevant to the importance function are:

$n^*(\mathbf{r}, E, \boldsymbol{\Omega}) = 0$ for $\mathbf{r}$ on the external boundary and $\boldsymbol{\Omega}$ directed outside (assumed having a convex external surface).

This condition is obtained considering that the contribution to the response from a neutron escaping from the system is clearly null.

In general, we may define the general principle of symmetry between the real flux and importance function, according to which all the properties valid for the flux are also valid for the asjoint function, provided that the sense of energy, angular, space and time variations are reversed.

Diffusion approximation

Let us consider now the multigroup equation in diffusion theory:

$$\frac{dn_i}{dt} = v_i D_i \nabla^2 n_i - v_i n_i \Sigma_{t,i} + \sum_{j=1}^{i} v_j n_j \Sigma^o_{s,j\to i} + \chi_i \sum_{j=1}^{N} v_j n_j \nu\Sigma_{f,j} \qquad (A.5)$$

Basing on the previous arguments, the importance function relevant to the corresponding response expressed in vector form as $Q = << \mathbf{h}^+, \mathbf{n} >>$, will be:

$$-\frac{dn_i^*}{dt} = v_i D_i \nabla^2 n_i^* - v_i n_i^* \Sigma_{t,i} + v_i \sum_{j=i}^{N} n_j^* \Sigma^o_{s,i\to j} + v_i \nu\Sigma_{f,i} \sum_{j=1}^{N} n_j^* \chi_j + h_i^+ \qquad (A.6)$$

In this case, since the laplacian ∇^2 corresponds do a double derivation in space, its sign doesn't change with respect to the real case, while in the terms of the sum their indeces i,j are exchanged.

A significant simplification of the notation is obtained by writing the equations in vector representation by introducing matrix operators. In particular, for equations (A.5) and (A.6), relevant to the real flux and the importance function, respectively, we may define the following operators:

$$B = A + F \qquad (A.7)$$

$$B^* = A^* + F^* \qquad (A.8)$$

where

$$A = \begin{vmatrix} (D_1\nabla^2 - \Sigma_{t,1} + \Sigma_{1\to 1}) & 0 & \dots & 0 \\ \Sigma_{1\to 2} & (D_2\nabla^2 - \Sigma_{t,2} + \Sigma_{2\to 2}) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \Sigma_{1\to N} & \Sigma_{2\to N} & \dots & (D_N\nabla^2 - \Sigma_{t,N} + \Sigma_{N\to N}) \end{vmatrix} \tag{A.9}$$

$$F = \begin{vmatrix} \nu\Sigma_{f,1}\chi_1 & \nu\Sigma_{f,2}\chi_1 & \dots & \nu\Sigma_{f,N}\chi_1 \\ \nu\Sigma_{f,1}\chi_2 & \nu\Sigma_{f,2}\chi_2 & \dots & \nu\Sigma_{f,N}\chi_2 \\ \dots & \dots & \dots & \dots \\ \nu\Sigma_{f,1}\chi_N & \nu\Sigma_{f,2}\chi_N & \dots & \nu\Sigma_{f,N}\chi_N \end{vmatrix} \tag{A.10}$$

$$A^* = \begin{vmatrix} (D_1\nabla^2 - \Sigma_{t,1} + \Sigma_{1\to 1}) & \Sigma_{1\to 2} & \dots & \Sigma_{1\to N} \\ 0 & (D_2\nabla^2 - \Sigma_{t,2} + \Sigma_{2\to 2}) & \dots & \Sigma_{2\to N} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & (D_N\nabla^2 - \Sigma_{t,N} + \Sigma_{N\to N}) \end{vmatrix} \tag{A.11}$$

$$F^* = \begin{vmatrix} \nu\Sigma_{f,1}\chi_1 & \nu\Sigma_{f,1}\chi_2 & \dots & \nu\Sigma_{f,1}\chi_N \\ \nu\Sigma_{f,2}\chi_1 & \nu\Sigma_{f,2}\chi_2 & \dots & \nu\Sigma_{f,2}\chi_N \\ \dots & \dots & \dots & \dots \\ \nu\Sigma_{f,N}\chi_1 & \nu\Sigma_{f,N}\chi_2 & \dots & \nu\Sigma_{f,N}\chi_N \end{vmatrix}, \tag{A.12}$$

and the following vectors

$$\mathbf{n} = \begin{vmatrix} n_1 \\ n_2 \\ : \\ n_N \end{vmatrix}, \qquad \mathbf{n}^* = \begin{vmatrix} n_1^* \\ n_2^* \\ : \\ n_N^* \end{vmatrix}. \tag{A.13}$$

Equations (A.5) e (A.6) may then be written in the compact form

$$\frac{d\mathbf{n}}{dt} = BV\mathbf{n} \tag{A.14}$$

$$-\frac{d\mathbf{n}^*}{dt} = VB^*\mathbf{n}^* + \mathbf{h}^+ . \tag{A.15}$$

To note that the elements off the diagonal of matrices (A.9) and (A.11) correspond to scattering transfer macroscopic cross-sections, while the elements of (A.10) e (A.12) correspond to fission macroscopic ones, multiplied by the number of secondaries per fission. To note also that for obtaining matrix A^* from A, matrix F^* from F, and then matrix B^* from B, rows and columns are exchanged, which corresponds to exchanging group indices i,j.

In this case the boundary condition for the importance function remains the same as that for the real flux, i.e., it vanishes at the extrapolated length.

Writing the above equations in terms of the neutron flux $\boldsymbol{\phi}$ (=V$\mathbf{n}$), we have

$$V^{-1}\frac{d\boldsymbol{\phi}}{dt} = B\boldsymbol{\phi} \qquad (A.16)$$

$$-V^{-1}\frac{d\mathbf{n}^*}{dt} = B^*\mathbf{n}^* + V^{-1}\mathbf{h}^+. \qquad (A.17)$$

To note that equations (A.15) and (A.17), relevant to the importance function, are equivalent.

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